ON THE $W_5$-CURVATURE TENSOR OF GENERALIZED SASAKIAN-SPACE-FORMS

D. G. PRAKASHA, VASANT CHAVAN AND KAKASAB MIRJI

ABSTRACT. The object of the paper is to characterize generalized Sasakian-space-forms satisfying certain curvature conditions on $W_5$-curvature tensor. We characterize $W_5$-flat, $\phi$-$W_5$-flat and $\phi$-$W_5$-semisymmetric generalized Sasakian-space-forms.

1. Introduction

Generalized Sasakian-space-forms have become today a rather special topic in contact Riemannian geometry, but many contemporary works are concerned with the study of its properties and their related curvature tensor. The study of generalized Sasakian-space-forms was initiated by Algre et al., in [1] and then it was continued by many other authors. A generalized Sasakian-space-form is an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ whose curvature tensor $R$ is given by

$$R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\},$$

where $f_1$, $f_2$, $f_3$ are differentiable functions on $M$ and $X, Y, Z$ are vector fields on $M$. In such case we will write the manifold as $M(f_1, f_2, f_3)$. This kind of manifolds appears as a natural generalization of the Sasakian-space-forms by taking $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$, where $c$ denotes constant $\phi$-sectional curvature. The $\phi$-sectional curvature of generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$. Moreover, cosymplectic space-form and Kenmotsu space-form are also considered as particular types of generalized Sasakian-space-form. Generalized Sasakian-space-forms have
been studied in a number of papers from several points of view (for instance, [2]-[4], [6]-[8], [9]-[11], [13]-[17], etc).

In the context of generalized Sasakian-space-forms, Kim [11] studied conformally flat and locally symmetric generalized Sasakian-space-forms. Some symmetric properties of generalized Sasakian-space-forms with projective curvature tensor were studied by De and Sarkar [6] and Sarkar and Akbar [16]. In [13], Prakasha shown that every generalized Sasakian space-form is Weyl-pseudosymmetric. Hui [10] studied $W_2$-curvature tensor in generalized Sasakian-space-forms. Also, Prakasha and Nagaraja [14] studied quasi-conformally flat and quasi-conformally semisymmetric generalized Sasakian-space-forms. In a recent paper [8], De and Majhi studied $\phi$-Weyl semisymmetric and $\phi$-projectively semisymmetric generalized Sasakian-space-forms. Conharmonically flat generalized Sasakian-space-forms and conharmonically locally $\phi$-symmetric generalized Sasakian-space-forms were studied in [17]. In a recent paper, Hui and Prakasha [9] studied certain properties on the C-Bochner curvature tensor of generalized Sasakian-space-forms. As a continuation of this study, in this paper we plan to characterize flatness and symmetry property of generalized Sasakian-space-forms regarding $W_5$-curvature tensor.

The paper is organized as follows: after preliminaries in Section 3, we study the $W_5$-flat generalized Sasakian space-forms. We prove that a generalized Sasakian-space-form is $W_5$-flat if and only if $f_1 = 3f_2/1 - 2n = f_3$. In section 4, we study $\phi$-$W_5$-flat generalized Sasakian-space-form and obtain that a generalized Sasakian-space-form of dimension greater than three is $\phi$-$W_5$-flat if and only if it is conformally flat. In the last section, we prove that a generalized Sasakian-space-form is $\phi$-$W_5$-semisymmetric if and only if it is $W_5$-flat.

2. Preliminaries

An odd-dimensional Riemannian manifold $(M, g)$ is said to be an almost contact metric manifold [5] if there exist on $M$ a $(1, 1)$ tensor field $\phi$, a vector field $\xi$ (called the structure vector field) and a 1-form $\eta$ such that

\[
\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0,
\]
\[
g(X, \xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]
for arbitrary vector fields $X$ and $Y$. In view of (2.1) and (2.2), we have
\[
g(\phi X, Y) = -g(X, \phi), \quad g(\phi X, X) = 0.
\]
\[
(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y).
\]

Again, we know that in a generalized Sasakian space-form

\[
(2\mathcal{R})(X, Y) Z = f_1\{g(Y, Z)X - g(X, Z)Y\}
+ f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\}
+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}
\]

for any vector fields $X, Y, Z$ on $M$, where $R$ denotes the curvature tensor of $M$ and $f_1, f_2, f_3$ are smooth functions on the manifold. The Ricci operator $Q$ and Ricci tensor $S$ of the manifold of dimension $(2n + 1)$ are respectively given by

\[
Q X = (2nf_1 + 3f_2 - f_3)X - \{3f_2 + (2n - 1)f_3\}\eta(X)\xi,
\]
\[
S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - \{3f_2 + (2n - 1)f_3\}\eta(X)\eta(Y).
\]
In addition to the relation (2.3)-(2.5), for an $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form $M(f_1, f_2, f_3)$ the following relations also hold [1]:

\begin{align}
(2.6) \quad & \eta(R(X, Y)Z) = (f_1 - f_3)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \\
(2.7) \quad & R(X, Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \\
(2.8) \quad & R(\xi, X)Y = (f_1 - f_3)\{g(X, Y)\xi - \eta(Y)X\}.
\end{align}

The $W_5$-curvature tensor on a $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is given by [12]

\begin{equation}
(2.9) \quad W_5(X, Y, Z, U) = R(X, Y, Z, U) + \frac{1}{2n} \{g(X, Z)S(Y, U) - g(Y, U)S(X, Z)\}.
\end{equation}

For $n \geq 1$, $M(f_1, f_2, f_3)$ is locally $W_5$-flat if and only if the $W_5$-curvature tensor vanishes. Also, notice that $W_5$-curvature tensor is symmetric with change of pairs of the vector fields and does not satisfies the cyclic property. A relativistic significance of $W_5$-curvature tensor has been explored by Pokhariyal [12].

In view of (2.6)-(2.8), it can be easily construct that in a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form $M(f_1, f_2, f_3)$, the $W_5$-curvature tensor satisfies the following conditions:

\begin{align}
(2.10) \quad & \eta(W_5(X, Y)Z) = (f_1 - f_3)\{g(Y, Z)\eta(X)\} - \frac{1}{2n} \eta(Y)S(X, Z), \\
(2.11) \quad & W_5(X, Y)\xi = (f_1 - f_3)\{\eta(Y)X - 2\eta(X)Y\} + \frac{1}{2n} \eta(X)QY, \\
(2.12) \quad & \eta(W_5(X, Y)\xi) = 0.
\end{align}

3. $W_5$-flat generalized Sasakian-space-forms

**Definition 3.1.** A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form is called $W_5$-flat if it satisfies the condition

\[W_5(X, Y)Z = 0,\]

for any vector fields $X, Y$ and $Z$ on the manifold.

Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional $(n > 1)$ $W_5$-flat generalized Sasakian space-form. Then, by Definition 3.1) and (2.9), we get

\begin{equation}
(3.1) \quad R(X, Y)Z = \frac{1}{2n} \{S(X, Z)Y - g(X, Z)QY\}.
\end{equation}

In view of (2.6) and (2.7), the above equation takes the form

\begin{equation}
(3.2) \quad R(X, Y)Z = -\frac{1}{2n} \{3f_2 + (2n-1)f_3\} \{\eta(X)\eta(Z)Y - g(X, Z)\eta(Y)\xi\}.
\end{equation}

Using (2.3) in (3.2) yields

\begin{align}
& f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\
& \quad + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \\
& \quad - \frac{1}{2n} \{3f_2 + (2n-1)f_3\} \{\eta(X)\eta(Z)Y - g(X, Z)\eta(Y)\xi\}.
\end{align}
Taking $Z = \phi Z$ in (3.3), we have
\[
f_1\{g(Y, \phi Z)X - g(X, \phi Z)Y\} + f_2\{g(X, \phi^2 Z)\phi Y - g(Y, \phi^2 Z)\phi X + 2g(X, \phi Y)\phi^2 Z\} + f_3\{g(X, \phi Z)\eta(Y)\xi - g(Y, \phi Z)\eta(X)\xi\} = \frac{1}{2n}[3f_2 + (2n - 1)f_3]\{g(X, \phi Z)\eta(Y)\xi\}.
\]
If we take $Y = \xi$, then we obtain from the above equation
\[(3.4) \quad -2n(f_1 - f_3)g(X, \phi Z)\xi = [3f_2 + (2n - 1)f_3]g(X, \phi Z)\xi.
\]
Since $g(X, \phi Z)\xi \neq 0$, in general. Thus from (3.4), it follows that
\[(3.5) \quad 2nf_1 + 3f_2 - f_3 = 0.
\]
Again, we take $X = \xi$ in (3.3), we obtain
\[
(f_1 - f_3)\{g(Y, Z)\xi - \eta(Z)Y\} = \frac{1}{2n}[3f_2 + (2n - 1)f_3]\{\eta(Z)Y - \eta(Y)\xi\}.
\]
Taking inner product with $\xi$ of (3.6), we obtain
\[(3.7) \quad (f_1 - f_3)\{g(Y, Z) - \eta(Z)\eta(Y)\} = 0.
\]
This implies that
\[(3.8) \quad f_1 = f_3.
\]
Since $g(Y, Z) \neq \eta(Y)\eta(Z)$, in general. From (3.5) and (3.8), it is easy to see that
\[(3.9) \quad f_3 = \frac{3f_2}{1 - 2n}.
\]
Thus in view of (3.8) and (3.9), we have
\[(3.10) \quad f_1 = \frac{3f_2}{1 - 2n} = f_3.
\]
Conversely, suppose that (3.10) holds. Then from (2.4) and (2.5), we have $QX = 0$ and $S(X, Y) = 0$, respectively. 
Making use of this in (2.9), we get
\[(3.11) \quad W'_5(X, Y, Z, U) = R'(X, Y, Z, U),
\]
where $W'_5(X, Y, Z, U) = g(W_5(X, Y)Z, U)$ and $R'(X, Y, Z, U) = g(R(X, Y)Z, U)$. 
Putting $Y = Z = e_i$ in (3.11) and taking summation over $i$, $1 \leq i \leq 2n + 1$, we get
\[(3.12) \quad \sum_{i=1}^{2n+1} W'_5(X, e_i, e_i, U) = S(X, U).
\]
Next, because of (2.3) and (3.11), we have
\[
(3.13) \quad W'(X, Y, Z, U) = f_1\{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\} + f_2\{g(X, \phi Z)g(\phi Y, U) - g(Y, \phi Z)g(\phi X, U) + 2g(X, \phi Y)g(\phi Z, U)\} + f_3\{\eta(X)\eta(Z)g(Y, U) - \eta(Y)\eta(Z)g(X, U)\} + g(X, Z)\eta(Y)\eta(U) - g(Y, Z)\eta(X)\eta(U)\}.
\]
Now, putting $Y = Z = e_i$ in (3.13) and taking summation over $i$, $1 \leq i \leq 2n + 1$, we get

\begin{equation}
W_5'(X, e_i, e_i, U) = 2nf_1g(X, U) + 3f_2g(\phi X, \phi U) - f_3((2n - 1)\eta(X)\eta(U) + g(X, U)).
\end{equation}

By virtue of $S(X, U) = 0$, (3.12) and (3.14) we have

\begin{equation}
2nf_1g(X, U) + 3f_2g(\phi X, \phi U) - f_3((2n - 1)\eta(X)\eta(U) + g(X, U)) = 0.
\end{equation}

Putting $X = U = e_i$ in (3.15) and taking summation over $i$, $1 \leq i \leq 2n + 1$, we get $f_1 = 0$. Then in view of (3.10), $f_2 = f_3 = 0$. Therefore, we obtain from (2.3) that

\begin{equation}
R(X, Y)Z = 0.
\end{equation}

Using (2.4) and (2.5) in (4.2), we get

\begin{equation}
W_5(\phi X, \phi Y)\phi Z = R(\phi X, \phi Y)\phi Z.
\end{equation}

In virtue of (2.3), we get from above equation

\begin{equation}
W_5(\phi X, \phi Y)\phi Z = f_1\{g(Y, Z)\phi X - \eta(Y)\eta(Z)\phi X - g(X, Z)\phi Y + \eta(X)\eta(Z)\phi Y\} + f_2\{g(X, \phi Z)\phi^2 Y - g(Y, \phi Z)\phi^2 X + 2g(X, \phi Y)\phi^2 Z\}.
\end{equation}

Applying $\phi^2$ to both sides of (4.3), we have

\begin{equation}
\phi^2 W_5(\phi X, \phi Y)\phi Z = \phi^2 f_1\{g(Y, Z)\phi X - \eta(Y)\eta(Z)\phi X - g(X, Z)\phi Y + \eta(X)\eta(Z)\phi Y\} + f_2\{g(X, \phi Z)\phi^2 Y - g(Y, \phi Z)\phi^2 X + 2g(X, \phi Y)\phi^2 Z\}.
\end{equation}

**4. $\phi$ -W$5$-flat generalized Sasakian-space-forms**

**Definition 4.1.** A $(2n + 1)$-dimensional $(n > 1)$ generalized Sasakian-space-form is called $\phi$-W$5$-flat if it satisfies the condition

\begin{equation}
\phi^2 W_5(\phi X, \phi Y)\phi Z = 0,
\end{equation}

for any vector fields $X$, $Y$ and $Z$ on the manifold.

First, taking $X = \phi X$, $Y = \phi Y$ and $Z = \phi Z$ in (2.9), we have

\begin{equation}
W_5(\phi X, \phi Y)\phi Z = R(\phi X, \phi Y)\phi Z + \frac{1}{2n}\{g(\phi X, \phi Z)Q\phi Y - S(\phi X, \phi Z)\phi Y\}.
\end{equation}

Using (2.4) and (2.5) in (4.2), we get

\begin{equation}
W_5(\phi X, \phi Y)\phi Z = R(\phi X, \phi Y)\phi Z.
\end{equation}

In virtue of (2.3), we get from above equation

\begin{equation}
W_5(\phi X, \phi Y)\phi Z = f_1\{g(Y, Z)\phi X - \eta(Y)\eta(Z)\phi X - g(X, Z)\phi Y + \eta(X)\eta(Z)\phi Y\} + f_2\{g(X, \phi Z)\phi^2 Y - g(Y, \phi Z)\phi^2 X + 2g(X, \phi Y)\phi^2 Z\}.
\end{equation}

Applying $\phi^2$ to both sides of (4.3), we have

\begin{equation}
\phi^2 W_5(\phi X, \phi Y)\phi Z = \phi^2 f_1\{g(Y, Z)\phi X - \eta(Y)\eta(Z)\phi X - g(X, Z)\phi Y + \eta(X)\eta(Z)\phi Y\} + f_2\{g(X, \phi Z)\phi^2 Y - g(Y, \phi Z)\phi^2 X + 2g(X, \phi Y)\phi^2 Z\}.
\end{equation}
Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional $(n > 1)$ $\phi$-$W_5$-flat generalized Sasakian-space-form. Then, by Definition 4.1 and (4.4), we get

\begin{equation}
\phi^2[f_1(g(Y, Z)\phi X - \eta(Y)\eta(Z)\phi X - g(X, Z)\phi Y + \eta(X)\eta(Z)\phi Y] + f_2\{g(X, \phi Z)\phi^2 Y - g(Y, \phi Z)\phi^2 X + 2g(X, \phi Y)\phi^2 Z] = 0.
\end{equation}

By virtue of (2.1) and (2.2), the above equation yields

\begin{equation}
\begin{align*}
&f_1\{g(Y, Z)\phi X - \eta(Y)\eta(Z)\phi X - g(X, Z)\phi Y + \eta(X)\eta(Z)\phi Y]
+ f_2\{g(X, \phi Z)\phi^2 Y - g(Y, \phi Z)\phi^2 X + 2g(X, \phi Y)\phi^2 Z] = 0.
\end{align*}
\end{equation}

Taking inner product with $Y$ in (4.6), we obtain

\begin{equation}
\begin{align*}
&f_1\{g(Y, Z)\phi X - \eta(Y)\eta(Z)\phi X - g(X, Z)\phi Y + \eta(X)\eta(Z)\phi Y]
+ f_2\{g(X, \phi Z)\phi^2 Y - g(Y, \phi Z)\phi^2 X + 2g(X, \phi Y)\phi^2 Z] = 0.
\end{align*}
\end{equation}

Putting $Y = Z = e_i$ in (4.7) and taking summation over $i$, $1 \leq i \leq 2n + 1$, we get $3f_2g(X, \phi U) = 0$. Since $g(X, \phi U) \neq 0$, in general. Hence, it follows that

\begin{equation}
\begin{align*}
&f_2 = 0.
\end{align*}
\end{equation}

In (4.7) again putting $Y = U = e_i$, and taking summation over $i$, $1 \leq i \leq 2n + 1$, we get

\begin{equation}
\begin{align*}
&\{f_1 + (2n + 1)f_2\}g(\phi X, Z) - f_1\{g(X, Z) - \eta(X)\eta(Z)\} = \psi = 0,
\end{align*}
\end{equation}

where $\psi = \text{Trace of } \phi$. Plugging $X = Z = e_i$ in (4.9), and taking summation over $i$, $1 \leq i \leq 2n + 1$, we obtain $\{(2n - 1)f_1 + (2n + 1)f_2\} = 0$. Which in view of (4.8) yields $f_1 = 0$. Hence, we have $f_1 = f_2 = 0$.

Conversely, if $f_1 = f_2 = 0$ then from (4.4) it follows that

\begin{equation}
\phi^2W_5(\phi X, \phi Y)\phi Z = 0.
\end{equation}

That is, $M(f_1, f_2, f_3)$ is $\phi - W_5$-flat. Therefore, the converse holds when $f_1 = f_2 = 0$. Thus we are able to state the following:

**Theorem 4.1.** A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form is $\phi - W_5$-flat if and only if $f_1 = f_2 = 0$ holds.

In [11], U. K. Kim proved that for a $(2n+1)$-dimensional generalized Sasakian-space-form the following holds:

(i) If $n > 1$, then $M$ is conformally flat if and only if $f_2 = 0$.

(ii) If $M$ is conformally flat and $\xi$ is a Killing vector field, then $M$ is locally symmetric and has constant $\phi$-sectional curvature.

In view of the first part of the above theorem of Kim we immediately obtain the following:

**Theorem 4.2.** A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form is $\phi - W_5$-flat if and only if it is conformally flat.

Also, in view of the second part of the above theorem of Kim we get the following:

**Theorem 4.3.** A $(2n+1)$-dimensional $(n > 1)$ $\phi - W_5$-flat generalized Sasakian-space-form with $\xi$ as a Killing vector field is locally symmetric and has constant $\phi$-sectional curvature.
5. $\phi$-$W_5$-semisymmetric generalized Sasakian-space-forms

**Definition 5.1.** A $(2n + 1)$-dimensional $(n > 1)$ generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is called $\phi$-$W_5$-semisymmetric if it satisfies the condition

\[(5.1) \quad W_5(X, Y) \cdot \phi = 0,\]

for any vector fields $X, Y$ on the manifold.

Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$-dimensional $(n > 1)$ $\phi$-$W_5$-semisymmetric generalized Sasakian-space-form. The condition $W_5(X, Y) \cdot \phi = 0$ implies that

\[(5.2) \quad (W_5(X, Y) \cdot \phi) Z = W_5(X, Y) \phi Z - \phi W_5(X, Y) Z = 0,\]

for any vector fields $X, Y$ and $Z$. Now,

\[(5.3) \quad W_5(X, Y) \phi Z = R(X, Y) \phi Z + \frac{1}{2n}(g(X, \phi Z)QY - S(X, \phi Z)Y).\]

Using (2.3), (2.6) and (2.7) in (5.3), we get

\[(5.4) \quad W_5(X, Y) \phi Z = f_1\{g(Y, \phi Z)X - g(X, \phi Z)Y\} + f_2\{g(Y, Z)\phi X - g(X, Z)\phi Y + \eta(X)\eta(Z)\phi Y\} - \eta(Y)\eta(Z)\phi X - 2g(X, \phi Y)Z + 2g(X, \phi Y)\eta(Z)\xi\}
\[\quad + f_3\{g(X, \phi Z)\eta(Y)\} - \left[\frac{3f_2 + (2n - 1)f_3}{2n}\right] g(X, \phi Z)\eta(Y)\xi.\]

Similarly,

\[(5.5) \quad \phi W_5(X, Y) Z = \phi R(X, Y) Z + \frac{1}{2n}(g(X, Z)\phi QY - S(X, Z)\phi Y).\]

By virtue of (2.3), (2.6) and (2.7) we obtain from (5.5) that

\[(5.6) \quad \phi W_5(X, Y) Z = f_1\{g(Y, Z)\phi X - g(X, Z)\phi Y\} + f_2\{g(Y, \phi Z)X - g(X, \phi Z)Y + g(X, \phi Z)\eta(Y)\} - g(Y, \phi Z)\eta(X)\xi - 2g(X, \phi Y)Z + 2g(X, \phi Y)\eta(Z)\xi\}
\[\quad + f_3\{\eta(X)\eta(Z)\phi X\} + \left[\frac{3f_2 + (2n - 1)f_3}{2n}\right] \eta(X)\eta(Z)\phi Y.\]

Substituting (5.3) and (5.5) in (5.2) yields

\[(5.7) \quad (f_1 - f_2)\{g(Y, \phi Z)X - g(X, \phi Z)Y - g(Y, Z)\phi X + g(X, Z)\phi Y\} + (f_2 - f_3)\{\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X - g(X, \phi Z)\eta(Y)\xi + g(Y, \phi Z)\eta(X)\xi\}
\[\quad - \left[\frac{3f_2 + (2n - 1)f_3}{2n}\right] g(X, \phi Z)\eta(Y)\xi - \eta(X)\eta(Z)\phi Y\} = 0.\]

Putting $Y = \xi$ in (5.7), we obtain

\[(5.8) \quad \left[\frac{f_3 - 3f_2 - 2nf_1}{2n}\right] g(X, \phi Z)\xi = (f_1 - f_3)\eta(Z)\phi X.\]

Taking inner product with $U$, we get from (5.8)

\[(5.9) \quad \left[\frac{f_3 - 3f_2 - 2nf_1}{2n}\right] g(X, \phi Z)\eta(U) = (f_1 - f_3)\eta(Z)g(\phi X, U).\]

Putting $X = U = e_i$ in (5.9), and then taking summation over $i$, $1 \leq i \leq 2n + 1$, we get

\[(5.10) \quad (f_1 - f_3)\eta(Z)\psi = 0,\]
where $\psi = \text{Trace of } \phi$. From (5.10), we get

\begin{equation}
(5.11) \quad f_1 = f_3.
\end{equation}

Making use of (5.11) in (5.8), we obtain

\begin{equation}
(5.12) \quad [(1-2n)f_3 - 3f_2g(X, \phi Z)]\xi = 0,
\end{equation}

which implies that

\begin{equation}
(5.13) \quad f_3 = \frac{3f_2}{1-2n}.
\end{equation}

Thus in view of (5.11) and (5.13), we have

\begin{equation}
(5.14) \quad f_1 = \frac{3f_2}{1-2n} = f_3.
\end{equation}

Conversely, suppose (5.13) holds. Then in view of Theorem 3.1, we have $W_5 = 0$ and hence $W_5(X, Y).\phi = 0$. Thus we can state the following:

**Theorem 5.1.** A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space-form is $\phi$-$W_5$-semisymmetric if and only if $f_1 = \frac{3f_2}{1-2n} = f_3$.

In [7], De et al., proved the following result:

**Theorem 5.2.** A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space-form is conharmonically flat if and only if $f_1 = \frac{3f_2}{1-2n} = f_3$.

Taking into account of Theorem 3.1, Theorem 5.1 and Theorem 5.2, now we may present the following theorem:

**Theorem 5.3.** Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space-form. Then the following statements are equivalent:

1. $M(f_1, f_2, f_3)$ is $W_5$-flat;
2. $M(f_1, f_2, f_3)$ is $\phi$-$W_5$-semisymmetric;
3. $M(f_1, f_2, f_3)$ is conharmonically flat;
4. $f_1 = \frac{3f_2}{1-2n} = f_3$.

**References**


Department of Mathematics, Karnatak University, Dharwad - 580 003, INDIA.

E-mail address: prakashadg@gmail.com, kcvasant@gmail.com, mirjikk@gmail.com