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Some properties of $q-$ close-to-convex functions

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Abstract

Quantum calculus had been used first time by M.E.H.Ismail, E.Merkes and D.Steyr in the theory of univalent functions [5].
In this present paper we examine the subclass of univalent functions which is defined by quantum calculus.

Keywords: $q-$ convex function, $q-$ close to convex function, growth theorem, distortion theorem, coefficient inequality.

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1. Introduction

Let Ω be the family of functions $\phi(z)$ regular in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Denote by $\mathcal{P}(q)$ the family of functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ which are regular in the open unit disc $\mathbb{D}$ and satisfying

$$(1.1) \qquad \left| p(z) - \frac{1}{1-q} \right| < \frac{1}{1-q}, \quad z \in \mathbb{D}$$

where $q \in (0, 1)$ is a fixed real number. Let A be the family of functions $f(z)$ which are regular in the open unit disc $\mathbb{D}$ and satisfying the conditions $f(0) = 0$, $f'(0) = 1$ for every $z \in \mathbb{D}$. In other words; each f in A has the power series representation $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ Let $f_1(z)$ and $f_2(z)$ be an elements of A, if there exists a function $\phi(z) \in \Omega$, such that $f_1(z) = f_2(\phi(z))$ for all $z \in \mathbb{D}$, then we say that $f_1(z)$ is subordinate to $f_2(z)$ and we write $f_1(z) \prec f_2(z)$, thus $f_1(z) \prec f_2(z)$ if and

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only if $f_1(0) = f_2(0)$ and $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$ implies $f_1(\mathbb{D}_r) \subset f_2(\mathbb{D}_r)$, where $\mathbb{D}_r$ defined as $\mathbb{D}_r = \{z : |z| < r \ , 0 < r < 1\}$(Subordination principle[4]).

In this paragraph we will give the concept of the $q-$ calculus. Let $q \in (0,1)$ be a fixed number. A subset of $\mathbb{B}$ of $\mathbb{C}$ is called $q-$ geometric if $qz \in \mathbb{B}$ whenever $z \in \mathbb{B}$, if a subset $\mathbb{B}$ of $\mathbb{C}$ is a $q-$ geometric set, then it contains all geometric sequences $\{q^n z\}_0^\infty$, $zq \in \mathbb{B}$. Let f be a function (real or complex valued) defined on $q-$ geometric set $\mathbb{B}$, $|q| \neq 1$, the $q-$ difference operator which was introduced by Jackson F.H. and E.Heine or Euler([1],[2],[3],[7]) defined by

$$(1.2) \qquad D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, z \in \mathbb{B}\backslash\{0\}$$

The $q-$ difference operator (1.2) sometimes called Jackson $q-$difference operator. If $0 \in \mathbb{B}$, the $q-$ derivative at zero by $|q| < 1$

$$(1.3) \qquad D_q f(0) = \lim_{n \to \infty} \frac{f(zq^n) - f(0)}{zq^n}, \ z \in \mathbb{B}\backslash\{0\}$$

provided the limit exists, and does not depend on z in addition $q-$ derivative at zero defined by $|q| < 1$

$$(1.4) \qquad D_q f(0) = D_{q^{-1}} f(0)$$

Under the hypothesis of the definition of $q-$difference operator, then we have the following rules:

(1) For a function $f(z) = z^n$ we observe that $D_q f(z) = D_q z^n = \frac{1-q^n}{1-q} z^{n-1}$, therefore we have $f(z) = z + a_2 z^2 + a_3 z^3 \cdots + a_n z^n \cdots \Rightarrow$

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} a_n \frac{1-q^n}{1-q} z^{n-1}$$

(2) Let $f(z)$ and $g(z)$ be defined on a $q-$ geometric set $\mathbb{B} \subset \mathbb{C}$ such that q derivatives of f and g exist for all $z \in \mathbb{B}$, then

(i) $D_q(af(z) \pm bg(z)) = aD_q f(z) \pm bD_q g(z)$ where a and b are real or complex constants

(ii) $D_q(f(z)g(z)) = g(z)D_q f(z) + f(qz)D_q g(z)$

(iii) $D_q\left(\frac{g(z)}{h(z)}\right) = \frac{g(z)D_q h(z) - h(z)D_q g(z)}{h(z)h(qz)} = \frac{g(qz)D_q h(z) - h(qz)D_q g(z)}{h(z)h(qz)}$ where $h(z)h(qz) \neq 0$.

(iv) As a right inverse, Jackson([1],[2],[3],[7]) introduced the $q-$ integral

$$\int_0^z f(t) d_q t = z(1-q) \sum_{n=0}^{\infty} q^n f(zq^n)$$

provided that the series converges. The following theorem is an analogue of the fundamental theorem of calculus.

1.1. Theorem ([7])**.** Let f be a $q-$ regular at zero, defined on $q-$ geometric set $\mathbb{B}$ containing zero. Define

$$F(z) = \int_c^z f(\xi) d_q \xi \ , (z \in \mathbb{B})$$

where c is a fixed point in $\mathbb{B}$, then F is a regular at zero. Furthermore $D_qF(z)$ exists for every $z \in \mathbb{B}$ and

$$D_qF(z) = f(z)$$

for all $z \in \mathbb{B}$.

Conversely; If a and b are two points in $\mathbb{B}$, then

$$\int_a^b D_qf(z)d_qz = f(b) - f(a)$$

(3) The $q-$ differential is defined as

$$d_qf(z) = f(z) - f(qz)$$

therefore

$$D_qf(z) = \frac{d_qf(z)}{d_qz} = \frac{f(z) - f(qz)}{(1-q)z} \Rightarrow d_qf(z) = \frac{f(z) - f(qz)}{(1-q)z}d_qz.$$

(4) The partial $q-$ derivative of a multivariable real continuous function $f(x_1, x_2, x_3, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n)$ to a variable x_i defined by

$$D_{q,x_i}F(\vec{x}) = \frac{f(\vec{x}) - \varepsilon_{q,x_i}f(\vec{x})}{(1-q)x_i}, x_i \neq 0, q \in (0,1)$$

$$[D_{q,x_i}F(\vec{x})]_{x_i=0} = \lim_{x_i \to 0} D_{q,x_i}f(\vec{x})$$

where $\varepsilon_{q,x_i}f(\vec{x}) = f(x_1, x_2, x_3, ..., x_{i-1}, qx_i, x_{i+1}, ..., x_n)$ and we use $D^k_{k,x}$ instead of operator $\frac{\partial^k_q}{\partial_q x^k}$ for some simplification.

1.2. Lemma. ([6] JACK'S LEMMA) Let $\phi(z)$ be analytic in $\mathbb{D}$ with $\phi(0) = 0$. If the maximum value of the $|\phi(z)|$ on the circle $|z| = r$, $(0 < r < 1)$ is attained at $z = z_0$, then we have

$$z_0\phi'(z_0) = m\phi(z_0), m \geq 1$$

Making use of the $q-$ derivative $D_qf(z)$, we introduce the following classes.

$$S^*_q = \{f(z) \in \ A\Big| \quad z\frac{D_qf(z)}{f(z)} = p(z), p(z) \in \ P(q)\}$$

(The class of $q-$ starlike functions [5].)

$$C_q = \{f(z) \in \ A\Big| \quad \frac{D_q(zD_qf(z))}{D_qf(z)} = p(z), p(z) \in \ P(q)\}$$

(The class of $q-$ convex functions)

$$K_q = \{g(z) \in \ A\Big| \quad \frac{D_qg(z)}{D_qf(z)} = p(z), p(z) \in \ P(q), f(z) \in \ C_q\}$$

(The class of $q-$ close-to-convex functions.)

In the present paper we will investigate the class of K_q.

2. Main Results

2.1. Theorem ([8])**.** $p(z) \in P(q)$ if and only if

$$p(z) \prec \frac{1+z}{1-qz}$$

Proof. Let $p(z)$ be an element of $P(q)$ then we have

$$\left| p(z) - \frac{1}{1-q} \right| < \frac{1}{1-q} \Rightarrow |p(z) - m| < m$$

where

$$\frac{1}{1-q} = m \iff 1 - q = \frac{1}{m} \Rightarrow 1 - \frac{1}{m} = q.$$

Therefore the function

$$\psi(z) = \frac{1}{m}p(z) - 1$$

has modulus at most 1 in the open unit disc $\mathbb{D}$ and so

$$\phi(z) = \frac{\psi(z) - \psi(0)}{1 - \overline{\psi(0)}\psi(z)} = \frac{(\frac{1}{m}p(z) - 1) - (\frac{1}{m} - 1)}{1 - (\frac{1}{m} - 1)(\frac{1}{m}p(z) - 1)}$$

satisfies the conditions of Schwarz Lemma, this shows that

$$p(z) = \frac{1 + \phi(z)}{1 - (1 - \frac{1}{m})\phi(z)} \Rightarrow p(z) \prec \frac{1+z}{1-qz}.$$

Conversely; Suppose that the function $p(z)$ analytic in $\mathbb{D}$ and satisfies the condition $p(0) = 1$ and

$$p(z) \prec \frac{1+z}{1-qz}$$

then we have

$$p(z) \prec \frac{1+z}{1-qz} \Rightarrow p(z) = \frac{1 + \phi(z)}{1 - (1 - \frac{1}{m})\phi(z)} \Rightarrow p(z) - m = m\frac{\frac{1-m}{m} + \phi(z)}{1 + \frac{1-m}{m}\phi(z)}.$$

On the other hand the function $\left(\frac{\frac{1-m}{m} + \phi(z)}{1 + \frac{1-m}{m}\phi(z)}\right)$ maps the unit circle onto itself, then we have

$$|p(z) - m| = \left| m\frac{\frac{1-m}{m} + \phi(z)}{1 + \frac{1-m}{m}\phi(z)} \right| < m.$$

This shows that $p(z) \in P(q)$. □

2.2. Lemma ([8])**.** Let $f(z)$ be a function (real or complex valued) defined on $q-$ geometric set $\mathbb{B}$ with $|q| \neq 1$,then

$$D_q(log f(z)) = \frac{D_q f(z)}{f(z)} \tag{2.1}$$

Proof. Using the definition of $q-$ difference operator, then we have

$$D_q(log f(z)) = \frac{log f(z) - log f(qz)}{(1-q)z} = \log\left(1 + h\frac{D_q f(z)}{f(z)}\right)^{\frac{1}{h}}$$

Taking limit for $h \to 0$ we obtain (2.1) □

2.3. Lemma. ($q-$Jack's Lemma [8]) Let $\phi(z)$ be analytic in $\mathbb{D}$ with $\phi(0) = 0$. Then if $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{D}$, then we have

$$z_0 D_q\phi(z_0) = m\phi(z_0),$$

where $m \geq 1$ is a real number.

Proof. Using the definition of $q-$ difference operator and Jack's Lemma (Lemma 1.2) then we have

$$D_q\phi(z) = \frac{\phi(z)-\phi(qz)}{(1-q)z} = \frac{\phi(z)-\phi(z_0)}{z-z_0}, \quad qz = z_0$$

taking limit for $z \to z_0$ we get

$$\lim_{z\to z_0} D_q\phi(z) = D_q\phi(z_0) = \lim_{z\to z_0} \frac{\phi(z)-\phi(z_0)}{z-z_0} = \phi'(z_0)$$

Therefore we have $z_0 D_q\phi(z_0) = z_0\phi'(z_0) = m\phi(z_0)$ □

2.4. Theorem ([8])**.** Let f(z) be an element of C_q, then

$$z\frac{D_qf(z)}{f(z)} \prec \frac{1}{1-qz} \tag{2.2}$$

Proof. We define the function $\phi(z)$ by

$$z\frac{D_qf(z)}{f(z)} = \frac{1}{1-q\phi(z)} \tag{2.3}$$

Since $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, $zD_qf(z) = z + a_2\frac{1-q^2}{1-q}z^2 + a_3\frac{1-q^3}{1-q}z^3 + \cdots$, then $\phi(z)$ is well defined and analytic at the same time

$$z\frac{D_qf(z)}{f(z)}\bigg|_{z=0} = 1 = \frac{1}{1-q\phi(z)} \Rightarrow \phi(0) = 0$$

We need to show that $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Assume to the contrary, then there exists a $z_0 \in \mathbb{D}$ such that $|\phi(z_0)| = 1$. The definition of the class C_q and subordination principle, then we write

$$A(r) = \left\{ f(z) : \left|(1+qz)\frac{D_q(D_qf(z))}{D_qf(z)} - \frac{1+qr^2}{1-q^2r^2}\right| \leq \frac{(1+q)r}{1-q^2r^2}, f(z) \in \ C_q \right\} \tag{2.4}$$

On the other hand, using the definition $q-$ derivative, theorem 2.1 and the relation (2.3) and after the straightforward calculations we get

$$(1+qz)\frac{D_q(D_qf(z))}{D_qf(z)} = q\left(\frac{1}{1-q\phi(z)}\right) + \frac{logq^{-1}}{1-q}\frac{zD_q\phi(z)}{1-q\phi(z)} + \left(1 - q\frac{logq^{-1}}{1-q}\right) \tag{2.5}$$

Using $q-$Jack's Lemma in (2.5)

$$(1+qz_0)\frac{D_q(D_qf(z_0))}{D_qf(z_0)} = q\left(\frac{1}{1-q\phi(z_0)}\right) + \frac{logq^{-1}}{1-q}\frac{m\phi(z_0)}{1-q\phi(z_0)} + \left(1 - q\frac{logq^{-1}}{1-q}\right) \notin A(r).$$

But this is a contradict with (2.4). Therefore $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. □

2.5. Theorem ([8])**.** Let f(z) be an element of C_q, then

$$\left(\frac{r}{(1+qr)^{\frac{1+q}{q}}}\right)^{\frac{1-q}{logq^{-1}}} \leq |f(z)| \leq \left(\frac{r}{(1-qr)^{\frac{1+q}{q}}}\right)^{\frac{1-q}{logq^{-1}}} \tag{2.6}$$

These bounds are sharp because extremal function is the solution of the $q-$ differential equation

$$z\frac{D_qf(z)}{f(z)} = \frac{1}{1-qz}$$

Proof. Since the linear transformation $w = \dfrac{1}{1-qz}$ maps $|z| = r$ onto the disc with the centre $C(r) = \dfrac{1}{1-q^2r^2}$ and the radius $\rho(r) = \dfrac{qr}{1-q^2r^2}$. Using theorem 2.1 and subordination principle ,then we can write

$$\left| z\frac{D_q f(z)}{f(z)} - \frac{1}{1-q^2r^2} \right| \leq \frac{qr}{1-q^2r^2} \tag{2.7}$$

The inequality (2.7) can be written in the following form

$$\frac{1}{1+qr} \leq Re(re^{i\theta}\frac{D_q f(re^{i\theta})}{f(re^{i\theta})}) \leq \frac{1}{1-qr} \tag{2.8}$$

On the other hand we have (using the $q-$ partial rule)

$$Re\left(re^{i\theta}\frac{D_q f(re^{i\theta})}{f(re^{i\theta})} \right) = r\frac{\partial_q}{\partial_r} log|f(re^{i\theta})| \tag{2.9}$$

Considering (2.8) and (2.9) together we can write

$$\frac{1}{r(1+q)} \leq \frac{\partial_q}{\partial_r} log|f(re^{i\theta})| \leq \frac{1}{r(1-qr)} \tag{2.10}$$

If we take $q-$ integral both sides of (2.10) we get (2.6). □

2.6. Remark. Since $\lim_{q\to 1} \dfrac{1-q}{logq^{-1}} = 1$, then (2.6) reduces to

$$\frac{r}{1+r} \leq |f(z)| \leq \frac{r}{1-r}$$

This is the growth theorem for convex functions [4].

2.7. Theorem. Let f(z) be an element of C_q, then

$$(1+qr)^{-\left(\frac{1-q^2}{q^2 logq^{-1}}\right)} \leq |D_q f(z)| \leq (1-qr)^{-\left(\frac{1-q^2}{q^2 logq^{-1}}\right)} \tag{2.11}$$

These bounds are sharp, because extremal function is the solution of the $q-$ differential equation

$$1 + qz\frac{D_q(D_q f(z))}{D_q f(z)} = \frac{1+z}{1-qz}$$

Proof. Since the linear transformation $\left(\dfrac{1+z}{1-qz}\right)$ maps $|z| = r$ onto the disc with the centre $C(r) = \dfrac{1+qr^2}{1-q^2r^2}$ and the radius $\rho(r) = \dfrac{(1+q)r}{1-q^2r^2}$. Using the definition of C_q and subordination principle ,then we can write

$$\left| 1 + qz\frac{D_q(D_q f(z))}{D_q f(z)} - \frac{1+qr^2}{1-q^2r^2} \right| \leq \frac{(1+q)r}{1-q^2r^2}$$

Using the same technique of the proof theorem 2.5, we can obtain

$$\frac{1+q}{q}\frac{1}{1+qr} \leq \frac{\partial_q}{\partial_r} log|D_q f(re^{i\theta})| \leq \frac{1+q}{q}\frac{1}{1-qr} \tag{2.12}$$

Taking $q-$ integral both sides of (2.12) we get (2.11). □

2.8. Remark. Since $\lim_{q\to 1} \dfrac{1-q^2}{q^2 logq^{-1}} = 2$, then (2.11) gives

$$(1+r)^{-2} \leq |f'(z)| \leq (1-r)^{-2}$$

This is the distortion theorem of convex functions [5].

2.9. Theorem. Let $g(z)$ be an element of K_q, then

$$(1-r)(1+qr)^{-\left(\frac{1-q^2}{q^2 logq^{-1}}+1\right)} \leq |D_q g(z)| \leq (1+r)(1-qr)^{-\left(\frac{1-q^2}{q^2 logq^{-1}}+1\right)} \tag{2.13}$$

These bounds are sharp because the extremal function is the solution of the $q-$ differential equation

$$\frac{D_q g(z)}{D_q f(z)} = \frac{1+z}{1-qz}$$

under the provided that $f(z)$ is $q-$convex function.

Proof. Using the definition of the class K_q and theorem 2.1, then we can write

$$\frac{D_q g(z)}{D_q f(z)} = p(z) \Leftrightarrow \frac{D_q g(z)}{D_q f(z)} \prec \frac{1+z}{1-qz} \Rightarrow \left|\frac{D_q g(z)}{D_q f(z)} - \frac{1+qr^2}{1-q^2r^2}\right| \leq \frac{(1+q)r}{1-q^2r^2} \Rightarrow$$

$$\frac{1-r}{1+qr}|D_q f(z)| \leq |D_q g(z)| \leq \frac{1+r}{1-qr}|D_q f(z)|$$

which gives (2.13). □

2.10. Theorem. Let $g(z)$ be an element of K_q, then

$$\frac{g(z)}{f(z)} \prec \frac{1+z}{1-qz} \tag{2.14}$$

Proof. Since $g(z) \in K_q$,then we have

$$A(r) = \left\{\frac{D_q g(z)}{D_q f(z)} : \left|\frac{D_q g(z)}{D_q f(z)} - \frac{1+qr^2}{1-q^2r^2}\right| \leq \frac{(1+q)r}{1-q^2r^2}, f(z) \in \mathcal{C}_q, q \in (0,1)\right\} \tag{2.15}$$

□

Now we define the function $\phi(z)$ by

$$\frac{g(z)}{f(z)} = \frac{1+\phi(z)}{1-q\phi(z)},$$

thus $\phi(z)$ is analytic and $\phi(0) = 0$. Now we want to show that $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Assume to the contrary, then there exists $z_0 \in \mathbb{D}$ such that $|\phi(z_0)| = 1$. On the other hand we have

$$D_q\left(\frac{g(z)}{f(z)}\right) = D_q\left(\frac{1+\phi(z)}{1-q\phi(z)}\right) \Rightarrow$$

$$\frac{D_q g(z)}{D_q f(z)} = \frac{g(qz)}{f(qz)} + \frac{(1+q)D_q\phi(z)}{(1-q\phi(z))(1-q\phi(qz))}\frac{f(z)}{D_q f(z)}$$

or

$$\frac{D_q g(z_0)}{D_q f(z_0)} = \frac{1+\phi(qz_0)}{1-q\phi(qz_0)} + \frac{(1+q)z_0 D_q\phi(z_0)}{(1-q\phi(z_0))(1-q\phi(qz_0))}\frac{f(z_0)}{z_0 D_q f(z_0)} \tag{2.16}$$

In this step, if we use lemma 2.3 (q-Jack's Lemma) and theorem 2.4, then we have

$$\frac{D_q g(z_0)}{D_q f(z_0)} = \left(\frac{1+\phi(qz_0)}{1-q\phi(qz_0)} + \frac{(1+q)m\phi(z_0)}{(1-q\phi(z_0))(1-q\phi(qz_0))}\frac{(1-q^2r^2)}{1+qre^{i\theta}}\right) \notin A(r).$$

But this is a contradiction with (2.15). Therefore $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. We note that the factor $\left(\frac{1-q^2r^2}{1+qre^{i\theta}}\right)$ is the reciprocal value of the boundary value of $\left(\frac{zD_q f(z)}{f(z)}\right)$.

2.11. Corollary. If $g(z) \in K_q$, then

$$\left(\frac{r}{(1+qr)^{\frac{1+q}{q}}}\right)^{\frac{1-q}{logq^{-1}}}\frac{1-r}{1+qr} \leq |g(z)| \leq \left(\frac{r}{(1-qr)^{\frac{1+q}{q}}}\right)^{\frac{1-q}{logq^{-1}}}\frac{1+r}{1-qr} \tag{2.17}$$

Proof. Using the theorem 2.10, then we can write

$$\frac{1-r}{1+qr} \leq \left|\frac{g(z)}{f(z)}\right| \leq \frac{1+r}{1-qr} \Rightarrow$$

$$|f(z)|\frac{1-r}{1+qr} \leq |g(z)| \leq |f(z)|\frac{1+r}{1-qr}$$

In this step, if we use theorem 2.5 we get (2.17) □

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