RELATIONS AMONG HIGHER ORDER CROSSED MODULES OVER GROUPOIDS

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Abstract. In this work, we explored the categorical relationships between crossed modules, 2- and 3- crossed modules of groupoids.

INTRODUCTION

Crossed modules were introduced by Whitehead in [11] as algebraic models for homotopy 2-types. They can be seen as 2- dimensional version of groups. The category of crossed modules is equivalent to that of [9] and simplicial groups with Moore complex of length 1. Conduche [8] in 1984 described the notion of 2-crossed module as a model of connected 3-types. He proved that the category of 2-crossed modules over groups is equivalent to the category of simplicial groups with Moore complex of length 2. The relations among algebraic models for homotopy 3-types were explored by Arvasi and Ulualan in [1] explicitly. The notion of 3- crossed module was defined in [2] as model for homotopy connected 4-types. It can be said that the category of 3- crossed modules is equivalent to that of crossed 3-cubes and simplicial groups with Moore complex of length 3. This equivalence has been constructed in [2]. In this paper, we will consider these notion over groupoids. By using the definition of crossed modules over groupoids [6], the definition of 2-crossed modules can be extended to the notion of groupoids. Thus we can explore the relationship between 3-crossed modules over groups and 2- crossed modules over groupoids. Therefore, the results of this paper can be summarized by the following diagram:

\[
\begin{array}{ccc}
\text{XMod} & \longrightarrow & \text{Grpoid} \\
\downarrow & & \downarrow \\
\text{X}_2\text{Mod} & \longrightarrow & \text{X}_3\text{Mod} & \longrightarrow & \text{2CMod} & \longrightarrow & \text{CMod} \\
\end{array}
\]

with the arrows given below.

1. Preliminaries

In this section, we recall some basic definitions. The following definition is due to Whitehead [11].

Definition 1.1. [11] (Crossed Modules Over Groups): A crossed module is a group homomorphism \( \partial : M \rightarrow P \) together with an action of \( P \) on \( M \), written \( \triangleright m \) for \( p \in P \) and \( m \in M \), satisfying the conditions.

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CM1) \( \partial \) is \( P \)-equivariant, i.e, for all \( p \in P, m \in M \)
\[
\partial(pm) = p\partial(m)p^{-1}
\]
CM2) (Peiffer Identity) for all \( m, m' \in M \)
\[
\partial m m' = mm' m^{-1}
\]
We will denote such a crossed module by \((M, P, \partial)\). A morphism of crossed module from \((M, P, \partial)\) to \((M', P', \partial')\) is a pair of group homomorphisms
\[
\phi : M \longrightarrow M', \quad \psi : P \longrightarrow P'
\]
such that \( \phi(pm) = \psi(p) \phi(m) \) and \( \partial' \phi(m) = \psi \partial(m) \).

We thus get a category \( \text{XMod} \) of crossed modules.

 Examples of Crossed Modules
1) Any normal subgroup \( N \trianglelefteq P \) gives rise to a crossed module namely the inclusion map, \( i : N \hookrightarrow P \). Conversely, given any crossed module \( \partial : M \longrightarrow P \), \( \text{Im} \partial \) is a normal subgroup of \( P \).
2) Given any \( P \)-module \( M \), the trivial map \( 1 : M \longrightarrow P \) that maps everything to 1 in \( P \), is a crossed module. Conversely, if \( \partial : M \longrightarrow P \) is a crossed module, \( \ker \partial \) is central in \( M \) and inherits a natural \( P \)-module structure from the \( P \)-action on \( M \).

Conduche in [8] has defined 2-crossed module. The following definition of 2-crossed module is equivalent to that given by Conduche in [8]:

**Definition 1.2. (2-Crossed Modules Over Groups)** A 2-crossed module of groups consists of a complex of groups
\[
L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N
\]
with (a) actions of \( N \) on \( M \) and \( L \) so that \( \partial_2, \partial_1 \) are morphisms of \( N \)-groups, and (b) an \( N \)-equivariant function
\[
\{ , , \} : M \times M \longrightarrow L
\]
called a Peiffer lifting. This data must satisfy the following axioms:

2CM1) \( \{ \partial_2 m, m' \} = \{ \partial_1 m' \} mm'^{-1} m^{-1} \)
2CM2) \( \{ \partial_2 ll', l \} = \{ l', l \} \)
2CM3) (i) \( \{ mm'^{-1}, m'' \} = \{ \partial_2 m, m' \} \{ m, m'' m'^{-1} \} \)
(ii) \( \{ m, m' m'' \} = \{ m, m' \} m m'^{-1} \{ m, m'' \} \)
2CM4) \( \{ m, \partial_2 ll' \} = \{ l', l \} n \{ m, m' \} \)
2CM5) \( n \{ m, m' \} = \{ n m, n m' \} \)

for all \( l, l' \in L, m, m', m'' \in M \) and \( n \in N \).

We can give definition of 3-crossed module from [2] as follows:

**Definition 1.3.** [2] A 3-crossed module consists of a complex of groups
\[
K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N
\]
with an action of \( N \) on \( K, L, M \) and an action of \( M \) on \( K, L \) and an action of \( L \) on \( K \) so that \( \partial_3, \partial_2, \partial_1 \) are morphisms of \( N, M \)-groups and the \( M, N \)-equivariant liftings
\[
\{ , , \}(1,0) : L \times L \longrightarrow K, \quad \{ , , \}(0,2) : L \times L \longrightarrow K, \quad \{ , , \}(2,1) : L \times L \longrightarrow K,
\]
\[
\{ , , \}(1,0)(2) : M \times L \longrightarrow K, \quad \{ , , \}(2,0)(1) : M \times L \longrightarrow K,
\]
\[
\{ , , \}(0)(2,1) : L \times M \longrightarrow K, \quad \{ , , \} : M \times M \longrightarrow L
\]
called 3-dimensional Peiffer liftings. This data must satisfy the following axioms:
3CM1) \[ K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \] is a 2-crossed module with the Peiffer lifting \{ , \}_{(2,1)}
3CM2) \[ \{ m, \partial_3k \}_{(0,0)(2)} = \{ m, \partial_3k \}_{(2,0)(1)} \] for \( k \delta_m(k)^{-1} \)
3CM3) \[ \{ \partial_3k, m \}_{(0,2)(1)} = w^m(k)k^{-1} \]
3CM4) \[ \{ m, \partial_3k \}_{(1,0)(2)} = \{ m, \partial_3k \}_{(2,0)(3)} \] for \( k \delta_m(k)^{-1} \)
3CM5) \[ \{ l', \partial_3l' \}_{(0,2)(1)} = \{ l, l' \}_{(2)(1)} \] [\[ l', l' \]_{(1)(0)}]
3CM6) \[ \{ \partial_3l, l' \}_{(2)(0)(1)} = \{ l, l' \}_{(0)(2)} \] \[ [ l', l' \]_{(1)(0)}]
3CM7) \[ \{ \partial_3l, l' \}_{(1,0)(2)} = \{ l, l' \}_{(0)(2)} \] \[ (0,2) \]
3CM8) \[ \partial_3 \{ l, l' \}_{(1)(0)} = \{ l, l' \}_{(2)(1)} \] \[ (0,2) \]
3CM9) \[ \partial_3 \{ l, l' \}_{(0)(2)} = \partial_3 \{ l, l' \}_{(1,0)(2)} \] \[ (0,2) \]
3CM10) \[ \partial_3 \{ m, l \}_{(0,0)(1)} = \partial_3 \{ m, l \}_{(0,0)(2)} \] \[ (0,2) \]
3CM11) \[ \partial_3 \{ m, l \}_{(0,0)(1)} = \partial_3 \{ m, l \}_{(0,0)(2)} \] \[ (l, l') \]
3CM12a) \[ \partial_3 \{ m, l \}_{(1,0)(0)} = \partial_3 \{ m, l \}_{(1,0)(0)} \] \[ (l, l') \]
3CM12b) \[ \partial_3 \{ m, l \}_{(1,0)(0)} = \partial_3 \{ m, l \}_{(1,0)(0)} \] \[ (l, l') \]
3CM13) \[ \{ \partial_3k, \partial_3l' \}_{(1)(0)(0)} = \{ k', k \} \]
3CM14) \[ \{ \partial_3k, \partial_3l' \}_{(0)(0)(0)} = \{ k', k \} \]
3CM15) \[ \{ \partial_3k, \partial_3l' \}_{(1)(0)(0)} = \{ k, \partial_3l' \}_{(0)(0)} \]
3CM16) \[ \{ \partial_3k, \partial_3l' \}_{(2)(0)(1)} = \{ k, \partial_3l' \}_{(0)(0)} \]
3CM17) \[ \{ \partial_3k, \partial_3l' \}_{(0)(0)(1)} = \partial_3l' \] \[ (k, l') \]
3CM18) \[ \partial_3 \{ m, m' \} = mm' \delta_m(k)^{-1} \]

We denote such a 3-crossed module by \((K, L, M, N, \partial_3, \partial_2, \partial_1)\).

A morphism of 3-crossed modules of groups may be pictured by the diagram

\[
\begin{array}{cccccc}
L_3 & \xrightarrow{\partial_3} & L_2 & \xrightarrow{\partial_2} & L_1 & \xrightarrow{\partial_1} & L_0 \\
\downarrow{f_3} & & \downarrow{f_2} & & \downarrow{f_1} & & \downarrow{f_0} \\
L'_3 & \xrightarrow{\partial'_3} & L'_2 & \xrightarrow{\partial'_2} & L'_1 & \xrightarrow{\partial'_1} & L'_0
\end{array}
\]

where \[ f_1(m)^m = (f_0(n))f_1(m), \] \[ f_2(l)^m = (f_0(n))f_2(l), \] \[ f_3(m)^m = (f_0(n))f_3(k) \]
for \( , \}_{(0)(2)}, , \}_{(2)(1)}, , \}_{(1)(0)} \)

\[ \{ , \} f_2 \times f_2 = f_3 \{ , \} \]
for \( , \}_{(1)(0)(2)}, , \}_{(2)(0)(1)} \)

\[ \{ , \} f_1 \times f_2 = f_3 \{ , \} \]
for \( , \}_{(0)(2,1)} \)

\[ \{ , \} f_2 \times f_1 = f_3 \{ , \} \]
and for \( , \) \}

\[ \{ , \} f_1 \times f_1 = f_2 \{ , \} \]

for all \( k \in K, l \in L, m \in M, n \in N \). These compose in an obvious way. We thus can define the category of 3-crossed modules, denoting it by \( X_3\text{Mod} \).

2. Groupoids, Crossed Modules and 2-Crossed Modules over Groupoids

Recall that a groupoid is a small category in which every arrow is an isomorphism. That is, for any morphisms \( \alpha \) there exists a morphism \( \alpha^{-1} \) such that \( \alpha \circ \alpha^{-1} = e_{\alpha(\alpha)} \) and \( \alpha^{-1} \circ \alpha = e_{\alpha(\alpha)} \)
where \( e : C_0 \rightarrow C_1 \) gives the identity morphism at an object. We write a groupoid as \((C_1, C_0)\), where \( C_0 \) is the set of object and \( C_1 \) is the set of morphisms. For any groupoid \( C \), if \( C_1(x, y) \) is empty whenever \( x, y \) are distinct (that is, \( s = t \)), then \( C \) is called totally disconnected. You can
find detailed description of groupoids in [3, 4, 5, 7, 10]. A morphism of groupoid is \((f_1, f_0)\) in the following diagram

\[
\begin{array}{c}
C_1 \xrightarrow{f_1} C'_1 \\
\downarrow s_1 \quad \downarrow t_1 \\
C_0 \xrightarrow{f_0} C'_0
\end{array}
\]

and we denote the category of groupoids by \(\text{Grpoid}\). Now we recall the definition of crossed modules of groupoids. The basic reference is Brown-Higgins [9].

**Definition 2.1.** Let \(G, C\) be groupoids over same object set and let \(C\) be totally disconnected. Then an action of \(G\) on \(C\) is partially defined function

\[
C \times G \to C
\]

written \((c, a) \mapsto c^a\), which satisfies

I. \(c^a\) is defined if only if \(t(c) = s(a)\) and then \(t(c^a) = t(a)\)

II. \((c_1 \circ c_2)^a = c_1^a \circ c_2^a\)

III. \(c_1^{a b} = (c_1^b)^a\) and \(c_1^{e_0} = c_1\)

for all \(c_1, c_2 \in C(x, x)\) and \(a \in G(x, y), b \in G(y, z)\).

**Definition 2.2.** A crossed module of groupoids consists of a morphisms \(\delta : C \to G\) of groupoids \(C\) and \(G\) which is the identity on the object sets that \(C\) is totally disconnected, together with an action of \(G\) on \(C\) which satisfies

I. \(\delta(c^a) = a^{-1} \circ \delta(c) \circ a\)

II. \(c^{s_{c_1}} = c^{-1} \circ c \circ c_1\)

for all \(c, c_1 \in C(x, x)\) and \(a \in G(x, y)\). A crossed module of groupoids will be denoted by \((C, G, \delta)\). A morphism of crossed modules of groupoids from \((C, G, \delta)\) to \((C', G', \delta')\) is a pair of groupoid homomorphisms

\[
\alpha : C \to C', \beta : G \to G'
\]

such that \(\alpha(c^a) = \alpha(c)^{\beta(a)}\) and \(\delta'^{-1} \alpha(c) = \beta \delta(c)\) for all \(c \in C(x, x)\) and \(a \in G(x, y)\). We thus get a category \(\text{CMod}\) of crossed modules of groupoids.

Using the definition of crossed modules of groupoids, we can define 2-crossed modules of groupoid similarly.

**Definition 2.3.** \(L, M\) and \(N\) are groupoids, having the same set of objects, \(O\), and let \(L, M\) be totally disconnected groupoids.

A crossed modules of groupoids consists of a complex of groupoids

\[
\begin{array}{c}
L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N \\
\downarrow O \quad \downarrow O \quad \downarrow O
\end{array}
\]

together with an action of \(N\) on \(L\) and \(M\), so that both \(\partial_2\) and \(\partial_1\) are morphisms of groupoid. Also

\[
\begin{array}{c}
\{-, -\} : M \times M \to L
\end{array}
\]

called a Peiffer lifting, which satisfies the following axioms:

- **2CM1** \(\partial_2 \{m, m'\} = \{\partial_1 m' \circ m \circ m'^{-1}, m^{-1}\}\)
- **2CM2** \(\{\partial_2 l, \partial_2 l'\} = \{l', l\}\)
- **2CM3** \((i)\) \(\{m \circ m', m''\} = \partial_1 m \{m', m''\} \{m, m' \circ m'' \circ m'^{-1}\}\)
  \((ii)\) \(\{m, m' \circ m''\} = \{m, m'\} \{m, m''\} \{m, m''\} \{m, m''\} \{m, m''\} \{m, m''\}\)
- **2CM4** \(\{m, \partial_2 l\} \{\partial_2 l, m\} = \{\partial_1 m \circ l^{-1}\}\)
- **2CM5** \(n \{m, m'\} = \{m, m'\}\)
for all \(m, m', m'' \in M, l, l' \in L\) and \(n \in N\). A morphism of 2-crossed modules of groupoids is groupoid homomorphisms from \((L, M, N, \partial_2, \partial_1)\) to \((L', M', N', \partial'_2, \partial'_1)\) such that

\[
f_0 \partial_1 = \partial'_1 f_1, \quad f_1 (''m) = f_0 (m), \quad f_2 (''l) = f_0 (l)
\]

and

\[
\{-, -\} f_1 \times f_1 = f_2 \{-, -\}
\]

for all \(l \in L, m \in M\) and \(n \in N\). We thus can define the category of 2-crossed modules of groupoids, denoting it by \(2\text{CMod}\).

3. Relations Among Higher Order Crossed Modules

In this section, we would have given the following diagram by defining the functors between categories.

\[
\begin{array}{ccc}
\text{XMod} & \xrightarrow{F_1} & \text{Grpoid} \\
F_2 \downarrow & & \downarrow F_6 \\
\text{X}_2\text{Mod} & \xrightarrow{F_4} & \text{CMod} \\
F_3 \downarrow & & \downarrow F_7 \\
\text{X}_3\text{Mod} & \xrightarrow{F_5} & \text{2CMod}
\end{array}
\]

3.1. From crossed modules of groups to groupoids. For this functor see also [4]. Let \(\partial : M \to P\) be crossed module. In this case, there is an action of \(P\) on \(M\) so that we can define \(M \rtimes P\), which is called semi direct product group. The operation in \(M \rtimes P\) as follows:

\[
(m, p) \odot (m', p') = (mm', pp')
\]

Let \(C_0 = P\) and \(C_1 = M \rtimes P\). Now we must define \(s, t, e\) and \(\circ\) in \((C_1, C_0)\). The source, target maps and identity morphism are given by respectively

\[
s(m, p) = p, \quad t(m, p) = m \partial p, \quad e(p) = (1, p)
\]

The composition of two morphisms is given by

\[
(m', p') \circ (m, p) = (m'm, p)
\]

if \(S(m', p') = p' = \partial m. p = t(m, p)\).

In this case, we obtain

\[
s((m', p') \circ (m, p)) = s(m'm, p) = p = s(m, p)
\]

and

\[
t((m', p') \circ (m, p)) = t(m'm, p) = \partial(m) \partial(m') p = \partial(m') p' \quad (\because \partial(m)p = p') = t(m', p')
\]

Furthermore

\[
se(p) = s(1, p) = p \quad \text{and} \quad se = id : C_0 \to C_0
\]

\[
te(p) = t(1, p) = \partial(1)p = p \quad \text{and} \quad te = id : C_0 \to C_0
\]

and we can show inverse morphism of \((m, p) : p \to \partial(m)p\) as follows

\[
(m^{-1}, \partial(m)p) : \partial(m)p \to p.
\]
Then we obtain a groupoid \((C_1, C_0, s, t, e)\). Therefore, we can define a functor from \(\text{XMod}\) to \(\text{Grpoid}\). We denote this functor by

\[
F_1 : \text{XMod} \to \text{Grpoid}
\]

3.2. From 2 crossed modules of groups to crossed modules of groupoids. Let

\[
K \xrightarrow{\partial_2} L \xrightarrow{\partial_1} M
\]

be a 2-crossed module. There are actions of \(\text{Grpoid}\) From 2-crossed modules to 3-crossed modules of groups.

3.4. From 2 crossed modules of groups to crossed modules of groupoids. Let

\[
X \xrightarrow{\xi} L \to M
\]

This map clearly satisfies the required conditions. Therefore, we can define a functor from \(\text{XMod}\) to \(\text{Grpoid}\). We denote this functor by

\[
F_1 : \text{XMod} \to \text{Grpoid}
\]

\[
F_4 : \text{X}_2\text{Mod} \to \text{CMod}
\]

3.3. From crossed modules to 2-crossed modules of groups. Let \(M \xrightarrow{\partial_1} N\) be crossed module.

\[
\{-, -\} : M \times M \to \{1\}
\]

This map clearly satisfies the required conditions. Therefore, we can define a functor from \(\text{XMod}\) to \(\text{X}_2\text{Mod}\). We denote this functor by

\[
F_2 : \text{XMod} \to \text{X}_2\text{Mod}
\]

3.4. From 2-crossed modules to 3-crossed modules of groups. Let

\[
\{,\}^{(1)(0)} : L \times L \to \{1\}, \quad \{,\}^{(2)(0)} : L \times L \to \{1\}, \quad \{,\}^{(1)(2)} : L \times L \to \{1\},
\]

\[
\{,\}^{(1,0)(2)} : M \times L \to \{1\}, \quad \{,\}^{(2,0)(1)} : M \times L \to \{1\},
\]

\[
\{,\}^{(0)(2,1)} : L \times M \to \{1\}, \quad \{,\} : M \times M \to \{1\}
\]

be 3-dimensional Peiffer liftings.

3CM1. We showed that the structure

\[
\{1\} \xrightarrow{\partial_2} L \xrightarrow{\partial_2} M
\]
We can define Peiffer lifting map for the above structure by using \( \{-, -\} \). Totally disconnected groupoids.

In this structure, we obtain \( 3 \text{CM2}. \{m, \partial_31\}_{(1,0)(2)} = 1 \) and \( \{m, \partial_31\}_{(2,0)(1)} \cdot m_{1.1} = 1 \).

\( 3 \text{CM3}. \{\partial_31, m\}_{(0)(2,1)} = 1 \) and \( m_{1.1} = 1 \)

\( 3 \text{CM4}. \{m, \partial_31\}_{(1,0)(2)} = 1 \) and \( \{m, \partial_31\}_{(2,0)(1)} \cdot \{\partial_31, m\}_{(0)(2,1)} \cdot m_{1.1} = 1 \).

The other axioms of 3-crossed module can be showed similarly. Therefore, we can define a functor from \( \text{X}_2\text{Mod} \) to \( \text{X}_3\text{Mod} \). We denote this functor by

\[ F_3 : \text{X}_2\text{Mod} \rightarrow \text{X}_3\text{Mod} \]

3.5. From 3-crossed modules to 2-crossed modules of over groupoids. Let

\[ K \xrightarrow{o_2} L \xrightarrow{o_3} M \xrightarrow{o_1} N \]

be a 3-crossed module. We try to show that structure

\[ K \times N \xrightarrow{\delta_2} L \times N \xrightarrow{\delta_1} M \times N \]

\[ N \xrightarrow{id} N \xrightarrow{id} N \]

is a 2-crossed module over groupoid \( N \).

In this structure, we obtain

\[ \delta_1(l, n) = (\partial_2(l), n) \]

\[ \delta_2(k, n) = (\partial_3(k), n) \]

and \( s(l, n) = n, t(l, n) = \partial_1 \circ \partial_2(l)n = n \) and \( s(k, n) = n, t(k, n) = \partial_1 \circ \partial_2 \circ \partial_3(k)n = n \) then

\[ L \times N \xrightarrow{s,t} N, K \times N \xrightarrow{s,t} N \]

are totally disconnected groupoids.

We can define Peiffer lifting map for the above structure by using \( \{-, -\}_{(2)(1)} : L \times L \rightarrow K \), namely.

\( \{-, -\} : (l \times N) \times (l \times N) \rightarrow K \times N \)

\( \{(l, n), (l', n)\} \rightarrow (l', n) \circ (l, n)^{-1} \circ \delta_1(l, n) (l'^{-1}, n) \)

We can show that all axioms of Peiffer lifting map given in Definition 2.3 are satisfied.

**PL.1.** It must be that

\[ \delta_2\{(l, n), (l', n)\} = (l, n) \circ (l', n) \circ (l, n)^{-1} \circ \delta_1(l, n) (l'^{-1}, n) \]

For \((l, n), (l', n) \in L \times N\) we obtain the following equality;

\[ \delta_2\{(l, n), (l', n)\} = \delta_2\{(l, l')_{(2)(1)}, n\} = \{\partial_3(l, k'_{(2)(1)}, n) = (l, l')^{-1} \circ \partial_2(l) (l'^{-1}, n) = (l, n) \circ (l', n) \circ (l, n)^{-1} \circ \delta_1(l, n) (l'^{-1}, n) \]

**PL.2.** For \((k, n), (k', n) \in K \times N\). We have

\[ \{\delta_2(k, n), \delta_2(k', n)\} = \{\partial_3(k, n), (\partial_3(k'), n)\} = ((\partial_3(k), \partial_3(k'))_{(2)(1)}, n) = ([k, k'], n) = (k, n) \circ (k', n) \circ (k^{-1}, n) \circ (k'^{-1}, n) = [(k, n), (k', n)] \]

and this shows that the axiom **PL.2** is satisfied.

**PL.3.** For the elements \((l, n), (l', n), (l'', n) \in L \times N\), we calculate the following result:
\begin{align*}
(i, n, (l', n)) &= ((l, l')_1)_2, n) \\
&= ((l, l')_1)_2, n) \\
&= ((l, l')_1)_2, n) \\
&= ((l, l')_1)_2, n) \\
&= ((l, l')_1)_2, n). \\
\end{align*}

PL4. For \((l, n) \in L \times N\) and \((k, n) \in K \times N\), we have
\[
\{l, n, \delta_2(k, n)\}(\delta_2(k, n), (l, n)) = \{l, n, (\delta_2(k, n), (l, n)) = \{l, n, (\delta_2(k, n), (l, n)) = \delta_1, l, n, (k^{-1}, n).
\]

PL5. For \((l, n), (l', n) \in L \times N\) and \((m, n) \in M \times N\), we have
\[
\]

Therefore, we can define a functor from the category of 3-crossed module to that of 2-crossed module of groupoids. We denote this functor by
\[
F_5 : X_3\text{Mod} \to 2\text{Mod}
\]

3.6. From groupoids to crossed module of groupoids. Let \(N \xrightarrow{s.t} O\) be groupoid, so it can be clearly seen that
\[
\{e\} \xrightarrow{i} N \xrightarrow{s.t} O
\]
is crossed module over \(O\), where \(i\) is inclusion map. Therefore, we can define a functor from the category of groupoids to \(\text{CMod}\). We show this functor by
\[
F_6 : \text{Grpoid} \to \text{CMod}
\]

3.7. From crossed module of groupoids to 2-crossed module of groupoids. We can obtain a 2-crossed module of groupoids from crossed module over groupoids by using similar way in subsection 3.3.

References


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