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ON THE GEOMETRY OF THE TANGENT BUNDLE WITH VERTICAL RESCALED METRIC

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Abstract. Let (M, g) be a n-dimensional smooth Riemannian manifold. In the present paper, we introduce a new class of natural metrics denoted by G^f and called the vertical rescaled metric on the tangent bundle TM. We calculate its Levi-Civita connection and Riemannian curvature tensor. We study the geometry of (TM, G^f) and several important results are obtained on curvature, Einstein structure, scalar and sectional curvatures.

1. Introduction

In differential geometry, the tangent bundle TM have a great importance in smooth manifold M and in many areas of mathematics and physics. The metric g on M gives rise to several natural metrics on the tangent bundle.

In [17], Sasaki introduced a metric g^s on TM based on a natural splitting of the tangent bundle TTM into vertical and horizontal subbundles by means of the Levi-Civita connection ∇ on (M, g), but in most cases the study of some geometric properties of the tangent bundle endowed with this metric led to the flatness of the base manifold (see [10],[11] and [13]).

Cheeger-Gromoll in [6] introduced a natural metric on TM which it was expressed more explicitly in [13] and its geometry are studied in ([9],[10],[18] and [23]). The authors in ([1], [2] and [3]) defined and studied the g-natural metric on TM.

Zayatuev in [21] introduced a Riemannian metric on TM, given by

$$\begin{aligned} i)\ {}^s g_f(X^h, Y^h) &= f(p) g_p(X, Y) \\ ii)\ {}^s g_f(X^v, Y^h) &= 0 \\ iii)\ {}^s g_f(X^v, Y^v) &= g_p(X, Y) \end{aligned}$$

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for all vector fields X and Y on (M,g), where f is strictly positive smooth function on (M,g). In [19], J.Wang and Y. Wang called ${}^s g_f$ the rescaled Sasaki metric and studied the geometry of TM endowed with ${}^s g_f$.

Motivated by the above studies, we define a new class of naturally metric on TM, given by

$$\begin{aligned} i)\ G^f_{(p,u)}(X^h,Y^h) &= g_p(X,Y) \\ ii)\ G^f_{(p,u)}(X^v,Y^h) &= 0 \\ iii)\ G^f_{(p,u)}(X^v,Y^v) &= f(p)g_p(X,Y) \end{aligned}$$

for some strictly positive smooth function f in (M,g) and any vector fields X and Y on M. For $f \equiv 1$, the metric G^f is exactly the Sasaki metric. We call G^f the vertical rescaled metric.

This paper is presented as follow, In the section 2, we lay the basis for this work about basic notions and definitions of horizontal and vertical lifts.

In the section 3, we introduce by the Definition 2 a new class of natural metrics denoted by G^f and called the vertical rescaled metric on TM. Its Levi-Civita and the curvature tensor are given in the Propositions 3 and 4, respectively.

Finally, the section 4 is devoted to the geometry of TM endowed with the metric G^f and obtain several important geometry consequences on curvature properties, Einstein structure, sectional curvature and scalar Curvature.

2. Preliminaries

Let (M,g) be a n-dimensional pseudo-Riemannian manifold and ∇ be the Levi-Civita connection. We denote the natural projection of TM to M by $\pi(\xi) = \pi(p,u) = p$.

The tangent space $T_{\tilde{p}}TM$ of TM at $\xi = (p,u) \in TM$, splits into the horizontal and vertical subspaces H_ξ and V_ξ with respect to ∇,

$$T_\xi TM = H_\xi \oplus V_\xi$$

For a vector field X on M, the horizontal lift of X to a point p in TM is the unique vector $X^h \in H_\xi$ given by

$$\pi^*(X^h) = X$$

and the vertical lift of X to ξ is the unique vector $X^v \in V_\xi$ such that

$$X^v(df) = X(f)$$

for all smooth functions f on M. For each system of local coordinates $(x_i)_{i=1,..,n}$ in M, one defines, in the standard way, the system of local coordinates $(x_i,u_j)_{i,j=1,..,n}$ in TM. Let $X = X^i \frac{\partial}{\partial x_i}$ be a local vector fields on M, the vertical and the horizontal lifts of X are defined by

$$X^v = X^i \frac{\partial}{\partial u_i} \text{ and } X^h = X^i \frac{\partial}{\partial x_i} - X^i u^j \Gamma^k_{ij} \frac{\partial}{\partial u_k}.$$

The map $X \to X^h$ is an isomorphism between the vector spaces T_pM and $H_{(p,u)}$. Similarly, the map $X \to X^v$ is an isomorphism between T_pM and $V_{(p,u)}$ (for more details see [7], [11], [13] and [16]). Each tangent vector $\tilde{X} \in T_{(p,u)}(TM)$ can be written in the form

$$\tilde{X} = X^h + Y^v$$

where $X, Y \in \mathcal{X}(M)$ are uniquely determined vectors. The Lie bracket of vertical and horizontal vector fields on TM is given by the following proposition.

Proposition 1. *Let (M, g) be a pseudo-Riemannian manifold, ∇ be a Levi-Civita connection and R be a curvature tensor. Then the Lie bracket on the tangent bundle TM satisfies the following identities:*
i) $[X^v, Y^v] = 0$,
ii) $[X^h, Y^v] = (\nabla_X Y)^v$,
iii) $[X^h, Y^h] = ([X, Y])^h - (R(X, Y)u)^v$
for all $X, Y \in \mathfrak{X}(M)$ and $\xi = (p, u) \in TM$.

3. Vertical rescaled metric

In this section, we define a new class of a natural metric on the tangent bundle called the vertical rescaled metric and denoted by G^f where f is some a strictly positive smooth function in M. We calculate its Levi-Civita connection ∇^f and its Riemannian curvature tensor R^f.

Definition 1. *Let (M, g) be a Riemannian manifold and f be a strictly positive smooth function. We define the vertical rescaled metric G^f on the tangent bundle TM by*

$$\begin{cases} i)\ G^f_{(p,u)}(X^h, Y^h) = g_p(X, Y), \\ ii)\ G^f_{(p,u)}(X^v, Y^h) = 0, \\ iii)\ G^f_{(p,u)}(X^v, Y^v) = f(p) g_p(X, Y), \end{cases}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$ and $\xi = (p, u) \in TM$.

The vertical rescaled metric is contained in the class of g-natural metrics, for $f = 1$, it follows that $G^f = g^s$. The metric G^f is constructed in such a way that the vertical and horizontal sub-bundles are orthogonal and the bundle map $\pi : (TM, G^f) \to (M, g)$ is Riemannian submersion and induces a norm on each tangent space of TM denoted by $\|.\|$.

3.1. Levi-Civita connection of G^f.

Proposition 2. *Let (M, g) be a n-dimensional Riemannian manifold and ∇^f be a Levi-Civita connection of (TM, G^f). Then, we have*

$$\begin{aligned}
i)(\nabla^f_{X^h} Y^h)_{(p,u)} &= (\nabla_X Y)^h_{(p,u)} - \frac{1}{2}(R_p(X,Y)u)^v \\
ii)(\nabla^f_{X^h} Y^v)_{(p,u)} &= \frac{1}{2} f(p)(R_p(u,Y)X)^h + ((\nabla_X Y)_p + \frac{X(f)}{2f(p)} Y)^v \\
iii)(\nabla^f_{X^v} Y^h)_{(p,u)} &= \frac{1}{2} f(p)(R_p(u,X)Y)^h + \frac{Y(f)}{2f(p)} X^v \\
iv)(\nabla^f_{X^v} Y^v)_{(p,u)} &= -\frac{1}{2} g(X,Y)(\sharp df)^h
\end{aligned}$$

for any $X, Y \in C^\infty(TM)$, $\xi = (p, u) \in TM$ and $\sharp df$ is the gradient of f.

Proof. Using the Koszul formula for ∇^f

$$\begin{aligned}
2G^f(\nabla^f_{X_i} Y_j, Z_k) &= X_i(G^f(Y_j, Z_k)) + Y_j(G^f(Z_k, X_i)) - Z_k(G^f(X_i, Y_j)) \\
&\quad -G^f(X_i, [Y_j, Z_k]) + G^f(Y_j, [Z_k, X_i]) + G^f(Z_k, [X_i, Y_j])
\end{aligned}$$

for all vector fields $X, Y, Z \in C^\infty(TM)$ and $i, j, k \in \{h, v\}$, the Proposition 1 and the Definition 2, then we have for (ii)

$$\begin{aligned}
2G^f(\nabla^f_{X^h} Y^v, Z^h) &= X^h(G^f(Y^v, Z^h)) + Y^v(G^f(Z^h, X^h)) - Z^h(G^f(X^h, Y^v)) \\
&\quad -G^f(X^h, [Y^v, Z^h]) + G^f(Y^v, [Z^h, X^h]) + G^f(Z^h, [X^h, Y^v]) \\
&= G^f(Y^v, [Z, X]^h - (R(Z, X)u)^v) = f(p)G^f((R(u, Y)X)^h, Z^h)
\end{aligned}$$

and

$$\begin{aligned}
2G^f(\nabla^f_{X^h} Y^v, Z^v) &= X^h(G^f(Y^v, Z^v)) + Y^v(G^f(Z^v, X^h)) - Z^v(G^f(X^h, Y^v)) \\
&\quad -G^f(X^h, [Y^v, Z^v]) + G^f(Y^v, [Z^v, X^h]) + G^f(Z^v, [X^h, Y^v]) \\
&= X(f(p)g(Y, Z)) - G^f(Y^v, (\nabla_X Z)^v) + G^f(Z^v, (\nabla_X Y)^v) \\
&= \frac{X(f)}{f(p)} fg(Y, Z) + 2f(p)g(\nabla_X Y, Z) = G^f(\frac{X(f)}{f(p)} Y^v + 2(\nabla_X Y)^v, Z^v)
\end{aligned}$$

finaly we find (ii) as

$$(\nabla^f_{X^h} Y^v)_{(p,u)} = (\nabla_X Y)^v_{(p,u)} + \frac{X(f)}{2f(p)} Y^v + \frac{1}{2}(R_p(u, Y)X)^h.$$

For (iv)

$$\begin{aligned}
2G^f(\nabla^f_{X^v} Y^v, Z^h) &= -Z(f)g(X, Y) - f(p)[g(\nabla_Z X, Y) + g(X, \nabla_Z Y)] + f(p)g(X, (\nabla_Z Y)) \\
&\quad +f(p)g(Y, \nabla_Z X) \\
&= -g(\sharp df g(X, Y), Z) = -G^f(g(X, Y)(\sharp df)^h, Z^h)
\end{aligned}$$

and

$$2G^f(\nabla^f_{X^v} Y^v, Z^v) = 0$$

then

$$(\nabla^f_{X^v} Y^v)_{(p,u)} = -\frac{1}{2} g(X,Y)(\sharp df)^h$$

It's simylar and easy for (i) and (iii). $\square$

3.2. **Riemannian curvature tensor of G^f.**

Proposition 3. *Let R (resp. R^f) be a Riemannian curvature tensor of (M,g) (resp. (TM,G^f)). Then, the following formulas hold*

$$\begin{aligned}
1/\ R^f(X^v,Y^v)Z^v &= \frac{\| \sharp df \|^2}{4f(p)}[g(X,Z)Y - g(Y,Z)X]^v \\
&\quad + \frac{1}{4}[g(X,Z)R(u,Y)\sharp df - g(Y,Z)R(u,X)\sharp df]^h \\
2/\ R^f(X^v,Y^v)Z^h &= [f(p)R(X,Y)Z + \frac{1}{4}(R(u,X)R(u,Y)Z - \frac{1}{4}R(u,Y)R(u,X)Z)]^h \\
3/\ R^f(X^h,Y^v)Z^v &= -\frac{1}{4}[g(Y,Z)R(X,\sharp df)u]^v - \frac{1}{4}[2g(Y,Z)\nabla_X \sharp df \\
&\quad - \frac{X(f)}{f(p)} g(Y,Z)(\sharp df) + 2f(p)R(Y,Z)X + f(p)(R(u,Y)R(u,Z)X)]^h \\
4/\ R^f(X^h,Y^v)Z^h &= \frac{1}{4}[\frac{2X(Z(f))}{f(p)}Y - \frac{X(f)Z(f)}{f(p)^2}Y - \frac{2}{f(p)}(\nabla_X Z)(f)Y + 2R(X,Z)Y \\
&\quad - g(Y,R(X,Z)u)\sharp df + f(p)(R(R(u,Y)Z,X)u)]^v + \frac{1}{4}[\frac{Z(f)}{f(p)}R(u,Y)X \\
&\quad + 2X(f)R(u,Y)Z + 2f(p)((\nabla_X R)(u,Y)Z)]^h \\
5/\ R^f(X^h,Y^h)Z^v &= [R(X,Y)Z + \frac{1}{4}f(p)(R(Y,R(u,Z)X)u - R(X,R(u,Z)Y)u)]^v + \\
&\quad + \frac{1}{4}[X(f)R(u,Z)Y - Y(f)R(u,Z)X - 2g(R(X,Y)u,Z)(\sharp df) \\
&\quad + 2f(p)((\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X)]^h \\
6/\ R^f(X^h,Y^h)Z^h &= \frac{1}{2}[(\nabla_X R)(Y,Z)u - (\nabla_Y R)(X,Z)u - \frac{1}{2f}(Y(f)R(X,Z)u \\
&\quad - X(f)R(Y,Z)u - 2Z(f)R(X,Y)u)]^v \\
&\quad + \frac{1}{2}[2R(X,Y)Z + f(p)(R(u,R(X,Y)u)Z + \frac{1}{2}f(p)R(u,R(X,Z)u)Y \\
&\quad - \frac{1}{2}f(p)R(u,R(Y,Z)u)X]^h
\end{aligned}$$

for all $X,Y,Z \in \mathfrak{X}(M)$.

4. Geometric consequences of (TM, G^f)

Let (M, g) be a n-dimensional Riemannian manifold and TM be its tangent bundle endowed with the vertical rescaled metric G^f where f is strictly positive smooth function in (M, g). ∇^f and R^f are the Levi-Civita connection and the Riemannian curvature tensor, respectively, of (TM, G^f).

4.1. Curvature properties.

Theorem 1. *The tangent bundle (TM, G^f) is flat if and only if the following assertions hold*
1/ (M, g) is flat
2/ The function f satisfies the Eikonal equation $\| \sharp df \|^2 = 0$ and PDE

$$2f(p)H^f(X, Y) - X(f)Z(f) = 0,$$

for all $X, Y \in \mathfrak{X}(M)$ and $H^f(X, Y) = X(Y(f)) - (\nabla_X Y)(f)$ is the Hessian of f.

Proof. 1/ Let (TM, G^f) be a flat manifold (i.e $R^f \equiv 0$). If we assume that $u = 0$ in the Proposition 3 , we deduce

$$R^f_{(p,0)}(X^h, Y^h)Z^h = (R(X, Y)Z)^h_{(p,0)} = 0$$

for all $p \in M$ and all $X, Y, Z \in \mathfrak{X}(M)$. Then $R \equiv 0$.
2/ From the Proposition 3-(4 and 1), we have

$$\begin{aligned} R^f_{(p,0)}(X^h, Y^v)Z^h &= [\frac{X(Z(f))}{2f(p)} - \frac{X(f)Z(f)}{4f(p)^2} - \frac{1}{2f(p)}(\nabla_X Z)(f)]Y^v = 0 \\ R^f_{(p,0)}(X^v, Y^v)Z^v &= \frac{\| \sharp df \|^2}{4f(p)}[g(X, Z)Y^v - g(Y, Z)X^v] = 0 \end{aligned}$$

Consequently

$$\frac{X(Z(f))}{2f(p)} - \frac{X(f)Z(f)}{4f(p)^2} - \frac{1}{2f(p)}(\nabla_X Z)(f) = 0 \text{ and } \| \sharp df \|^2 = 0$$

□

4.2. Einstein structure.

Let $\{e_1, ..., e_m\}$ be an orthonormal basis of T_pM, then the family $\{e_1^h, ..., e_n^h, e_1^v, ..., e_n^v\}$ is an orthonormal basis of $T_{(p,u)}TM$. The Ricci-curvature of (TM, G^f) is given by

$$Ric^f_{(p,u)}(X^*, Y^*) = \sum_{i=1}^n G^f(R^f(X^*, e_i^h)Y^*, e_i^h) + \sum_{i=1}^n G^f(R^f(X^*, e_i^v)Y^*, e_i^v) \quad (4.1)$$

Using the Proposition 3 and the Eq.(4.1), we have

$$1/\ Ric^f_{(p,u)}(X^H,Y^H) = Ric_p(X,Y) + \frac{3}{4}f(p)\sum_{i=1}^{n} g(R(X,e_i)u, R(Y,e_i)u) + n(\frac{X(Y(f))}{2} \tag{4.2}$$

$$-\frac{X(f)Y(f)}{4f(p)} - \frac{1}{2}(\nabla_X Y)(f)) + \frac{1}{4}f(p)\sum_{i=1}^{n} g(R(X,Y)e_i,u)g(\nabla f, e_i)$$

$$2/\ Ric^f_{(p,u)}(X^H,Y^V) = -\sum_{i=1}^{n}\{\frac{3}{4}e_i(f)g(R(X,e_i)u,Y) + \frac{1}{2}f(p)g((\nabla_{e_i}R)(u,Y)X,e_i)\} \tag{4.3}$$

$$-\frac{1}{4}f(p)g(\nabla f, R(u,Y)X)$$

$$3/\ Ric^f_{(p,u)}(X^V,Y^V) = \frac{1}{4}(g(X,Y)\sum_{i=1}^{n}\{2H^f(e_i,e_i) - \frac{e_i(f)e_i(f)}{f(p)}\} \tag{4.4}$$

$$+ f(p)Tr(R(u,X)R(u,Y)) + \|\sharp df\|^2 g(X,Y)(n-1))$$

Proposition 4. *Let (TM, G^f) be an Einstein manifold.*
1/ (M,g) is Einstein manifold if f satisfies the PDE

$$2f(p)H^f(X,Y) - X(f)Y(f) = 0,$$

Moreover, for all X and $Y \in \mathfrak{X}(M)$,
2/ If $\| \nabla f \| = 0$ with $\nabla f \neq 0$, then, (M,g) is Ricci-Flat.
3/ If $\| \nabla f \| \neq 0$, then,

$$f(p) = \frac{(n-1)}{4\lambda} \| \nabla f \|^2 .$$

Proof. We suppose that (TM, G^f) is λ-Einstein, then

$$Ric^f_{(p,u)}(X^*,Y^*) = \lambda G^f_{(p,u)}(X^*,Y^*)$$

1/ If we take $u = 0$ in the Eq.(4.2), we have

$$\begin{aligned} 1/\ Ric^f_{(p,0)}(X^H,Y^H) &= Ric_p(X,Y) + n[\frac{X(Y(f))}{2} - \frac{X(f)Y(f)}{4f(p)} - \frac{1}{2}(\nabla_X Y)(f)] \\ &= \lambda g(X,Y) \end{aligned}$$

we obtain

$$Ric_p(X,Y) = \lambda g(X,Y) - n(\frac{H^f(X,Y)}{2} - \frac{X(f)Y(f)}{4f(p)})$$

when the function f satisfy the PDE $\frac{H^f(X,Y)}{2} = \frac{X(f)Y(f)}{4f(p)}$, then (M,g) is Einstein manifold. In this case, if we take $u = 0$ in the Eq.(4.4), we obtain the equation

$$4\lambda f(p) = (n-1) \| \nabla f \|^2 .$$

2/ If $\| \nabla f \| = 0$, we obtain $\lambda = 0$, then (M, g) is Ricci-flat.

3/ If $\| \nabla f \| \neq 0$, we obtain $\lambda \neq 0$, then $f(p) = \dfrac{(n-1)}{4\lambda} \| \nabla f \|^2$. □

4.3. **Sectional curvature.** The sectional curvatures of the tangent bundle (TM, G^f) is given by

$$\mathcal{K}^f(X^i, Y^j) = \frac{G^f(R^f(X^i, Y^j)Y^j, X^i)}{G^f(X^i, X^i)G^f(Y^j, Y^j) - G^f(X^i, Y^j)} \tag{4.5}$$

spanned by the two orthonormal tangent vectors $X^i, Y^j \in T_{(p,u)}TM$ and $i, j = \{h, v\}$.

Proposition 5. *The sectional curvatures $\mathcal{K}^f$ of (TM, G^f) is given as follow*

$$\begin{aligned}
1/\ \mathcal{K}^f(X^h, Y^h) &= \mathcal{K}(X, Y) - \frac{3}{4} f(p) \left\| R(X, Y)u \right\|^2 \\
2/\ \mathcal{K}^f(X^h, Y^v) &= -\frac{1}{2f(p)} H^f(X, X) + (\frac{X(f)}{2f(p)})^2 + \frac{1}{4} g(R(u, Y)X, R(u, Y)X) \\
3/\ \mathcal{K}^f(X^v, Y^v) &= 0
\end{aligned}$$

Proof. Using the Proposition 3 and the Eq.(4.5), we can calculate

$$\begin{aligned}
1/\ \mathcal{K}^f(X^h, Y^h) &= \frac{G^f(R^f(X^h, Y^h)Y^h, X^h)}{G^f(X^h, X^h)G^f(Y^h, Y^h) - G^f(X^h, Y^h)} \\
&= G^f((R(X, Y)Y)^h, X^h) + G^f(\frac{1}{2} f(p)(R(u, R(X, Y)u)Y)^h, X^h) \\
&\quad + G^f(\frac{1}{4} f(p)(R(u, R(X, Y)u)Y)^h, X^h) \\
&= g(R(X, Y)Y, X) - \frac{3}{4} f(p) g(R(X, Y)u, R(X, Y)u) \\
&= \mathcal{K}(X, Y) - \frac{3}{4} f(p) \left\| R(X, Y)u \right\|^2
\end{aligned}$$

and

$$\begin{aligned}
2/\ \mathcal{K}^f(X^h,Y^v) &= \frac{G^f(R^f(X^h,Y^v)Y^v,X^h)}{G^f(X^h,X^h)G^f(Y^v,Y^v)-G^f(X^h,Y^v)}\\
&= \frac{1}{f(p)}[-\frac{1}{2}G^f((\nabla_X\sharp df)^h,X^h)+\frac{X(f)}{4f(p)}G^f((\sharp df)^h,X^h)\\
&\quad -\frac{1}{4}f(p)G^f((R(u,Y)R(u,Y)X)^h,X^h)]\\
&= \frac{1}{f(p)}[-\frac{1}{2}g(\nabla_X\sharp df,X)+\frac{X(f)}{4f(p)}g(\sharp df,X)\\
&\quad +\frac{1}{4}f(p)g(R(u,Y)X,R(u,Y)X)]\\
&= -\frac{1}{2f(p)}H^f(X,X)+(\frac{X(f)}{2f(p)})^2+\frac{1}{4}g(R(u,Y)X,R(u,Y)X)
\end{aligned}$$

For (3), the sectional curvature for a plane spanned by two vertical vectors vanishes, then

$$\mathcal{K}^f(X^v,Y^v)=0$$

□

Corollary 1. *Let (M,g) be a n-dimensional Riemannian manifold of constant sectional curvature κ. Then*

$$\begin{aligned}
1/\ \mathcal{K}^f(X^h,Y^h) &= \kappa-\frac{3}{4}f(p)\kappa^2[g^2(Y,u)+g^2(X,u)]\\
2/\ \mathcal{K}^f(X^h,Y^v) &= -\frac{1}{2f(p)}H^f(X,X)+(\frac{X(f)}{2f(p)})^2+\frac{1}{4}\kappa^2f(p)g^2(u,X)\\
3/\ \mathcal{K}^f(X^v,Y^v) &= 0
\end{aligned}$$

Proof. Using the Proposition 5 and taking account that for a Riemannian manifold of constant sectional curvature κ, its Riemannian curvature R is

$$R(X,Y)Z=\kappa(g(Y,Z)X-g(X,Z)Y) \tag{4.6}$$

we can immediately get the result of corollary. □

Proposition 6. *If (TM,G^f) is Riemannian manifold of constant sectional curvature κ^f, then (M,g) is Riemannian manifold of negative constant sectional curvature with*

$$f(p)=(\frac{\|\ \sharp df\ \|^2}{-4\kappa^f})^{1/2}.$$

Proof. If κ^f is constant, From the Proposition 3 and the Eq.(4.6), we have

$$R^f_{(p,u)}(X^h,Y^h)Z^h=\kappa^f(G^f_{(p,u)}(Y^h,Z^h)X^h-G^f_{(p,u)}(X^h,Z^h)Y^h) \tag{4.7}$$

for $X, Y, Z \in \mathfrak{X}(M)$ et $(p, u) \in TM$.
If $u = 0$ in the Eq.(4.7), then

$$R^f_{(p,0)}(X^h, Y^h)Z^h = \kappa^f(g_p(Y, Z)X^h_{(p,0)} - g_p(X, Z)Y^h_{(p,0)})$$

and from the Proposition 3-(6), we obtain

$$R^f_{(p,0)}(X^h, Y^h)Z^h = (R(X, Y)Z)^h_{(p,0)}$$

after comparison and using the isomorphism between the vector space T_pM and $\mathcal{H}_{(p,0)}$, we have

$$R(X, Y)Z = \kappa^f(g_p(Y, Z)X - g_p(X, Z)Y)$$

Similarly, we calculate

$$\begin{aligned} R^f_{(p,u)}(X^v, Y^v)Z^v &= \kappa^f(G^f_{(p,u)}(Y^v, Z^v)X^v - G^f_{(p,u)}(X^v, Z^v)Y^v) \\ &= \kappa^f f(p)(g_p(Y, Z)X^v_{(p,u)} - g_p(X, Z)Y^v_{(p,u)}) \end{aligned}$$

and using the Proposition 3-(1), we get

$$R^f_{(p,u)}(X^v, Y^v)Z^v = \frac{\| \sharp df \|^2}{4f(p)}(g(X, Z)Y^v - g(Y, Z)X^v)$$

By comparison, we obtain

$$\kappa^f f(p) = -\frac{\| \sharp df \|^2}{4f(p)}.$$

Then $\kappa^f < 0$ and $f(p) = (\frac{\| \sharp df \|^2}{-4\kappa^f})^{1/2}$. □

4.4. **Scalar curvature.**

Proposition 7. *Let S (resp. S^f) be a scalar curvature of (M, g) (resp. (TM, G^f)). Then*

$$S^f_{(p,u)} = S_p - \frac{1}{4}f(p)\sum_{i,j=1}^{n}|R(X_i, X_j)u|^2 - \frac{n}{f(p)}[\Delta f - \frac{\| \sharp df \|^2}{2f(p)}] \tag{4.8}$$

where $\{X_1, ..., X_n\}$ is a local orthonormal frame of (TM, G^f) and Δ is the Laplacian of f.

Proof. For a local orthonormal frame $\{Y_1, ..., Y_{2n}\}$ for TM with $X_i^h = Y_i$ and $X_i^v = Y_{n+i}$; $i = 1, ..., n$, we get from Proposition 5

$$\begin{aligned} S^f_{(p,u)} &= \sum_{i,j=1}^{2n} \mathcal{K}^f(Y_i, Y_i) \\ &= \sum_{i,j=1}^{n} (\mathcal{K}^f(X_i^h, Y_j^h) + 2\mathcal{K}^f(X_i^h, Y_j^v) + \mathcal{K}^f(X_i^v, Y_j^v)) \\ &= \sum_{i,j=1}^{n} (\mathcal{K}(X_i, X_j) - \frac{3}{4} f(p) \left\| R(X_i, X_j)u \right\|^2 - \frac{1}{f(p)} H^f(X_i, X_i) \\ &\quad + 2(\frac{X_i(f)}{2f(p)})^2 + \frac{1}{2} f(p) \left\| R(u, X_j)X_i \right\|^2) \end{aligned}$$

In order to simplify this last expression, we put $u = \sum_{i=1}^{n} u_i X_i$, then

$$\sum_{i,j=1}^{n} \left\| R(X_j, u)X_i \right\|^2 = \sum_{i,j=1}^{n} \left\| R(X_j, X_i)u \right\|^2$$

and

$$\begin{aligned} S^f_{(p,u)} &= \sum_{i,j=1}^{n} (\mathcal{K}(X_i, Y_j) - \frac{1}{4} f(p) \left\| R(X_i, Y_j)u \right\|^2 - \frac{1}{f(p)} H^f(X_i, X_i) + \frac{1}{2f(p)^2} X_i(f)^2) \\ &= \sum_{i,j=1}^{n} (\mathcal{K}(X_i, X_j) - \frac{1}{4} f(p) \left\| R(X_i, X_j)u \right\|^2) - \frac{n}{f(p)} \sum_{i=1}^{n} (H(X_i, X_i) - \frac{X_i(f)^2}{2f(p)}) \\ &= S_p - \frac{1}{4} f(p) \sum_{i,j=1}^{n} \left\| R(X_i, X_j)u \right\|^2 - \frac{n}{f(p)} \sum_{i=1}^{n} (H(X_i, X_i) - \frac{X_i(f)^2}{2f(p)}) \end{aligned}$$

Since $\sum_{i=1}^{n} H(X_i, X_i) = \Delta f$ and $\sum_{i=1}^{n} X_i(f)^2 = \sum_{i=1}^{n} g(\nabla f, X_i) g(\nabla f, X_i) = g(\nabla f, \nabla f)$, then we can get the result. □

Proposition 8. *(TM, G^f) has constant scalar curvature if and only if (M, g) is flat and f satisfies the PDE*

$$[\Delta f - \frac{\| \sharp df \|^2}{2f(p)}] - \frac{a}{n} f(p) \ = 0 \ (a \textit{ is a constant})$$

Proof. If $S^f_{(p,0)} = S^f_0$ is constant scalar curvature of (TM, G^f), then the function $p \mapsto S^f_{(p,0)}$ is constant on (M, g), equal to S^f_0. From the Eq.(4.8), $S^f_{(p,0)}$ turn to

$$S^f_{(p,0)} = S_p - \frac{n}{f(p)} [\Delta f - \frac{\| \sharp df \|^2}{2f(p)}] = S^f_0$$

Then

$$S^f_{(p,u)} = S^f_0 - \frac{1}{4} f(p) \sum_{i,j=1}^{n} \|R(X_i, X_j)u\|^2$$

ho give $\sum_{i,j=1}^{n} \|R(X_i, X_j)u\|^2 = 0$ and $R \equiv 0$ (i.e. (M, g) is flat). The scalar curvature of (M, g) is $S_p = 0$, then

$$\begin{aligned} \frac{n}{f(p)}[\Delta f - \frac{\| \sharp df \|^2}{2f(p)}] &= a \text{ (constant) or} \\ [\Delta f - \frac{\| \sharp df \|^2}{2f(p)}] - \frac{a}{n} f(p) &= 0. \end{aligned}$$

The inverse is immediate. □

Proposition 9. *If (M, g) have a non null constant sectional curvature κ. Then the scalar curvature S^f of (TM, G^f) has the form*

$$S^f = -\frac{1}{2} f(p) K^2 (n-1) \|u\|^2 + n(n-1)K - \frac{n}{f(p)}[\Delta f - \frac{\| \sharp df \|^2}{2f(p)}]$$

Proof. Using the Proposition 7 and the Eq.(4.8), we get the result. □

Remark 1. *If the Lapolacian of f; $\Delta f = \sum_{i=1}^{n} H(X_i, X_i)$ satisfied*

$$2f(p)\Delta f - \| \sharp df \|^2 = 0,$$

then

$$S^f = (n-1)K(n - \frac{1}{2} f(p) K \|u\|^2).$$

Conclusion

Several authors have studied the geometry of the tangent bundle TM endowed with different metrics, of a Riemannian manifold (M, g). We introduce a new kind of metrics denoted G^f on TM as multiplication with strictly positive smooth function in M, in the vertical part of Sasaki metric. We call this metric the vertical rescaled metric. After, computing its Levi-Civita and the curvature tensor, we studied the geometry of (TM, G^f) by giving a relationships of the curvatures, Einstein structure, scalar and sectional curvatures between (TM, G^f) and (M, g). A non-flat metric G^f has been obtained if the function f does not satisfy the Eikonel equation and the metric G^f can be Einstein metric on TM without the basic metric g being flat.

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