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OSCILLATION OF A CLASS OF NONLINEAR DIFFERENCE EQUATIONS OF SECOND ORDER WITH OSCILLATING COEFFICIENTS

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Abstract. In this paper, we study asymptotic behaviour of solutions of the following second-order difference equation:

$$\Delta\big[a(n)\Delta\big[x(n)+r(n)F(x(n-\rho))\big]\big]+p(n)G\left(x(n-\tau)\right)-q(n)G\left(x(n-\sigma)\right)=s(n),$$

where $n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, $\{r(n)\}_{n\in\mathbb{N}_0}$ and $\{s(n)\}_{n\in\mathbb{N}_0}$ are sequences of real numbers, $\{p(n)\}_{n\in\mathbb{N}_0}$ and $\{q(n)\}_{n\in\mathbb{N}_0}$ are nonnegative sequences of real numbers, $\{a(n)\}_{n\in\mathbb{N}_0}$ is positive, $\rho,\tau,\sigma\geq 0$ are integers and F,G are continuous functions satisfying the usual sign condition; i.e., $F(u)/u, G(u)/u>0$ for $u\in\mathbb{R}\backslash\{0\}$. Various ranges of the sequence $\{r(n)\}_{n\in\mathbb{N}_0}$ are considered, and illustrating examples are provided to show applicability of the results.

1. Introduction

In the literature, all the papers concerning second-order equations deal with asymptotic behaviour of all solutions of delay difference equations have the following form:

$$\Delta\left[a(n)\Delta\left[x(n)+r(n)x(n-\rho)\right]\right]+p(n)x(n-\tau)=f(n),$$

where $n\in\mathbb{N}_0$, $\{r(n)\}_{n\in\mathbb{N}_0}$ is of single sign, $\{a(n)\}_{n\in\mathbb{N}_0}$ and $\{p(n)\}_{n\in\mathbb{N}_0}$ are nonnegative sequences of real numbers, $\rho,\tau\geq 0$ are integers and $\{f(n)\}_{n\in\mathbb{N}_0}$ is a sequence of real numbers (see [1, 2]). Here, the forward difference operator Δ is defined as $\Delta x(n):=x(n+1)-x(n)$ and $\Delta^2x(n):=\Delta\left[\Delta x(n)\right]$ for $n\in\mathbb{N}_0$.

In this paper, depending on the sign of the sequence $\{r(n)\}_{n\in\mathbb{N}_0}$, we investigate the oscillatory and asymptotic behavior of solutions of the second-order neutral

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nonlinear difference equation with positive and negative coefficients having the following form:

(1.1)

$$\Delta\left[a(n)\Delta\left[x(n)+r(n)F\left(x(n-\rho)\right)\right]\right]+p(n)G\left(x(n-\tau)\right)-q(n)G\left(x(n-\sigma)\right)=s(n),$$

where $n\in\mathbb{N}_0$, $\{r(n)\}_{n\in\mathbb{N}_0}$ and $\{s(n)\}_{n\in\mathbb{N}_0}$ are allowed to oscillate, $\{p(n)\}_{n\in\mathbb{N}_0}$ and $\{q(n)\}_{n\in\mathbb{N}_0}$ are nonnegative, $\{a(n)\}_{n\in\mathbb{N}_0}$ is positive, $\rho,\tau,\sigma\geq 0$ are integers. To the best of our knowledge, in the literature, there is no work done on second-order difference equations involving oscillating coefficients inside the neutral part, and positive and negative coefficients outside the neutral part. Moreover, some of our results are not restricted with boundedness of the solutions. Also the readers are referred to the paper [3] which introduces a new method for

$$\Delta\left[a(n)\Delta\left[x(n)+r(n)x(n-\rho)\right]\right]+p(n)x(n-\tau)-q(n)x(n-\sigma)=s(n).$$

In [4], the authors study the following difference equation

$$\Delta\left[a(n)\Delta\left[x(t)+\sum_{i\in R}r_i(n)x(n-\rho_i)\right]\right]+\sum_{i\in P}p_i(n)x(n-\tau_i)-\sum_{i\in Q}q_i(n)x(n-\sigma_i)=f(n),$$

and state new results depending on three different ranges of the sequence $\left\{\sum_{i\in R}r_i(n)\right\}_{n\in\mathbb{N}_0}$. Our results here extend the results of [4] for nonlinear equations, also see the results in the paper [5] where the author gives results for the existence of positive solutions.

For the fundamentals on the oscillation theory, the readers are referred to the books [6, 7, 8].

Let $\delta:=\max\{\rho,\tau,\sigma\}$. As is usual, a solution x of (1.1) is a sequence of real numbers defined for all integers satisfying $n\geq-\sigma$, and satisfies (1.1) identically for all $n\in\mathbb{N}_0$. It is also known that (1.1) has a unique solution x if an initial sequence x_0 is given to hold $x(n)=x_0(n)$ for $n=-\delta,-\delta+1,\ldots,1$. Throughout the paper, for convenience, we do not consider eventually null solutions of (1.1).

2. Main Results

In this section, we give sufficient conditions for (1.1) to be almost oscillatory, that is every solution of (1.1) oscillates or tends to zero at infinity. We state our primary assumptions as follows:

(H1) $0<F(u)/u\leq M$ and $N_1\leq G(u)/u\leq N_2$ for all $u\neq 0$ and some positive constants M, N_1, N_2,

(H2) There exists a pair of nonnegative real numbers r^-,r^+ such that either one the followings are true:
- {i} $-r^-\leq r(n)\leq r^+$ holds for all sufficiently large n , and that $[r^-+r^+]\,M<1$ holds,
- {ii} $r^-\leq r(n)\leq r^+$ holds for all sufficiently large n and, satisfying $Mr^->1$,
- {iii} $-r^-\leq r(n)\leq -r^+$ holds for all sufficiently large n and satisfying $Mr^+>1$,

(H3) $\sum_n^\infty(1/a(n))$ is divergent,

(H4)
- {i} $\delta\geq 1$ holds, where $\delta=\tau-\sigma$,
- {ii} $\{h(n)\}_{n\in\mathbb{N}_0}$ defined by $h(n):=p(n)-q(n-\delta)$ is an eventually positive sequence of reals,
- {iii} $\sum_n^\infty h(n)$ is divergent,

{iv} $\sum_{n}^{\infty}(1/a(n))\sum_{k=n-\delta}^{n-1} q(k)$ is convergent,

(H5) There exists a sequence $\{S(n)\}_{n\in\mathbb{N}_0}$ such that $\lim_{n\to\infty} S(n)$ exists and $\Delta\left((1/a(n))\Delta S(n)\right) = s(n)$ holds for all $n \in \mathbb{N}_0$.

Theorem 2.1. *Assume that* (H1)*,* (H2){i}*,* (H3)*,* (H4){i–iv} *and* (H5) *hold, then every solution of* (1.1) *oscillates or tends to zero at infinity.*

Proof. Let (1.1) have a nonoscillatory solution x, which does not tend to zero at infinity. Without loss in the generality, we may suppose that x is eventually positive, the case where x is eventually negative is very similar and thus we omit. There exists $n_1 \in \mathbb{N}_0$ such that $x(n) > 0$ for all $n \geq n_1$. From (H2){i} and (H4){iv}, we may find $n_2 \geq n_1 + \delta$ such that

$$N_2 \sum_{n=n_2}^{\infty} \frac{1}{a(n)} \sum_{k=n-\delta}^{n-1} q(k) < \frac{1}{2}\left(1 - r^-\right) \tag{2.1}$$

holds. For $n \geq n_2$, set

$$y(n) := x(n) + r(n)F\left(x(n-\rho)\right) \tag{2.2}$$

and

$$z(n) := y(n) - \sum_{k=n_2}^{n-1} \frac{1}{a(k)} \sum_{l=k-\delta}^{k-1} q(l)G\left(x(l-\sigma)\right) - S(n). \tag{2.3}$$

Using the fact that x is a solution of (1.1) and (H4){i,ii}, we have

$$\begin{aligned}
\Delta w(n) =&\Delta\left[a(n)\Delta y(n)\right] - \left[q(n)G\left(x(n-\sigma)\right) - q(n-\delta)G\left(x(n-\tau)\right)\right] - s(n)\\
=& - p(n)F(x(n-\tau)) + q(n-\delta)G(x(n-\tau))\\
\leq& - p(n)G(x(n-\tau)) + q(n-\delta)G(x(n-\tau))\\
=& - \left[p(n) - q(n-\delta)\right]G(x(n-\tau))\\
=& - h(n)G\left(x(n-\tau)\right) \leq 0
\end{aligned} \tag{2.4}$$

for all $n \geq n_2$, where w is defined by $w(n) := a(n)\Delta z(n)$ for $n \geq n_2$. Clearly, w is eventually nonincreasing. Then, from (H4){ii,iii} and (2.4), we have either $w < 0$ or $w > 0$ for all $n \geq n_3$ for some $n_3 \geq n_2$. Consider the following possible ranges:

(C1) $w(n) < 0$ holds for all $n \geq n_3$. We first claim that

$$\lim_{n\to\infty} z(n) = -\infty \tag{2.5}$$

holds. Considering the definition of w, we may write

$$\Delta z(n) \leq \frac{w(n_3)}{a(n)} < 0 \tag{2.6}$$

for all $n \geq n_3$, which proves that (2.5) is true by summing up from n_3 to ∞ because of (H3). Hence, (H5) and (2.5) implies that

$$\lim_{n\to\infty}\left[z(n) + S(n)\right] = -\infty \tag{2.7}$$

holds. Next, we claim that x is bounded. If it is not the case, from (2.7), there exists $T \geq n_3$,

$$x(T) = \max\{x(n) : n_3 \leq n \leq T\} \text{ and } z(T) + S(T) < 0. \tag{2.8}$$

Therefore, considering (H2){i}, (2.1), (2.3) and (2.8), we obtain the following contradiction:

$$\begin{aligned}0 >& z(T)+S(T)=y(T)-\sum_{k=n_2}^{N-1}\frac{1}{a(k)}\sum_{l=k-\delta}^{k-1}q(l)G\left(x(l-\sigma)\right)\\ \geq& x(T)-r^{-}x(T-\rho)-N_2\sum_{k=n_2}^{N-1}\frac{1}{a(k)}\sum_{l=k-\delta}^{k-1}q(l)x(l-\sigma)\\ \geq&\left(1-r^{-}-N_2\sum_{k=n_2}^{N-1}\frac{1}{a(k)}\sum_{l=k-\delta}^{k-1}q(l)\right)x(T)\\ \geq&\frac{1}{2}\left(1-r^{-}\right)x(T)\geq 0.\end{aligned}$$

Thus, by (H2){i}, (H5), (2.1)–(2.3), we see that z is bounded. This is a contradiction to (2.5). Hence, this case is not possible.

(C2) $w(n)>0$ for all $n\geq n_3$. In this case, we see that L is a nonnegative constant, where $L:=\lim_{n\to\infty}w(n)$. Considering (H4){iii} and summing up (2.4) from n_3 to ∞, we obtain

$$\infty > w(n_3)-L=N_1\sum_{n=n_3}^{\infty}h(n)x(n-\tau), \tag{2.9}$$

which implies that $\liminf_{n\to\infty}x(n)=0$ and $\ell\in(0,\infty)$ are true, where $\ell:=\limsup_{n\to\infty}x(n)$. Note that, z has limit at infinity because $\Delta z>0$ holds since $a>0$ holds. Because of the boundedness of x, monotonicity of z, (H4){iv}, (H5) and (2.3), we infer that y has a finite limit at infinity. Now, we prove the contradiction that $\ell=0$ holds. For this purpose, pick two increasing divergent sequences of integers $\{\zeta_n\}_{n\in\mathbb{N}_0},\{\xi_n\}_{n\in\mathbb{N}_0}$ such that $\lim_{n\to\infty}x(\zeta_n)=\ell$ and $\lim_{n\to\infty}x(\xi_n)=0$ hold. Without loss in the generality, we may suppose that $\lim_{n\to\infty}x(\zeta_n-\rho)$ and $\lim_{n\to\infty}x(\xi_n-\rho)$ exist because of the boundedness of x, and it is trivial that all these limits are not greater than ℓ. From (2.2), we can estimate that

$$\begin{aligned}y(\zeta_n)-y(\xi_n)=& x(\zeta_n)+r(\zeta_n)F\left(x(\zeta_n-\rho)\right)-\left[x(\xi_n)+r(\xi_n)F\left(x(\xi_n-\rho)\right)\right]\\ \geq& x(\zeta_n)-r^{-}F\left(x(\zeta_n-\rho)\right)-\left[x(\xi_n)+r^{+}F\left(x(\xi_n-\rho)\right)\right]\\ \geq& x(\zeta_n)-r^{-}Mx(\zeta_n-\rho)-x(\xi_n)-r^{+}Mx(\xi_n-\rho)\end{aligned}$$

is true for all $n\in\mathbb{N}_0$, which yields the inequality

$$0\geq\left(1-\left(r^{-}+r^{+}\right)M\right)\ell$$

by letting n tend to infinity, and this implies that $\ell=0$ holds by (H2){i}. A contradiction.

Contradictions appear in both possible distinct cases. Hence, every solution of (1.1) oscillates or tends to zero at infinity. □

Theorem 2.2. *Assume that* (H1), (H2){ii}, (H3), (H4){i–iv}, *and* (H5) *hold, then every solution of* (1.1) *oscillates or tends to zero at infinity.*

Proof. Assume that (1.1) has an eventually positive solution x, which does not tend to zero at infinity. Pick $n_1 \in \mathbb{N}_0$ such that $x(n) > 0$ for all $n \geq n_1$. From (H2){ii} and (H4){iv}, we may find $n_2 \geq n_1 + \delta$ such

$$M \sum_{n=n_2}^{\infty} \frac{1}{a(n)} \sum_{k=n-\delta}^{n-1} q(k) < \frac{1}{2} \tag{2.10}$$

holds. Set y, z and w as in the proof of Theorem 2.1, then we have (2.4) for all $n \geq n_2$ for some $n_2 \geq n_1$. It is not hard to prove that $w < 0$ is not possible by following the steps in (C1) of the proof of Theorem 2.1. Then, by following the steps in (C2) of the proof of Theorem 2.1, we learn that $\liminf_{n\to\infty} x(n) = 0$ and $\ell \in (0, \infty)$ are true, where ℓ is the superior limit of x, and y has a finite limit at infinity. Now, we show that $\ell = 0$ holds. Pick two increasing divergent sequences of integers $\{\xi_n\}_{n\in\mathbb{N}_0}, \{\zeta_n\}_{n\in\mathbb{N}_0}$ as in the proof of Theorem 2.1. Without loss in the generality, we may suppose that $\lim_{n\to\infty} x(\xi_n+\rho)$ and $\lim_{n\to\infty} x(\zeta_n+\rho)$ exist. We can estimate that

$$\begin{aligned} y(\xi_n+\rho) - y(\zeta_n+\rho) =& x(\xi_n+\rho) + r(\xi_n+\rho)F\left(x(\xi_n)\right) - \left[x(\zeta_n+\rho) + r(\zeta_n+\rho)F\left(x(\zeta_n)\right)\right] \\ \leq& x(\xi_n+\rho) + r^+ M x(\xi_n) - r^- M x(\zeta_n) \end{aligned}$$

is true for all $n \in \mathbb{N}_0$, which yields to the inequality

$$0 \leq \left(1 + r^- M\right) \ell$$

by letting n tend to infinity, and this implies that $\ell = 0$ by (H2){ii}. This is a contradiction. Hence, every solution of (1.1) oscillates or tends to zero at infinity. □

The proof of the following theorem is very similar to that of Theorem 2.2, and thus we omit.

Theorem 2.3. *Assume that* (H3), (H2){iii}, (H3), (H4){i–iv} *and* (H5) *hold, then every bounded solution of* (1.1) *oscillates or tends to zero at infinity.*

3. Applications

To illustrate the applicability of our main results in § 2, we give the following examples.

Example 3.1. Consider the following neutral nonlinear difference equation:

$$\begin{aligned} \Delta^2 \left[x(n) + \frac{2}{5}(-1)^n \frac{x(n-2)|x^3(n-2)|}{|x^3(n-2)|+1}\right] \quad &+ \quad \frac{n}{n^2+1} \frac{x(n-3)\left(|x^3(n-3)|+1\right)}{|x^3(n-3)|+3} \\ &- \quad \frac{1}{3^n} \frac{x(n-1)\left(|x^3(n-1)|+1\right)}{|x^3(n-1)|+3} = \frac{2}{(n+1)(n+2)(n+3)}. \end{aligned} \tag{3.1}$$

For this equation, we see that $a(n) \equiv 1$, $r(n) = 2(-1)^n/5$, $\rho = 2$, $p(n) = 1/(n+1)$, $\tau = 3$, $q(n) = 1/3^n$, $\sigma = 1$, Hence, we have $r^- = r^+ = 2/5$, $r^- + r^+ = 4/5 < 1$, $\delta = \tau - \sigma = 3 - 1 = 2$, $h(n) = p(n) - q(n-\delta) = 1/(n+1) - 1/3^{n-2} \to 0^+$ as $n \to \infty$ and $S(n) = 1/\left(n+1\right)$ for $n \in \mathbb{N}_0$. It is not hard to see that

$$\sum_{n}^{\infty} h(n) = \sum_{n}^{\infty} \left(\frac{n}{n^2+1} - \frac{1}{3^{n-2}}\right) = \infty,$$

and

$$\sum_{n=2}^{\infty}\left(\sum_{k=n-\delta}^{n-1} q(k)\right)=\sum_{n=2}^{\infty}\left(\sum_{k=n-2}^{n-1} \frac{1}{3^k}\right)=2$$

are true. Therefore, all conditions of Theorem 2.1 are satisfied, and thus every solution of (3.1) oscillates or tends to zero at infinity. The following graphic belongs to the solution with the initial conditions $x(-3)=x(-2)=x(-1)=x(0)=x(1)=1$ and of 70 iterates:

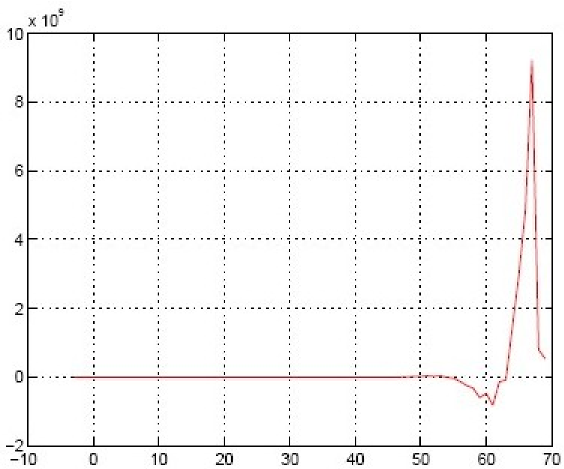

Figure 1. Graphic of $(n, x(n))$

Next, we give another example.

Example 3.2. Consider the following neutral nonlinear difference equation:

$$\begin{aligned}\Delta\left[\frac{1}{n}\Delta\left[x(n)+\frac{x(n-3)|x(n-3)|}{|x(n-3)|+1}\right]\right] &+ \frac{n^2}{n^3+1}\frac{x(n-2)\left(|x(n-2)|+3\right)}{|x(n-2)|+5} \\ &- \frac{1}{7^n}\frac{x(n-1)\left(|x(n-1)|+3\right)}{|x(n-1)|+5}=\frac{2}{(n+2)(n+3)(n+4)}.\end{aligned}\tag{3.2}$$

For this equation, we see that $a(n)=1/n$, $r(n)=3$, $\rho=3$, $p(n)=n^2/(n^3+1)$, $\tau=2$, $q(n)=1/7^n$, $\sigma=1$. Hence, we have $r^-=r^+=3>1$, $\delta=\tau-\sigma=2-1=1$, $h(n)=p(n)-q(n-\delta)=n^2/(n^3+1)-1/7^{n-1}\to 0^+$ as $n\to\infty$ and $S(n)=1/(n+2)$ for $n\in\mathbb{N}_0$. It is not hard to see that

$$\sum_{n}^{\infty} h(n)=\sum_{n}^{\infty}\left(\frac{n^2}{n^3+1}-\frac{1}{7^{n-1}}\right)=\infty,$$

and

$$\sum_{n=1}^{\infty}\left(\frac{1}{a(n)}\sum_{k=n-\delta}^{n-1} q(k)\right)=\sum_{n=1}^{\infty}\frac{n}{7^{n-1}}=\frac{49}{36}$$

are true. Therefore, all conditions of Theorem 2.2 are satisfied, and thus every solution of (3.2) oscillates or tends to zero at infinity. The following graphic belongs to the solution with the initial conditions $x(-3)=x(-2)=x(-1)=x(0)=x(1)=1$ and of 70 iterates:

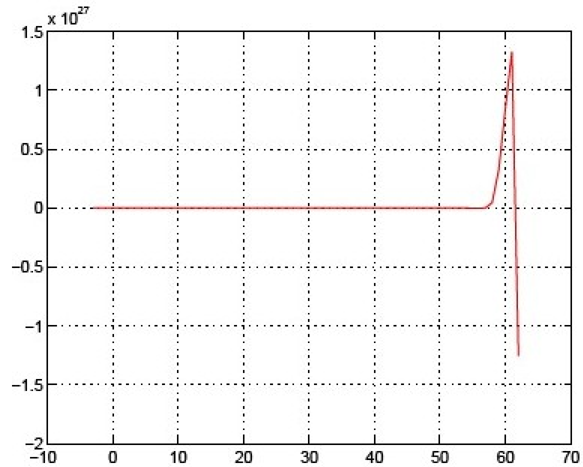

Figure 2. Graphic of $(n, x(n))$

Next, we give another example.

Example 3.3. Consider the following neutral nonlinear difference equation:

$$\begin{aligned}(3.3)\ \Delta\left[\frac{1}{n}\Delta\left[x(n)-2\frac{x^3(n-1)}{x^2(n-1)+1}\right]\right] \quad &+ \quad \frac{n^2}{n^3+1}\frac{x(n-3)(x^2(n-3)+2)}{x^2(n-3)+3} \\ &- \quad \frac{1}{5^n}\frac{x(n-2)(x^2(n-2)+2)}{x^2(n-2)+3}=0.\end{aligned}$$

For this equation, we see that $a(n) = 1/n$, $r(n) = 1/4$, $p(n) = n^2/(n^3 + 1)$, $\tau = 3$, $q(n) = 1/5^n$, $\sigma = 1$. Hence, we have $r^+ = 2$, $r_1^- = 2$, $\delta = \tau - \sigma = 3 - 2 = 1$, $h(n) = p(n) - q(n - \delta) = n^2/(n^3 + 1) - 1/5^{n-1} \to 0^+$ as $n \to \infty$ and $S(n) \equiv 0$ for $n \in \mathbb{N}_0$. Also, one can shown that

$$\sum_n^{\infty} h(n) = \sum_n^{\infty}\left(\frac{n^2}{n^3+1}-\frac{1}{5^{n-1}}\right)=\infty,$$

and

$$\sum_{n=1}^{\infty}\left(\frac{1}{a(n)}\sum_{k=n-\delta}^{n-1} q(k)\right)=\sum_{n=1}^{\infty}\frac{n}{5^{n-1}}=\frac{25}{16}$$

hold. Therefore, all bounded solutions of (3.3) oscillate or tend to zero at infinity by Theorem 2.3.

The following graphic probably belongs to an unbounded solution with the initial conditions $x(-3) = x(-2) = x(-1) = x(0) = x(1) = 1$ and of 75 iterates:

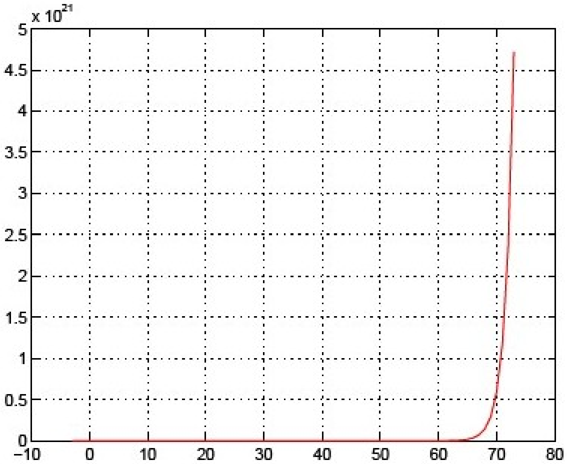

Figure 3. Graphic of $(n, x(n))$

Thus, the equation may also admit unbounded solutions.

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