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ON GENERALIZED f-HARMONIC MAPS AND LIOUVILLE TYPE THEOREM

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Abstract. In this paper, we prove that every semi-conformal harmonic map between Riemannian manifolds is a generalized f-harmonic map. We also prove a Liouville type theorem for f-harmonic maps in general sense from IR^m onto a Riemannian manifold N with non-positive sectional curvature, where $f \in C^\infty(IR^m \times N)$ is a smooth positive function which satisfies some suitable conditions.

1. Introduction

Liouville type theorems for harmonic maps between complete smooth Riemannian manifolds have been done by many authors. Eells-Sampson [9] proved that any (bounded) harmonic map from a compact Riemannian manifold with positive Ricci curvature into a complete manifold with non-positive curvature is a constant map. Schoen-Yau [15] also proved that any harmonic map with finite energy from a complete smooth Riemannian manifold with non-negative Ricci curvature into a complete manifold with non-positive curvature is a constant map. Cheng [3] showed that any harmonic map with sublinear growth from a complete Riemannian manifold with non-negative Ricci curvature into an Hadamard manifold is a constant map. Liu [8] proved the Liouville-type theorem for p-harmonic maps with free boundary values. Bair-Fardoun-Ouakkas [1] proved the Liouville-type theorem for bi-harmonic maps

The purpose of this paper is to provide a proof of the Liouville type theorem for f-harmonic maps in generalized sense from IR^m onto a Riemannian manifold N with non-positive sectional curvature, where $f \in C^\infty(IR^m \times N)$ is a smooth positive function which satisfies some suitable conditions.

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Consider a smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ between Riemannian manifolds and let $f : M \times N \longrightarrow (0, +\infty)$ be a smooth positive function. The map φ is said to be a generalized f-harmonic map, if it is a critical point of the f-energy functional

$$E_f(\varphi) = \frac{1}{2}\int_K f(x, \varphi(x))\, |d\varphi|^2\, v_g. \tag{1.1}$$

on any compact subset $K \subset M$. The Euler-Lagrange equation associated to the f-energy functional is

$$\tau_f(\varphi) \quad \equiv \quad f_\varphi\, \tau(\varphi) + d\varphi(\mathrm{grad}^M f_\varphi) - e(\varphi)(\mathrm{grad}^N f) \circ \varphi = 0, \tag{1.2}$$

where:
$f_\varphi : M \longrightarrow (0, +\infty)$ is the positive function defined by

$$f_\varphi(x) = f(x, \varphi(x)), \quad x \in M, \tag{1.3}$$

$$(grad^M f)_{(x,y)} = \sum_{i,j=1}^{m} g^{ij} \frac{\partial f}{\partial x^i}(x, y) \frac{\partial}{\partial x^j}, \quad (x, y) \in M \times N,$$
$$(grad^N f)_{(x,y)} = \sum_{i,j=1}^{n} h^{ij} \frac{\partial f}{\partial y^i}(x, y) \frac{\partial}{\partial y^j}, \quad (x, y) \in M \times N,$$

$\tau(\varphi) = \mathrm{trace}_g \nabla d\varphi$ is the tension field of φ, and $e(\varphi) = \frac{1}{2}|d\varphi|^2$ is the energy density of φ.

$\tau_f(\varphi)$ is called the f-tension field of φ ([4]) [11]).

2. Semi-conformal maps and f-harmonicity

Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds. Let $x \in M$, the tangent space at x splits $T_x M = H_x \oplus V_x$ where $V_x = \mathrm{Ker}\, d_x\varphi$ and $H_x = V_x^\perp$ is the orthogonal complement of the vertical space V_x. The map φ is called semi-conformal if for each $x \in M$ where $d_x\varphi \neq 0$ the restriction $d_x\varphi : H_x \longrightarrow T_{\varphi(x)}N$ is conformal and surjective. On setting $\lambda(x) = 0$ at points x where $d_x\varphi = 0$, we obtain a continuous function $\lambda : M \longrightarrow I\!R_+$ such that for any $X, Y \in H_x$

$$h(d_x\varphi(X), d_x\varphi(Y)) = \lambda^2(x) g(X, Y),$$

the function λ is called the dilation of φ. Note that the generalized conformal maps is discussed in [13].

Let M^m be a Riemannian manifold and N^n be a Riemannian submanifold of $I\!R^k$. Then, we have

Theorem 2.1. *Any semi-conformal harmonic map $\varphi : M^m \longrightarrow N^n$ is f-harmonic with $f(x, y) = F\big(2\, y + (n - 2)\, \varphi(x)\big)$ for all $(x, y) \in M \times N$ where $F \in C^\infty(I\!R^k)$ is a smooth positive function.*

Proof. A semi-conformal harmonic map φ is f-harmonic if and only if

$$\tau_f(\varphi) \quad = \quad d\varphi(\mathrm{grad}^M f_\varphi) - e(\varphi)(\mathrm{grad}^N f) \circ \varphi = 0,$$

where $f_\varphi : M \longrightarrow (0, +\infty)$ is a smooth positive function given by

$$f_\varphi(x) = f(x, \varphi(x)) = F\big(n\, \varphi(x)\big).$$

Let us choose $\{e_1, ..., e_m\}$ an orthonormal frame on a domain of M such that the vectors $\{e_1, ..., e_n\}$ are horizontal and the vectors $\{e_{n+1}, ..., e_m\}$ are vertical, so that $d\varphi(e_i) = \lambda\,(\widetilde{e}_i \circ \varphi)$ for $i = 1, ..., n$ where $\{\widetilde{e}_1, ..., \widetilde{e}_n\}$ is an orthonormal frame on a domain of N. Then, we have

$$\begin{aligned}
d\varphi(\operatorname{grad}^M f_\varphi) &= \sum_{i=1}^{m} e_i(f_\varphi)\, d\varphi(e_i) \\
&= n \sum_{i=1}^{n} d\varphi(e_i)(F)\, d\varphi(e_i) \\
&= n\,\lambda^2 \sum_{i=1}^{n} (\widetilde{e}_i \circ \varphi)(F)\,(\widetilde{e}_i \circ \varphi) \\
&= n\,\lambda^2 (\operatorname{grad}^N F) \circ \varphi. \\
(\operatorname{grad}^N f) \circ \varphi &= \sum_{i=1}^{n} (\widetilde{e}_i \circ \varphi)(f)\,(\widetilde{e}_i \circ \varphi) \\
&= 2 \sum_{i=1}^{n} \sum_{\alpha=1}^{k} (\widetilde{e}_i \circ \varphi)(y^\alpha)\,\partial_\alpha(F)\,(\widetilde{e}_i \circ \varphi) \\
&= 2 \sum_{i=1}^{n} (\widetilde{e}_i \circ \varphi)(F)(\widetilde{e}_i \circ \varphi) \\
&= 2\,(\operatorname{grad}^N F) \circ \varphi.
\end{aligned}$$

Since $e(\varphi) = \frac{n}{2}\lambda^2$, we get

$$e(\varphi)(\operatorname{grad}^N f) \circ \varphi = n\,\lambda^2 (\operatorname{grad}^N F) \circ \varphi.$$

□

If $n = 1$, we arrive at the following corollary

Corollary 2.1. *Let $F \in C^\infty(IR)$ be a smooth positive function and $f(x, y) = F\big(2\,y - \varphi(x)\big)$ for all $(x, y) \in M \times IR$. Then $\varphi \in C^\infty(M)$ is f-harmonic map if and only if is harmonic.*

Proof. From the formula (1.2) we have

$$\tau_f(\varphi) \quad = \quad f_\varphi\,\tau(\varphi) + d\varphi(\operatorname{grad}^M f_\varphi) - e(\varphi)(\operatorname{grad}^N f) \circ \varphi, \tag{2.1}$$

with

$$f_\varphi(x) = f(x, \varphi(x)) = F(\varphi(x)),$$

for all $x \in M$. By calculating the terms of equation 2.1, we obtain

$$\begin{aligned} d\varphi(\operatorname{grad}^M f_\varphi) &= \sum_{i=1}^{m} e_i(f_\varphi) d\varphi(e_i) \\ &= \sum_{i=1}^{m} e_i(F \circ \varphi) e_i(\varphi) \\ &= \sum_{i=1}^{m} e_i(\varphi)(F' \circ \varphi) e_i(\varphi) \\ &= (F' \circ \varphi) |\operatorname{grad}^M \varphi|^2, \qquad (2.2) \\ -e(\varphi)(\operatorname{grad}^{IR} f) \circ \varphi &= -\frac{1}{2} \sum_{i=1}^{m} < d\varphi(e_i), d\varphi(e_i) > (\frac{\partial f}{\partial y}) \circ \varphi \\ &= -\frac{1}{2} \sum_{i=1}^{m} e_i(\varphi)^2 [2(F' \circ \varphi)] \\ &= -|\operatorname{grad}^M \varphi|^2 (F' \circ \varphi), \qquad (2.3) \end{aligned}$$

where $\{e_i\}$ is an orthonormal frame in M, $F' = dF/dt$ and $e_i(\varphi) = d\varphi(e_i)$.
Substituting (2.2) and, (2.3) in (2.1), we obtain

$$\tau_f(\varphi) = f_\varphi \, \tau(\varphi).$$

□

Example 2.1. Let $F \in C^\infty(IR)$ be a smooth positive function. The map

$$\begin{aligned} \varphi : IR^2 &\longrightarrow IR \\ (x_1, x_2) &\longrightarrow x_1^2 - x_2^2 \end{aligned}$$

is f-harmonic with $f(x_1, x_2, y) = F(2\,y - x_1^2 + x_2^2)$ for all $(x_1, x_2, y) \in IR^2 \times IR$.

Note that φ is harmonic, from the formula (1.2), we deduce that φ is f-harmonic if and only if

$$d\varphi(\operatorname{grad}^{IR^2} f_\varphi) - e(\varphi)(\operatorname{grad}^{IR} f) \circ \varphi = 0. \qquad (2.4)$$

We have

$$\begin{aligned}
f_\varphi(x_1,x_2) &= f(x_1,x_2,\varphi(x_1,x_2))\\
&= F(2\varphi(x_1,x_2)-x_1^2+x_2^2)\\
&= F(2x_1^2-2x_2^2-x_1^2+x_2^2)\\
&= F(x_1^2-x_2^2).\\
\operatorname{grad}^{IR^2} f_\varphi &= \frac{\partial f_\varphi}{\partial x_1}\frac{\partial}{\partial x_1}+\frac{\partial f_\varphi}{\partial x_2}\frac{\partial}{\partial x_2}\\
&= 2x_1F'(x_1^2-x_2^2)\frac{\partial}{\partial x_1}-2x_2F'(x_1^2-x_2^2)\frac{\partial}{\partial x_2}.\\
d\varphi(\operatorname{grad}^{IR^2} f_\varphi) &= 2x_1F'(x_1^2-x_2^2)\frac{\partial\varphi}{\partial x_1}-2x_2F'(x_1^2-x_2^2)\frac{\partial\varphi}{\partial x_2}\\
&= 4x_1^2F'(x_1^2-x_2^2)+4x_2^2F'(x_1^2-x_2^2). \qquad (2.5)\\
e(\varphi) &= \frac{1}{2}\Big(\frac{\partial\varphi}{\partial x_1}\Big)^2+\frac{1}{2}\Big(\frac{\partial\varphi}{\partial x_2}\Big)^2\\
&= 2x_1^2+2x_2^2.\\
(\operatorname{grad}^{IR} f)\circ\varphi &= \Big(\frac{\partial f}{\partial y}\Big)\circ\varphi\\
&= 2F'(x_1^2-x_2^2).\\
e(\varphi)(\operatorname{grad}^{IR} f)\circ\varphi &= 4x_1^2F'(x_1^2-x_2^2)+4x_2^2F'(x_1^2-x_2^2). \qquad (2.6)
\end{aligned}$$

From (2.5) and (2.6) we obtain (2.4).

Example 2.2. The radial projection $\varphi : x \in IR^{m+1}\backslash\{0\} \longrightarrow \varphi(x) = \frac{x}{|x|} \in S^m$ is f-harmonic, where $F \in C^\infty(IR^{m+1}\backslash\{0\})$ is a smooth positive function and $f(x,y) = F\Big(\frac{2\,y\,|x|+(m-2)\,x}{|x|}\Big)$ for all $(x,y) \in (IR^{m+1}\backslash\{0\}) \times S^m$.

Indeed; the radial projection φ is a semi-conformal harmonic maps (see [6]), so from Theorem 2.1, we deduce that φ is f-harmonic with

$$\begin{aligned}
f(x,y) &= F\Big(2y+(m-2)\frac{x}{|x|}\Big)\\
&= F\Big(\frac{2\,y\,|x|+(m-2)\,x}{|x|}\Big).
\end{aligned}$$

Remark 2.1. Using Theorem 2.1, we can construct many examples for f-harmonic maps in a generalized sense.

Theorem 2.2. *Let $f_1 \in C^\infty(M)$ and $f_2 \in C^\infty(N)$ be two smooth functions and $f = e^{f_1 f_2}$. A semi-conformal map $\varphi : M^m \longrightarrow N^2$ from a Riemannian manifold M of dimension m to a Riemannian manifold N of dimension 2, is f-harmonic if and only if*

$$\tau(\varphi) + (f_2\circ\varphi)\,d\varphi(\operatorname{grad}^M f_1) = 0.$$

Proof. We have

$$\tau_f(\varphi) = f_\varphi\,\tau(\varphi) + d\varphi(\operatorname{grad}^M f_\varphi) - e(\varphi)(\operatorname{grad}^N f)\circ\varphi, \qquad (2.7)$$

where $f_\varphi(x) = f(x,\varphi(x)) = e^{f_1(x)\,f_2(\varphi(x))}$.
Let $\{e_1,...,e_m\}$ be an orthonormal frame on a domain of M such that the vectors

$\{e_1, e_2\}$ are horizontal and the vectors $\{e_3, ..., e_m\}$ are vertical, so that $d\varphi(e_i) = \lambda\,(\widetilde{e}_i \circ \varphi)$ for $i = 1, 2$ where $\{\widetilde{e}_1, \widetilde{e}_2\}$ is an orthonormal frame on a domain of N. By calculating the terms of equation 2.7, we obtain

$$\begin{aligned}
d\varphi(\mathrm{grad}^M f_\varphi) &= \sum_{i=1}^{m} e_i(f_\varphi)\, d\varphi(e_i)\\
&= \sum_{i=1}^{m} e^{f_1(f_2\circ\varphi)}\, e_i\big(f_1(f_2\circ\varphi)\big)\, d\varphi(e_i)\\
&= e^{f_1(f_2\circ\varphi)}\big\{(f_2\circ\varphi)\, d\varphi(\mathrm{grad}^M f_1) + f_1\, d\varphi(\mathrm{grad}^M(f_2\circ\varphi))\big\},\\
d\varphi(\mathrm{grad}^M(f_2\circ\varphi)) &= \sum_{i=1}^{m} e_i(f_2\circ\varphi)\, d\varphi(e_i)\\
&= \sum_{i=1}^{2} d\varphi(e_i)(f_2)\, d\varphi(e_i)\\
&= \sum_{i=1}^{2} \lambda^2\,(\widetilde{e}_i\circ\varphi)(f_2)\,(\widetilde{e}_i\circ\varphi)\\
&= \lambda^2\,(\mathrm{grad}^N f_2)\circ\varphi,
\end{aligned}$$

$$d\varphi(\mathrm{grad}^M f_\varphi) = e^{f_1(f_2\circ\varphi)}\big\{(f_2\circ\varphi)\, d\varphi(\mathrm{grad}^M f_1) + f_1\,\lambda^2\,(\mathrm{grad}^N f_2)\circ\varphi\big\}. \tag{2.8}$$

$$\begin{aligned}
(\mathrm{grad}^N f)\circ\varphi &= \sum_{i=1}^{2}(\widetilde{e}_i\circ\varphi)(f)(\widetilde{e}_i\circ\varphi)\\
&= \sum_{i=1}^{2}(\widetilde{e}_i\circ\varphi)(f_1\, f_2)\, e^{f_1\,(f_2\circ\varphi)}\,(\widetilde{e}_i\circ\varphi)\\
&= f_1\, e^{f_1\,(f_2\circ\varphi)}\,(\mathrm{grad}^N f_2)\circ\varphi,
\end{aligned}$$

As $e(\varphi) = \lambda^2$, then

$$e(\varphi)(\mathrm{grad}^N f)\circ\varphi = \lambda^2 f_1\, e^{f_1(f_2\circ\varphi)}\,(\mathrm{grad}^N f_2)\circ\varphi. \tag{2.9}$$

Substituting (2.8) and (2.9) in (2.7), we obtain

$$\begin{aligned}
\tau_f(\varphi) &= e^{f_1\,(f_2\circ\varphi)}\big[\tau(\varphi) + (f_2\circ\varphi)d\varphi(\mathrm{grad}^M f_1) + f_1\lambda^2(\mathrm{grad}^N f_2)\circ\varphi\\
&\quad -f_1\lambda^2(\mathrm{grad}^N f_2)\circ\varphi\big]\\
&= e^{f_1\,(f_2\circ\varphi)}\big[\tau(\varphi) + (f_2\circ\varphi)\, d\varphi(\mathrm{grad}^M f_1)\big]. \qquad (2.10)
\end{aligned}$$

From the formula (2.10), the Theorem 6 follows. □

Example 2.3. Let $M = (IR^2\backslash\{0\}) \times IR$ and let $\varphi : M \longrightarrow IR^2$ defined by

$$\varphi(x_1, x_2, x_3) = (\sqrt{x_1^2 + x_2^2}, x_3).$$

The map φ is semi-conformal with dilation $\lambda = 1$. The tension field of φ is

$$\tau(\varphi)(x_1, x_2, x_3) = \Big(\frac{1}{\sqrt{x_1^2 + x_2^2}}, 0\Big).$$

According to Theorem 2.2 the map φ is f-harmonic with $f = e^{f_1 f_2}$ where

$$f_1(x_1, x_2, x_3) = \frac{1}{\sqrt{x_1^2 + x_2^2}} \quad and \quad f_2(y_1, y_2) = y_1.$$

3. A Liouville Type Theorem for f-Harmonic Maps

Theorem 3.1. *Let (N, h) be a Riemannian manifold with non-positive sectional curvature $\mathrm{Sect}^N \leq 0$. Consider an f-harmonic map $\varphi : IR^m \longrightarrow N$ with finite f-energy $E_f(\varphi) = \frac{1}{2}\int_{IR^m} f_\varphi\, |d\varphi|^2\, dx < \infty$, where $f \in C^\infty(IR^m \times N)$ is a smooth positive function such that $Hess(f_\varphi) \leq 0$. If $\nabla^\varphi e(\varphi)(\mathrm{grad}^N \ln f) \circ \varphi \geq 0$ and $\mathrm{Vol}_f(IR^m) \equiv \int_{IR^m} f_\varphi\, dx = \infty$, then φ is constant.*

We need the following lemmas to prove Theorem 3.1.

Lemma 3.1 ([14]). *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ a smooth mapping between Riemannian manifolds and let $f \in C^\infty(M)$, then*

$$\big\langle d\varphi, \nabla^\varphi d\varphi(\mathrm{grad}^M f)\big\rangle = \frac{1}{2}\big(\mathrm{grad}^M f\big)\big(|d\varphi|^2\big) + \big\langle d\varphi, d\varphi(\nabla^M \mathrm{grad}^M f)\big\rangle.$$

Proof. Let $\{e_1, ..., e_m\}$ be an orthonormal frame such that $\nabla^M_{e_i} e_j = 0$ at $x \in M$ for all $i, j = 1, ..., m$. Then calculating at x

$$\big\langle d\varphi, \nabla^\varphi d\varphi(\mathrm{grad}^M f)\big\rangle = \sum_{i=1}^m h(d\varphi(e_i), \nabla^\varphi_{e_i} d\varphi(\mathrm{grad}^M f)).$$

For all $i = 1, ..., m$, we have

$$\begin{aligned}
\nabla^\varphi_{e_i} d\varphi(\mathrm{grad}^M f) &= \sum_{j=1}^m \nabla^\varphi_{e_i}\big(e_j(f)\, d\varphi(e_j)\big) \\
&= \sum_{j=1}^m e_j(f)\, \nabla^\varphi_{e_i} d\varphi(e_j) + \sum_{j=1}^m e_i(e_j(f))\, d\varphi(e_j) \\
&= \sum_{j=1}^m e_j(f)\, \nabla^\varphi_{e_j} d\varphi(e_i) + \sum_{j=1}^m e_i(e_j(f))\, d\varphi(e_j),
\end{aligned}$$

we conclude that

$$\begin{aligned}
\big\langle d\varphi, \nabla^\varphi d\varphi(\mathrm{grad}^M f)\big\rangle &= \sum_{i,j=1}^m e_j(f)\, h(d\varphi(e_i), \nabla^\varphi_{e_j} d\varphi(e_i)) \\
&\quad + \sum_{i,j=1}^m e_i(e_j(f))\, h(d\varphi(e_i), d\varphi(e_j)).
\end{aligned}$$

By noticing that

$$\frac{1}{2}\big(\mathrm{grad}^M f\big)\big(|d\varphi|^2\big) = \sum_{i,j=1}^m e_j(f)\, h(d\varphi(e_i), \nabla^\varphi_{e_j} d\varphi(e_i)),$$

and

$$\big\langle d\varphi, d\varphi(\nabla^M \operatorname{grad}^M f)\big\rangle = \sum_{i,j=1}^m h(d\varphi(e_i), d\varphi(\nabla^M_{e_i} e_j(f) e_j))$$
$$= \sum_{i,j=1}^m e_i(e_j(f))\, h(d\varphi(e_i), d\varphi(e_j)),$$

the Lemma 3.1 follows. □

Lemma 3.2. *Let (N, h) be a Riemannian manifold and $f \in C^\infty(IR^m \times N)$ be a smooth positive function. Consider an f-harmonic map $\varphi : IR^m \longrightarrow N$, then we have*

$$\frac{1}{2}\Delta^{IR^m}|d\varphi|^2 = |\nabla d\varphi|^2 + \frac{1}{f_\varphi^2}|d\varphi(\operatorname{grad}^{IR^m} f_\varphi)|^2 + \big\langle d\varphi, \nabla^\varphi e(\varphi)(\operatorname{grad}^N \ln f)\circ\varphi\big\rangle$$
$$- \frac{1}{2f_\varphi}\big(\operatorname{grad}^{IR^m} f_\varphi\big)\big(|d\varphi|^2\big) - \frac{1}{f_\varphi}\big\langle d\varphi, d\varphi(\nabla^{IR^m}\operatorname{grad}^{IR^m} f_\varphi)\big\rangle$$
$$- \sum_{i,j=1}^m h\big(R^N\big(d\varphi(e_i), d\varphi(e_j)\big)d\varphi(e_j), d\varphi(e_i)\big)$$

where $\{e_1, ..., e_m\}$ be an orthonormal frame on IR^m.

Proof. We start recalling the standard Bochner formula for the smooth map φ. Let $\{e_1, ..., e_m\}$ be an orthonormal frame on IR^m, we have

$$\frac{1}{2}\Delta^{IR^m}|d\varphi|^2 = |\nabla d\varphi|^2 + \big\langle d\varphi, \nabla^\varphi \tau(\varphi)\big\rangle + \sum_{i=1}^m h\big(d\varphi\big(\operatorname{Ricci}^{IR^m} e_i\big), d\varphi(e_i)\big)$$
$$- \sum_{i,j=1}^m h\big(R^N\big(d\varphi(e_i), d\varphi(e_j)\big)d\varphi(e_j), d\varphi(e_i)\big) \tag{3.1}$$

where

$$|\nabla d\varphi|^2 = \sum_{i,j=1}^m h\big(\nabla d\varphi(e_i, e_j), \nabla d\varphi(e_i, e_j)\big),$$

and

$$\big\langle d\varphi, \nabla^\varphi \tau(\varphi)\big\rangle = \sum_{i=1}^m h\big(d\varphi(e_i), \nabla^\varphi_{e_i}\tau(\varphi)\big).$$

Since

$$\tau_f(\varphi) = f_\varphi\,\tau(\varphi) + d\varphi(\operatorname{grad}^{IR^m} f_\varphi) - e(\varphi)(\operatorname{grad}^N f)\circ\varphi = 0,$$

we obtain

$$\tau(\varphi) = -\frac{1}{f_\varphi}d\varphi(\operatorname{grad}^{IR^m} f_\varphi) + \frac{1}{f_\varphi}e(\varphi)(\operatorname{grad}^N f)\circ\varphi$$
$$= -\frac{1}{f_\varphi}d\varphi(\operatorname{grad}^{IR^m} f_\varphi) + e(\varphi)(\operatorname{grad}^N \ln f)\circ\varphi,$$

then we get

$$\begin{aligned} \big\langle d\varphi, \nabla^{\varphi}\tau(\varphi)\big\rangle &= \frac{1}{f_\varphi^2}|d\varphi(\operatorname{grad}^{IR^m} f_\varphi)|^2 - \frac{1}{f_\varphi}\big\langle d\varphi, \nabla^{\varphi} d\varphi(\operatorname{grad}^{IR^m} f_\varphi)\big\rangle \\ &\quad + \big\langle d\varphi, \nabla^{\varphi} e(\varphi)(\operatorname{grad}^{N}\ln f)\circ\varphi\big\rangle. \end{aligned} \tag{3.2}$$

By the Lemma 3.1, the second term on the right-hand side of (3.2) is

$$\begin{aligned} -\frac{1}{f_\varphi}\big\langle d\varphi, \nabla^{\varphi} d\varphi(\operatorname{grad}^{IR^m} f_\varphi)\big\rangle &= -\frac{1}{2f_\varphi}\big(\operatorname{grad}^{IR^m} f_\varphi\big)\big(|d\varphi|^2\big) \\ &\quad - \frac{1}{f_\varphi}\big\langle d\varphi, d\varphi(\nabla^{IR^m}\operatorname{grad}^{IR^m} f_\varphi)\big\rangle. \end{aligned} \tag{3.3}$$

Since $\operatorname{Ricci}^{IR^m} = 0$, by (3.1), (3.2) and (3.3), we have

$$\begin{aligned} \frac{1}{2}\Delta^{IR^m}|d\varphi|^2 &= |\nabla d\varphi|^2 + \frac{1}{f_\varphi^2}|d\varphi(\operatorname{grad}^{IR^m} f_\varphi)|^2 + \big\langle d\varphi, \nabla^{\varphi} e(\varphi)(\operatorname{grad}^{N}\ln f)\circ\varphi\big\rangle \\ &\quad - \frac{1}{2f_\varphi}\big(\operatorname{grad}^{IR^m} f_\varphi\big)\big(|d\varphi|^2\big) - \frac{1}{f_\varphi}\big\langle d\varphi, d\varphi(\nabla^{IR^m}\operatorname{grad}^{IR^m} f_\varphi)\big\rangle \\ &\quad - \sum_{i,j=1}^{m} h\big(R^N\big(d\varphi(e_i), d\varphi(e_j)\big)d\varphi(e_j), d\varphi(e_i)\big). \end{aligned}$$

□

Proof. of Theorem 3.1. By the Lemma 3.2, we have

$$\begin{aligned} \frac{1}{2}f_\varphi\Delta^{IR^m}|d\varphi|^2 &= f_\varphi|\nabla d\varphi|^2 + \frac{1}{f_\varphi}|d\varphi(\operatorname{grad}^{IR^m} f_\varphi)|^2 - \frac{1}{2}\big(\operatorname{grad}^{IR^m} f_\varphi\big)\big(|d\varphi|^2\big) \\ &\quad + f_\varphi\big\langle d\varphi, \nabla^{\varphi} e(\varphi)(\operatorname{grad}^{N}\ln f)\circ\varphi\big\rangle - \big\langle d\varphi, d\varphi(\nabla^{IR^m}\operatorname{grad}^{IR^m} f_\varphi)\big\rangle \\ &\quad - f_\varphi\sum_{i,j=1}^{m} h\big(R^N\big(d\varphi(e_i), d\varphi(e_j)\big)d\varphi(e_j), d\varphi(e_i)\big). \end{aligned}$$

If we denote $\Delta_f^{IR^m}\rho \equiv f_\varphi\Delta^{IR^m}\rho + \big(\operatorname{grad}^{IR^m} f_\varphi\big)\big(\rho\big)$ for all $\rho \in C^\infty(IR^m)$, then

$$\begin{aligned} \frac{1}{2}\Delta_f^{IR^m}|d\varphi|^2 &= f_\varphi|\nabla d\varphi|^2 + f_\varphi\big\langle d\varphi, \nabla^{\varphi} e(\varphi)(\operatorname{grad}^{N}\ln f)\circ\varphi\big\rangle \\ &\quad + \frac{1}{f_\varphi}|d\varphi(\operatorname{grad}^{IR^m} f_\varphi)|^2 - \big\langle d\varphi, d\varphi(\nabla^{IR^m}\operatorname{grad}^{IR^m} f_\varphi)\big\rangle \\ &\quad - f_\varphi\sum_{i,j=1}^{m} h\big(R^N\big(d\varphi(e_i), d\varphi(e_j)\big)d\varphi(e_j), d\varphi(e_i)\big). \end{aligned} \tag{3.4}$$

Since $\operatorname{Sect}^N \leq 0$, $\operatorname{Hess}^{IR^m} f_\varphi \leq 0$ and $\nabla^{\varphi} e(\varphi)(\operatorname{grad}^{N}\ln f)\circ\varphi \geq 0$, by (3.4) we obtain

$$\frac{1}{2}\Delta_f^{IR^m}|d\varphi|^2 \geq f_\varphi\,|\nabla d\varphi|^2. \tag{3.5}$$

Since

$$\frac{1}{2}\Delta_f^{IR^m}|d\varphi|^2 = |d\varphi|\Delta_f^{IR^m}|d\varphi| + f_\varphi\,|\operatorname{grad}^{IR^m}|d\varphi||^2$$

by (3.5) and the Kato's inequality (see [2] [7]) we get

$$|d\varphi|\Delta_f^{IR^m}|d\varphi| \geq f_\varphi\big(|\nabla d\varphi|^2 - |\operatorname{grad}^{IR^m}|d\varphi||^2\big) \geq 0. \tag{3.6}$$

Let $\rho : IR^m \longrightarrow IR$ be a smooth function with compact support, then

$$\begin{aligned}\rho^2|d\varphi|\Delta_f^{IR^m}|d\varphi| &= \rho^2|d\varphi|\operatorname{div}^{IR^m}\big(f_\varphi \operatorname{grad}^{IR^m}|d\varphi|\big)\\ &= \operatorname{div}^{IR^m}\big(\rho^2|d\varphi|f_\varphi \operatorname{grad}^{IR^m}|d\varphi|\big) - f_\varphi\rho^2|\operatorname{grad}^{IR^m}|d\varphi||^2\\ &\quad - 2f_\varphi\rho|d\varphi| < \operatorname{grad}^{IR^m}\rho, \operatorname{grad}^{IR^m}|d\varphi| >_{IR^m} .\end{aligned} \tag{3.7}$$

By (3.6), (3.7) and the Stokes theorem, we deduce

$$\begin{aligned} 0 &\leq \int_{IR^m}\rho^2|d\varphi|\Delta_f^{IR^m}|d\varphi|dx\\ 0 &\leq -\int_{IR^m} f_\varphi\rho^2|\operatorname{grad}^{IR^m}|d\varphi||^2dx\\ &\quad - 2\int_{IR^m} f_\varphi\rho|d\varphi| < \operatorname{grad}^{IR^m}\rho, \operatorname{grad}^{IR^m}|d\varphi| >_{IR^m} dx.\end{aligned} \tag{3.8}$$

By the Young inequality (see [17]) we have

(3.9)

$$-2 < |d\varphi|\operatorname{grad}^{IR^m}\rho, \rho\operatorname{grad}^{IR^m}|d\varphi| >_{IR^m} \leq \frac{1}{\epsilon}|d\varphi|^2|\operatorname{grad}^{IR^m}\rho|^2 + \epsilon\rho^2|\operatorname{grad}^{IR^m}|d\varphi||^2.$$

Substituting (3.9) in (3.8), we obtain

$$\begin{aligned} 0 \leq &-\int_{IR^m} f_\varphi\rho^2|\operatorname{grad}^{IR^m}|d\varphi||^2dx + \frac{1}{\epsilon}\int_{IR^m} f_\varphi|d\varphi|^2|\operatorname{grad}^{IR^m}\rho|^2dx\\ &+ \epsilon\int_{IR^m} f_\varphi\rho^2|\operatorname{grad}^{IR^m}|d\varphi||^2dx,\end{aligned}$$

then

$$(1-\epsilon)\int_{IR^m} f_\varphi\rho^2|\operatorname{grad}^{IR^m}|d\varphi||^2dx \leq \frac{1}{\epsilon}\int_{IR^m} f_\varphi|d\varphi|^2|\operatorname{grad}^{IR^m}\rho|^2dx, \tag{3.10}$$

for any $\epsilon > 0$. Choose the smooth cut-off $\rho = \rho_R$, i.e $\rho \leq 1$ on M, $\rho = 1$ on the ball $B(0,R)$, $\rho = 0$ on $IR^m\backslash B(0,2R)$ and $|\operatorname{grad}^{IR^m}\rho| \leq \frac{2}{R}$. Let $0 < \epsilon < 1$, replacing $\rho = \rho_R$ in (3.10) we obtain

$$0 \leq (1-\epsilon)\int_{IR^m} f_\varphi\rho^2|\operatorname{grad}^{IR^m}|d\varphi||^2dx \leq \frac{4}{\epsilon R^2}\int_{IR^m} f_\varphi|d\varphi|^2dx. \tag{3.11}$$

Since $E_f(\varphi) = \frac{1}{2}\int_{IR^m} f_\varphi\,|d\varphi|^2\,dx < \infty$, when $R \to \infty$, we have

$$\frac{4}{\epsilon R^2}\int_{IR^m} f_\varphi|d\varphi|^2dx \to 0.$$

Thus, by (3.11), we have $|\operatorname{grad}^{IR^m}|d\varphi|| = 0$, i.e $|d\varphi| = c$ constant. If $c > 0$,

$$E_f(\varphi) = \frac{c^2}{2}\int_{IR^m} f_\varphi dx = \frac{c^2}{2}\operatorname{Vol}_f(IR^m) < \infty.$$

But $\operatorname{Vol}_f(IR^m) = \infty$ then $c = 0$, i.e φ is constant. □

If $f(x,y) = 1$ for all $(x,y) \in IR^m \times N$, we recover the following classical result:

Corollary 3.1. *Let* (N,h) *be Riemannian manifold with non-positive sectional curvature* $\mathrm{Sect}^N \leq 0$. *Consider an harmonic map* $\varphi : IR^m \longrightarrow N$ *with finite energy* $E(\varphi) = \frac{1}{2}\int_{IR^m} |d\varphi|^2\, dx < \infty$, *then* φ *is constant.*

Let $f_1 : IR^m \longrightarrow (0,\infty)$ be a smooth function. If $f(x,y) = f_1(x)$ for all $(x,y) \in IR^m \times N$. We recover the following result obtained in Theorem 3.3 of [14] and Theorem 1.2 of [16]:

Corollary 3.2. *Let* (N,h) *be Riemannian manifold with non-positive sectional curvature* $\mathrm{Sect}^N \leq 0$ *and let* $f_1 : IR^m \longrightarrow (0,+\infty)$ *be a smooth positive function with non-positive hessien* $\mathrm{Hess}^{IR^m} f_1 \leq 0$. *Consider an* f_1*-harmonic map* $\varphi : IR^m \longrightarrow N$ *with finite* f_1*-energy* $E_{f_1}(\varphi) = \frac{1}{2}\int_{IR^m} f_1\, |d\varphi|^2\, dx < \infty$. *If*

$$\mathrm{Vol}_{f_1}(IR^m) = \int_{IR^m} f_1\, dx = \infty,$$

then φ *is constant.*

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