HYERS-ULAM-RASSIAS TYPE STABILITY OF POLYNOMIAL EQUATIONS

N. EGHBALI

Abstract. In this paper we introduce the concept of Hyers-Ulam-Rassias stability of polynomial equations and then we show that if \( x \) is an approximate solution of the equation \( a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \), then there exists an exact solution of the equation near to \( x \).

1. Introduction

The basic problem of the stability of functional equations had been first raised by Ulam [7] which Hyers in [3] gave a partial solution of Ulam’s problem for the case of approximately additive mappings. And then Rassias provided a generalization of the Hyers theorem for additive and linear mappings in [6].

Moreover the approximately mappings have been studied extensively in several papers (See for instance [4], [5]).

Li and Hua [2] investigated the Hyers-Ulam stability of the polynomial equation \( x^n + \alpha x + \beta = 0 \) on \([-1, 1]\). Later Bikhdam et al. in [1] proved that if \(|a_1|\) is large and \(|a_0|\) is small enough, then every approximate zero of the polynomial of degree \( n \), \( a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a = 0 \), can be approximated by a true zero within a good error bound.

In this paper, we prove the Hyers-Ulam-Rassias stability for the following two equations

\[
a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a = 0
\]

\[
e^x + \alpha x + \beta = 0
\]

on a Banach space \( X \) with real coefficients.

Date: August 1, 2015 and, in revised form, October 4, 2015.
2000 Mathematics Subject Classification. Primary 46S40; Secondary 39B52, 39B82, 26E50.
Key words and phrases. Hyers-Ulam stability, Hyers-Ulam-Rassias stability; Polynomial equation; power series equation.
2. Preliminaries

In this section, we provide a collection of definitions and related results which are essential and used in the next discussions.

Definition 2.1. One says that the equation
\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a = 0 \]
has the Hyers-Ulam stability if there exists a constant \( K > 0 \) with the following property:
for every \( \varepsilon > 0, y \in [-1, 1] \), if
\[ |a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a| \leq \varepsilon \]
then there exists some \( y \in [-1, 1] \) satisfying
\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a = 0 \]
such that \( |x - y| \leq K \varepsilon \). One call such \( K \) a Hyers-Ulam stability constant for 2.1.

Theorem 2.1. For a given integer \( n > 1 \), let the constants \( a_0, a_1, \ldots, a_n \in \mathbb{R} \) satisfy
\[ |a_1| > 2|a_2| + 3|a_3| + \ldots + (n-1)|a_{n-1}| + n|a_n| \]
and \( |a_0| < |a_1| - (|a_2| + |a_3| + \ldots + |a_n|) \).
If \( v \in [-1, 1] \) satisfies the inequality
\[ |a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0| \leq \varepsilon \]
for some \( \varepsilon > 0 \), then there exists a zero \( y \in [-1, 1] \) of polynomial 2.1 such that
\[ |y - v| \leq K \varepsilon . \]

Proof. [1] \( \square \)

3. Hyers-Ulam-Rassias Stability of power series equations

We start our work with definition of Hyers-Ulam-Rassias stability of power series equations.

Definition 3.1. Let \( X \) be a complex Banach algebra with unit. The equation
\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a = 0 \]
from \( X \) into \( X \) with constant coefficient, has the Hyers-Ulam-Rassias stability if there exists a constant \( K > 0 \) with the following property:
for every \( \varepsilon > 0, p \in \mathbb{R} \) and \( y \in X \), if
\[ |a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0| \leq \varepsilon ||X||^n p \]
then there exists some \( x \in X \) satisfying
\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a = 0 \]
such that \( ||x - y|| \leq K \varepsilon \). One call such \( K \) a Hyers-Ulam-Rassias stability constant for 2.1.

Theorem 3.1. Let \( X \) be a complex Banach algebra with unit, the constants \( a_0, a_1, \ldots, a_n \in \mathbb{R} \) and \( r \in \mathbb{R}^+ \) satisfy
\[ |a_1| r > |a_0| + r^2 |a_2| + \ldots + nr^{n-1} |a_n| , \]
and
\[ |a_0| < r^2 |a_2| + 3r^3 |a_3| + \ldots + (n-1)r^n |a_n| ). \]
Let for some polynomial 2.1 such that

\[ \|a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0\| \leq \varepsilon \|x\|^{np} \]

for some \( \varepsilon > 0 \) and \( p \in \mathbb{R} \), then there exists a zero \( x_0 \in \{ x \in X; \|x\| \leq r \} \) of polynomial 2.1 such that

\[ \|x - x_0\| \leq \frac{\varepsilon r^{np}}{(1 - \lambda)|a_1|} \]

where \( \lambda = \frac{2|a_2| + 3|a_3| + \ldots + (n-1)|a_{n-1}| + n|a_n|}{|a_1|} \) is a positive constant less than 1 and it is independent of \( \varepsilon \) and \( x_0 \).

Proof. If we set \( g(x) = \frac{-1}{a_1} (a_0 + a_2 x^2 + a_3 x^3 + \ldots + a_{n-1} x^{n-1} + a_n x^n) \), for \( x \in \{ x \in X; \|x\| \leq r \} \), then we have

\[ \|g(x)\| = \frac{1}{|a_1|} \|a_0 + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n\| \leq r. \]

Now, we show that \( g \) is a contraction map:

\[ \|g(x) - g(y)\| = \left| \frac{1}{a_1} (a_0 + a_2 x^2 + \ldots + a_n x^n) - \frac{1}{a_1} (a_0 + \ldots + a_n y^n) \right| \leq \frac{1}{|a_1|} |x - y| \left\{ |a_2| \|x + y\| + \ldots + |a_n| \|x^{n-1} + \ldots + y^{n-1}\| \right\} \leq \frac{1}{|a_1|} |x - y| \left\{ 2r |a_2| + 3r^2 |a_3| + \ldots + nr^{n-1} |a_n| \right\}. \]

Here, with \( \lambda = \frac{2|a_2| + 3|a_3| + \ldots + (n-1)|a_{n-1}| + n|a_n|}{|a_1|} < 1 \), \( g \) is a contraction map. By the Banach contraction mapping theorem, there exists a unique \( x_0 \in \{ x \in X; \|x\| \leq 1 \} \) such that \( g(x_0) = x_0 \). It follows from (3.1) and (3.2) that

\[ \|x - x_0\| \leq \|x - g(x)\| + \|g(x) - g(x_0)\| \leq |x - \frac{1}{a_1} (-a_0 - a_2 x^2 - \ldots - a_1 x + a_0) + \lambda \|x - x_0\| = \frac{1}{|a_1|} \|a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0\| + \lambda \|x - x_0\|. \]

Thus we have

\[ \|x - x_0\| \leq \frac{1}{|a_1| |1 - \lambda|} \varepsilon \|x\|^{np} \leq \frac{\varepsilon r^{np}}{(1 - \lambda)|a_1|}. \]

\[ \square \]

**Theorem 3.2.** Let \( X \) be a complex Banach algebra with unit, \( r \in \mathbb{R}^+ \), \( |\alpha| > Max\{\sum_{n=1}^{\infty} \frac{n \alpha^n}{n!}, \frac{e^x + |\beta|}{r}\} \) and \( x \in \{ x \in X; \|x\| \leq r \} \) satisfies

\[ |e^x + \alpha x + \beta| \leq \varepsilon \|x\|^{np}. \]

Then \( e^x + \alpha x + \beta = 0 \) has a unique solution \( x_0 \in \{ x \in X; \|x\| \leq r \} \), such that

\[ \|x - x_0\| \leq \frac{\varepsilon r^{np}}{|\alpha||1 - \lambda|}, \] where \( \lambda = \sum_{n=1}^{\infty} \frac{n \alpha^n}{n!} \).

Proof. We define the function \( g(x) = \frac{-1}{\alpha} (e^x + \beta) \). It follows that \( \|g(x)\| \leq \frac{1}{|\alpha| |1 + |\beta||} (e^x + |\beta|) \leq r. \) Now, we have

\[ \|g(x) - g(y)\| \leq \frac{1}{|\alpha|} \|e^x - e^y\| \leq \frac{1}{|\alpha|} \sum_{n=1}^{\infty} \frac{x^n - y^n}{n!} \leq \frac{1}{|\alpha|} \sum_{n=2}^{\infty} \frac{x - y}{x - y} \|x^{n-1} + x^n - y + \ldots + xy^{n-2} + y^n - 1\| \leq \frac{1}{|\alpha|} \sum_{n=2}^{\infty} \frac{n \alpha^{n-1}}{n!} \|x - y\|. \]

By putting \( \lambda = \sum_{n=1}^{\infty} \frac{n \alpha^{n-1}}{n!} \) and Banach's contraction mapping theorem, \( g \) has a unique fixed point. So

\[ \|x - x_0\| \leq \|x - g(x)\| + \|g(x) - g(x_0)\| \leq \frac{1}{|\alpha|} \varepsilon \|x\|^{np} + \lambda \|x - x_0\|. \]
Therefore $\|x - x_0\| \leq \frac{\epsilon r^n p}{|\alpha|(1-\lambda)}$.

References


