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ON INVARIANT SUBMANIFOLDS OF ALMOST α-COSYMPLECTIC f-MANIFOLDS

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Abstract. In this paper, we investigate some properties of invariant submanifolds of almost α-cosymplectic f- manifolds. We show that every invariant submanifold of an almost α-cosymplectic f- manifold with Kaehlerian leaves is also an almost α-cosymplectic f- manifold with Kaehlerian leaves. Moreover, we give a theorem on minimal invariant submanifold and obtain a necessary condition on a invariant submanifold to be totally geodesic. Finally, we study some properties of the curvature tensors of M and $\widetilde{M}$.

1. Introduction

In 1963, Yano [13] introducted an f-structure on a C^∞ m-dimensional manifold M, defined by a non-vanishing tensor field φ of type $(1,1)$ which satisfies $\varphi^3+\varphi=0$ and has constant rank r. It is know that in this case r is even, $r=2n$. Moreover. TM splits into two complemantary subbundles $Im\varphi$ and $ker\varphi$ and the restriction of φ to $Im\varphi$ determines a complex structure on such subbundle. It is know that the exixtence of an f-structure on M is equivalent to a reduction of the structure group to $U(n)\times O(s)$ [2], where $s=m-2n$. The geometry of invariant submanifolds of a Riemannian manifold was studied by many geometers (see [3], [4], [6], [7], [8], [9], [10]). In general, the geometry an invariant submanifold inherits almost all properties of the ambient manifold. In 2014, Öztürk et.al. introduced and studied almost α-cosymplectic f-manifold [7] defined for any real number α which is defined a metric f-manifold with f-structure (φ,ξ_i,η^i,g) satisfying the condition $d\eta^i=0$, $d\Omega=2\alpha\overline{\eta}\wedge\Omega$.

In this paper, we introduce properties of invariant submanifolds of an almost α-cosymplectic f-manifold. In Section 2, we review basic formulas and definitions for almost α-cosymplectic f-manifolds. In Section 3, we show that every invariant submanifold of an almost α-cosymplectic f- manifold with Kaehlerian leaves is also an almost α-cosymplectic f- manifold with Kaehlerian leaves. Further, we give

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a theorem on minimal invariant submanifold and obtain a necessary condition on a invariant submanifold to be totally geodesic. In last section, we obtain some relations of curvature tensors M and $\widetilde{M}$.

2. Preliminaries

Let $\widetilde{M}$ be a real $(2n+s)$-dimensional framed metric manifold [12] with a framed $(\varphi, \xi_i, \eta^i, g)$, $i \in \{1, ..., s\}$, that is, φ is a non-vanishing tensor field of type (1,1) on $\widetilde{M}$ which satisfies $\varphi^3 + \varphi = 0$ and has constant rank $r = 2n$; $\xi_1, ...\xi_s$ are s vector fields; $\eta^1, ..., \eta^s$ are 1-forms and g is a Riemannian metric on $\widetilde{M}$ such that

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \tag{2.1}$$

$$\eta^i(\xi_j) = \delta^i_j, \ \varphi(\xi_i) = 0, \ \eta^i o \varphi = 0, \tag{2.2}$$

$$\eta^i(X) = g(X, \xi_i), \tag{2.3}$$

$$g(X, \varphi Y) + g(\varphi X, Y) = 0, \tag{2.4}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y) \tag{2.5}$$

for all $X, Y \in \Gamma(T\widetilde{M})$ and $i, j \in \{1, ..., s\}$. In above case, we say that $\widetilde{M}$ is a metric f-manifold and its associated structure will be denoted by $\widetilde{M}(\varphi, \xi_i, \eta^i, g)$ [12].
A 2-form Ω is defined by $\Omega(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(T\widetilde{M})$, is called the fundamental 2-form. A framed metric structure is called normal [12] if

$$[\varphi, \varphi] + 2d\eta^i \otimes \xi_i = 0$$

where $[\varphi, \varphi]$ is denoting the Nijenhuis tensor field associated to φ. Throughout this paper we denote by $\overline{\eta} = \eta^1 + \eta^2 + ... + \eta^s$, $\overline{\xi} = \xi_1 + \xi_2 + ... + \xi_s$ and $\overline{\delta}^j_i = \delta^1_i + \delta^2_i + ... + \delta^s_i$. In the sequel, from [7] we give the following definition.

Definition 2.1. Let $\widetilde{M}(\varphi, \xi_i, \eta^i, g)$ be a $(2n+s)$-dimensional a metric f-manifold for each $\eta^i, (1 \leq i \leq s)$ 1-forms and each 2-form Ω, if $d\eta^i = 0$ and $d\Omega = 2\alpha\overline{\eta} \wedge \Omega$ satisfy, then $\widetilde{M}$ is called almost α-cosymplectic f-manifold [7].

Let $\widetilde{M}$ be an almost α-cosypmlectic f-manifold. Since the distribution D is integrable, we have $L_{\xi_i}\eta^j = 0$, $[\xi_i, \xi_j] \in D$ and $[X, \xi_j] \in D$ for any $X \in \Gamma(D)$. Then the Levi-Civita connection is given by [7]:

$$2g((\widetilde{\nabla}_X \varphi)Y, Z) = 2\alpha g\left(\sum_{i=1}^{s}(g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X), Z\right) + g(N(Y, Z), \varphi X) \tag{2.6}$$

for any $X, Y \in \Gamma(T\widetilde{M})$. Putting $X = \xi_i$ we obtain $\widetilde{\nabla}_{\xi_i}\varphi = 0$ which implies $\widetilde{\nabla}_{\xi_i}\xi_j \in D^\perp$ and then $\widetilde{\nabla}_{\xi_i}\xi_j = \widetilde{\nabla}_{\xi_j}\xi_i$, since $[\xi_i, \xi_j] = 0$. We put $A_i X = -\widetilde{\nabla}_X \xi_i$ and $h_i =$

$\frac{1}{2}(L_{\xi_i}\varphi)$, where L denotes the Lie derivative operator. If $\widetilde{M}$ is almost α-cosymplectic f-manifold with Kaehlerian leaves [6], we have

$$(\widetilde{\nabla}_X\varphi)Y = \sum_{i=1}^{s}\left[-g(\varphi A_iX, Y)\xi_i + \eta^i(Y)\varphi A_iX\right]$$

or

$$(\widetilde{\nabla}_X\varphi)Y = \sum_{i=1}^{s}\left[\alpha\left(g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\right) + g(h_iX, Y)\xi_i - \eta^i(Y)h_iX\right]. \tag{2.7}$$

Proposition 2.1. *([7]) For any $i \in \{1, ..., s\}$ the tensor field A_i is a symmetric operator such that*

(i) $A_i(\xi_j) = 0$, *for any* $j \in \{1, ..., s\}$
(ii) $A_i o\varphi + \varphi o A_i = -2\alpha\varphi$
(iii) $tr(A_i) = -2\alpha n$
(iv) $\widetilde{\nabla}_X\xi_i = -\alpha\varphi^2X - \varphi h_iX.$

for any $X \in \Gamma(T\widetilde{M})$.

Proposition 2.2. *([2]) For any $i \in \{1, ..., s\}$ the tensor field h_i is a symmetric operator and satisfies*

(i) $h_i(\xi_j) = 0$, *for any* $j \in \{1, ..., s\}$
(ii) $h_i o\varphi + \varphi o h_i = 0$
(iii) $trh_i = 0$
(iv) $tr(\varphi h_i) = 0.$

Let $\widetilde{M}$ be an almost α-cosymplectic f-manifold with respect to the curvature tensor field $\widetilde{R}$ of $\widetilde{\nabla}$, the following formulas are proved in [7], for all $X, Y \in \Gamma(T\widetilde{M})$, $i, j \in \{1, ..., s\}$.

$$\begin{aligned}\widetilde{R}(X, Y)\xi_i = {} & \alpha^2\sum_{k=1}^{s}(\eta^k(Y)\varphi^2X - \eta^k(X)\varphi^2Y) \\ & - \alpha\sum_{k=1}^{s}(\eta^k(X)\varphi h_kY - \eta^k(Y)\varphi h_kX) \\ & + (\widetilde{\nabla}_Y\varphi h_i)X - (\widetilde{\nabla}_X\varphi h_i)Y,\end{aligned} \tag{2.8}$$

$$\begin{aligned}\widetilde{R}(X, \xi_j)\xi_i = {} & \sum_{k=1}^{s}\delta_j^k(\alpha^2\varphi^2X + \alpha\varphi h_kX) \\ & + \alpha\varphi h_iX - h_ih_jX + \varphi(\widetilde{\nabla}_{\xi_j}h_i)X,\end{aligned} \tag{2.9}$$

$$\widetilde{R}(\xi_j, X)\xi_i - \varphi\widetilde{R}(\xi_j, \varphi X)\xi_i = 2(-\alpha^2\varphi^2X + h_ih_jX). \tag{2.10}$$

Moreover, by using the above formulas, in [7] it is obtained that

$$\widetilde{S}(X, \xi_i) = -2n\alpha^2\sum_{k=1}^{s}\eta^k(X) - (div\varphi h_i)X, \tag{2.11}$$

$$\widetilde{S}(\xi_i, \xi_j) = -2n\alpha^2 - tr(h_jh_i) \tag{2.12}$$

for all $X, Y \in \Gamma(T\widetilde{M})$, $i, j \in \{1, ..., s\}$, where $\widetilde{S}$ denote, the Ricci tensor field of the Riemannian connection.

From [7], we have the following result.

Proposition 2.3. *Let $\widetilde{M}$ be an almost α-cosymplectic f-manifold and M be an integral manifold of D. Then*

(i) *when $\alpha = 0$, M is totally geodesic if and only if all the operators h_i vanish;*
(ii) *when $\alpha \neq 0$, M is totally umbilic if and only if all the operators h_i vanish.*

Theorem 2.1. [2] *A C-manifold $\widetilde{M}^{2n+s}$ is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold $\widetilde{M}_1^{2n}$ and an Abelian Lie group $\widetilde{M}_2^s$.*

3. On Invariant Submanifold Of Almost α -Cosymplectic f-Manifolds

Let M be a submanifold of the a $(2n+s)$-dimensional almost α-cosymplectic f-manifold $\widetilde{M}$. If $\varphi(T_pM) \subset T_pM$, for any point $p \in M$ and ξ_i are tangent to M for all $i \in \{1, ..., s\}$, the M is called an invariant submanifold of $\widetilde{M}$.
Let ∇ be the Levi-Civita connection of M with respect to the induced metric g. Then Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X, Y) \tag{3.1}$$

$$\widetilde{\nabla}_X N = \nabla^{\perp}_X N - A_N X \tag{3.2}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM)^{\perp}$. $\nabla^{\perp}$ is the connection in the normal bundle, B is the second fundamental form of M and A_N is the Weingarten endomorfhism associated with N. The second fundamental form B and the shape operator A related by

$$g(B(X, Y), N) = g(A_N X, Y). \tag{3.3}$$

The curvature transformattion of M and $\widetilde{M}$ will be denote by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{3.4}$$

and

$$\widetilde{R}(X, Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z, \tag{3.5}$$

respectively. Using (3.1) and (3.2) in (3.4) and (3.5), we obtain

$$\begin{aligned} \widetilde{R}(X, Y)Z = {} & R(X, Y)Z - A_{B(Y,Z)} X + A_{B(X,Z)} Y \\ & + (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \end{aligned} \tag{3.6}$$

for any $X, Y, Z \in \Gamma(TM)$. Then, if W is tangent to M, then using (3.6), we get

$$\begin{aligned} g(\widetilde{R}(X, Y)Z, W) = {} & g(R(X, Y)Z, W) + g(B(Y, W), B(X, Z)) \\ & - g(B(X, W), B(Y, Z)). \end{aligned} \tag{3.7}$$

Proposition 3.1. *Let M be an invariant submanifold of the almost α-cosymplectic f-manifold $\widetilde{M}$. Then we have*

$$(\widetilde{\nabla}_X \varphi) Y = (\nabla_X \varphi) Y \tag{3.8}$$

and

$$B(X, \varphi Y) = \varphi B(X, Y) = B(\varphi X, Y) \tag{3.9}$$

for any $X, Y \in \Gamma(TM)$.

Proof. For any $X, Y \in \Gamma(TM)$, using (3.1) we get

$$\begin{aligned}(\widetilde{\nabla}_X \varphi)Y &= \widetilde{\nabla}_X \varphi Y - \varphi \widetilde{\nabla}_X Y \\ &= \nabla_X \varphi Y + B(X, \varphi Y) - \varphi \nabla_X Y - \varphi B(X, Y) \\ &= (\nabla_X \varphi)Y + B(X, \varphi Y) - \varphi B(X, Y)\end{aligned}$$

In above equation, comparing the tangential and normal part of last equation, we obtain $B(X, \varphi Y) = \varphi B(X, Y)$. Then (3.9) follows in both cases by the symmetry of B. □

From (3.9) and using symmetry of B, we have the following result.

Corollary 3.1. *Let M be an invariant submanifold of the almost α-cosymplectic f-manifold $\widetilde{M}$. Then we get*

$$B(\varphi X, \varphi Y) = -B(X, Y) \tag{3.10}$$

for any $X, Y \in \Gamma(TM)$.

Definition 3.1. A submanifold of an almost α-cosymplectic f-manifold is called totally geodesic if $B(X, Y)$=0, for any $X, Y \in \Gamma(TM)$.

Proposition 3.2. *Let M be an invariant submanifold of the almost α-cosymplectic f-manifold. Then we have*

$$\widetilde{\nabla}_X \xi_j = \nabla_X \xi_j \tag{3.11}$$

and

$$B(X, \xi_j) = 0 \tag{3.12}$$

for any $X \in \Gamma(TM)$.

Proof. From (3.8), we obtain

$$\begin{aligned}(\widetilde{\nabla}_X \varphi)\xi_j = (\nabla_X \varphi)\xi_j \quad &\Rightarrow \varphi \widetilde{\nabla}_X \xi_j = \varphi \nabla_X \xi_j \\ &\Rightarrow \widetilde{\nabla}_X \xi_j = \nabla_X \xi_j.\end{aligned}$$

Then, using (3.11) we have

$$\begin{aligned}\widetilde{\nabla}_X \xi_j \quad &= \nabla_X \xi_j + B(X, \xi_j) \\ &\Rightarrow B(X, \xi_j) = 0.\end{aligned}$$

□

Proposition 3.3. *An invariant submanifold of an almost α-cosymplectic f-manifold with Kaehlerian leaves is also almost α-cosymplectic f-manifold with Kaehlerian leaves.*

Proof. For any $X, Y \in \Gamma(TM)$, using (3.1) we get

$$\begin{aligned}(\widetilde{\nabla}_X \varphi)Y &= \widetilde{\nabla}_X \varphi Y - \varphi(\widetilde{\nabla}_X Y) \\ &= \nabla_X \varphi Y + B(X, \varphi Y) - \varphi(\nabla_X Y) - \varphi B(X, Y).\end{aligned}$$

From (2.7) and the above equation, we get by considering the submanifold as invariant and comparing tangential and normal compenents, we obtain

$$(\nabla_X\varphi)Y = \sum_{i=1}^{s}\left[\alpha\left(g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\right) + g(h_iX, Y)\xi_i - \eta^i(Y)h_iX\right]. \tag{3.13}$$

From (3.13), we get the proof. □

Theorem 3.1. *Each invariant submanifold of almost α-cosymplectic f- manifold is minimal.*

Proof. Suppose that M minimal submanifold of an almost α-cosymplectic f-manifold and $\dim M = 2m + s (m < n)$. From (3.3), one can write,

$$\begin{aligned}(2m+s)tr(A_N) &= \sum_{i=1}^{m} g(B(e_i, e_i), N)\\ &+ \sum_{i=1}^{m} g(B(\varphi e_i, \varphi e_i), N)\\ &+ \sum_{i=1}^{s} g(B(\xi_i, \xi_i), N)\\ &= 0.\end{aligned}$$

Hence from above calculations, mean curvature of M, so $tr(A_N) = 0$. □

4. Curvature properties

Proposition 4.1. *Let M be an invariant submanifold of the almost α-cosymplectic f-manifold $\widetilde{M}$. Then we*

$$\widetilde{R}(X, Y)\xi_i = R(X, Y)\xi_i \tag{4.1}$$

for any $X, Y \in \Gamma(TM)$.

Proof. For any $X, Y \in \Gamma(TM)$, using (3.1) in (3.6) we get

$$\begin{aligned}\widetilde{R}(X, Y)\xi_i &= R(X, Y)\xi_i - A_{B(Y,\xi_i)}X + A_{B(X,\xi_i)}Y + (\nabla_X B)(Y, \xi_i) - (\nabla_Y B)(X, \xi_i)\\ &= R(X, Y)\xi_i - B(Y, \nabla_X\xi_i) + B(X, \nabla_Y\xi_i)\\ &= R(X, Y)\xi_i + \alpha\varphi^2 B(Y, X) - \alpha\varphi^2 B(X, Y) + \varphi B(Y, h_iX) - \varphi B(X, h_i, Y)\\ &= R(X, Y)\xi_i.\end{aligned}$$

□

Corollary 4.1. *Let M be an invariant submanifold of the almost α-cosymplectic f-manifold $\widetilde{M}$. Then $\widetilde{R}(X, Y)\xi_i$ is tangent to M for any $X, Y \in \Gamma(TM)$ and $i = 1, ..., s$.*

Proposition 4.2. *Let M be an invariant submanifold of the almost α-cosymlectic f- manifold $\widetilde{M}$. Then, we have*

$$\widetilde{R}(\xi_j, X)\xi_i = R(\xi_j, X)\xi_i, \tag{4.2}$$

$$\widetilde{R}(X, \xi_j)\xi_i = R(X, \xi_j)\xi_i, \tag{4.3}$$

$$\widetilde{R}(\xi_k, \xi_j)\xi_i = R(\xi_k, \xi_j)\xi_i = 0, \tag{4.4}$$

$$\widetilde{R}(\xi_j, X)Y = R(\xi_j, X)Y \tag{4.5}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Using (4.1), we obtain (4.2), (4.3), (4.4) and (4.5). □

Proposition 4.3. *Let M be an invariant submanifold of the almost α-cosymlectic f-manifold $\widetilde{M}$. Then, following relations hold*

$$\varphi(A_N X) = A_{\varphi N} X = -A_N \varphi X \tag{4.6}$$

for any $X \in \Gamma(TM)$,

Proof. For any $X, Y \in \Gamma(TM)$, using (3.3) and (3.9) we have

$$\begin{aligned} g(\varphi(A_N X), Y) &= -g(A_N X, \varphi Y) \\ &= -g(B(X, \varphi Y), N) \\ &= -g(B(\varphi X, Y), N) \\ &= -g(A_N \varphi X, Y) \end{aligned}$$

and then,

$$\varphi(A_N X) = -A_N \varphi X.$$

Moreover, we have

$$\begin{aligned} g(\varphi(A_N X), Y) &= -g(B(X, \varphi Y), N) \\ &= -g(\varphi B(X, Y), N). \end{aligned}$$

On the other hand, using (3.3) we have

$$g(A_{\varphi N} X, Y) = g(B(X, Y), \varphi N) = -g(\varphi B(X, Y), N)$$

and then we get

$$\varphi(A_N X) = A_{\varphi N} X.$$

□

Proposition 4.4. *Let M be an invariant submanifold of the almost α-cosymplectic f-manifold $\widetilde{M}$. Then we have*

$$\begin{aligned} g(R(X, \varphi X)\varphi X, X) &= g(\widetilde{R}(X, \varphi X)\varphi X, X) \\ &\quad - 2g(B(X, X), B(X, X)). \end{aligned} \tag{4.7}$$

for any $X \in \Gamma(TM)$.

Proof. In (3.6), if we take $Z = Y = \varphi X$ and $W = X$, then we obtain (4.7). □

Proposition 4.5. *Let M be an invariant submanifold of the almost α-cosymplectic f-manifold $\widetilde{M}$. And let $\widetilde{M}$ be of constant φ sectional curvature* [2]. *Then M is totally geodesic if only if M has constant φ sectional curvature.*

Proof. Let M be totally geodesic then from (4.7), the sectional curvature of M is the same as $\widetilde{M}$. Vice versa we suppose that the sectional φ-curvature determined by $\{X, \varphi X\}$ is the same for M and $\widetilde{M}$ for any $X \in \Gamma(TM)$. Hence from (4.7), we get that $B(X,X) = 0$ and $B = 0$. □

Proposition 4.6. *Let M be an invariant submanifold of the almost α cosymplectic f-manifold $\widetilde{M}$ and let $\alpha = 0$. Then B is parallel if only if M is totally geodesic.*

Proof. An easy calculation, we get

$$(\nabla_X B)(Y, \xi_i) = -\alpha B(Y,X) + h_i \varphi B(Y,X).$$

for any $X, Y \in \Gamma(TM)$. Hence, if B is parallel, then $B(Y,X) = 0$, for any $X, Y \in \Gamma(TM)$. Vice versa, it is clear that if $B = 0$, then $\nabla B = 0$, so B is parallel. □

Let M be a submanifold of a Rieamannian manifold $\widetilde{M}$. An isometric immersion $i : M \longrightarrow \widetilde{M}$ is semi- parallel if

$$\widetilde{R}(X,Y)B = \widetilde{\widetilde{\nabla}}_X(\widetilde{\widetilde{\nabla}}_Y B) - \widetilde{\widetilde{\nabla}}_Y(\widetilde{\widetilde{\nabla}}_X B) - \widetilde{\widetilde{\nabla}}_{[X,Y]} B = 0$$

where $\widetilde{R}$ is the curvature tensor of $\widetilde{\widetilde{\nabla}}$ [3], where $\widetilde{R}$ curvature tensor of the Van der Waerden-Bortolotti connection $\widetilde{\widetilde{\nabla}}$ and B the second fundamental from. In([1]), K. Arslan et al. defined and studied 2-semi parallel submanifold if

$$R(X,Y)\nabla B = 0$$

for any $X, Y \in \Gamma(TM)$.

Theorem 4.1. *Let M be an invariant submanifold of the α-cosymplectic f-manifold. If $\widetilde{M}$ is semi-parallel, then*
1) *When $\alpha = 0$, M totally geodesic and $\widetilde{M}$ is a locally decomposable Riemannian manifold which is locally the product of a kaehler manifold M_1^{2n} and an Abelian Lie group M_2^s.*
2) *When $\alpha \neq 0$, M totally geodesic.*

Proof. $\widetilde{\widetilde{\nabla}}$ is the connection in $TM \oplus TM^{\perp}$ built with ∇ and $\nabla^{\perp}$, where R $(resp. R^{\perp})$ denotes curvature tensor of the connection $\nabla(resp. \nabla^{\perp})$. If $R^{\perp}$ denotes the curvature tensor of $\nabla^{\perp}$ then we have

$$\begin{aligned}(\widetilde{R}(X,Y)B)(Z,U) &= R^{\perp}(X,Y)B(Z,U) \\ &\quad - B(R(X,Y)Z,U) \\ &\quad - B(Z,R(X,Y)U)\end{aligned} \tag{4.8}$$

for any $X, Y, Z, U \in \Gamma(TM)$. Now, we suppose that M is semi-parallel. Then $\widetilde{R}(X,Y)B = 0$ for any $X, Y \in \Gamma(TM)$. Using (4.8), we get

$$R^{\perp}(X,Y)B(Z,K) - B(R(X,Y)Z,K) - B(Z,R(X,Y)K) = 0.$$

If we take $X = \xi_i$, $K = \xi_j$, then we obtain,

$$R^{\perp}(\xi_i,Y)B(Z,\xi_j) - B(R(\xi_i,Y)Z,\xi_j) - B(Z,R(\xi_i,Y)\xi_j) = 0.$$

From (3.12), we have

$$B(Z,R(\xi_i,Y)\xi_j) = 0$$

and from the above equation, we arrive

$$\alpha^2 B(Z,Y) = 0.$$

So, we get $\alpha = 0$ or $B = 0$. □

References

[1] Arslan K., Lumiste C., Murathan C. and Özgür C., *2- semiparallel Surfaces in Space Forms.* I. Two Particular Cases, Proc. Estonian Acad. Sci Phys. Math., 49(3), (2000), 139-148.

[2] Blair D.E., *Geometry of manifolds with structural group $U(n) \times O(s)$*, J. Differential Geometry, 4(1970), 155-167.

[3] Chen B.Y., *Geometry of submanifolds*, Marcel Dekker Inc., New York, (1973).

[4] Chinea D., Prestelo P.S., *Invariant submanifolds of a trans-Sasakian manifolds.* Publ. Mat. Debrecen, 38/1-2 (1991), 103-109.

[5] Endo H., *Invariant submanifolds in contact metric manifolds*, Tensor (N.S.) 43 (1) (1886), pp. 193-202.

[6] Erken K.I, Dacko P. and Murathan C., *Almost α-paracosymplectic manifolds*, arxiv: 1402.6930v1 [$Math.DG$] 27 Feb 2014.

[7] Öztürk H., Murathan C., Aktan N., Vanli A.T., *Almost α-cosymplectic f-manifolds* Analele stııntıfice ale unıversıtatıı 'AI.I Cuza' Dı ıaşı (S.N.) Matematica, Tomul LX, f.1., (2014).

[8] Kon M., *Invariant submanifolds of normal contact metric manifolds*, Kodai Math. Sem. Rep., 27, (1973), 330-336.

[9] Terlizi L. D., *On invariant submanifolds of C and S-manifolds.* Acta Math. Hungar. 85(3), (1999), 229-239.

[10] Sarkar A. and Sen M., *On invariant submanifold of trans- sasakian manifolds*, Proceedings of the Estonian Academy of Sciences, 61(1), (2012), 29-37.

[11] De A., *Totally geodesic submanifolds of a trans-Sasakian manifold*, Proceedings of the Estonian Academy of Sciences, 62(4), (2013), 249-257.

[12] Yano K. and Kon M., *Structures on manifolds.* World Scientific, Singapore (1984).

[13] Yano K., *On a structure defined by a tensor f of type $(1,1)$ satisfying* $\varphi^3 + \varphi = 0$, tensor N S., 14, (1963), 99-109.

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