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ON SOME IDENTITIES FOR k-FIBONACCI DIFFERENCE SEQUENCE

EUDES ANTONIO COSTA, ÉLIS G. C. MESQUITA, AND PAULA M. M. C. CATARINO

0000-0001-6684-9961, 0000-0003-2385-4108, and 0000-0001-6917-5093

ABSTRACT. In this paper, several new identities are given for the k-Fibonacci difference sequence. This is accomplished by solving a class of non-homogeneous, linear recurrence relations. We address a sequence of the k-Fibonacci type, called the k-Fibonacci difference sequence. This sequence is obtained by applying the finite difference operator to the k-Fibonacci sequence a finite number of times. By taking advantage of the strong dependence this sequence has on its initial terms, as well as on the k-Fibonacci and k-Lucas sequences, we were able to derive a wide range of properties, including classical identities such as those of Tagiuri-Vajda, Catalan, and Cassini identities, among others. Moreover, we obtained its extension to negative indexes with explicit expressions. Several types of generating functions were also derived, including exponential and Poisson-type generating functions. In addition, we present some results concerning the limit of the ratio involving terms of the k-Fibonacci difference sequence for both positive and negative indexes, along with various identities for partial sums.

1. INTRODUCTION

For a positive real number k, the k-Fibonacci sequence was introduced by Falcón and Plata in [8], in which each new element is defined by the recurrence relation

$$f_{k,n+1} = kf_{k,n} + f_{k,n-1} \tag{1.1}$$

for all integers $n \geq 1$, with initial values $f_{k,0} = 0$ and $f_{k,1} = 1$. The first few elements are 0, 1, k, k^2+1, k^3+2k, k^4+3k^2+1, $k^5+4k^3+3k, \ldots$ In [3], Falcón presents the k-Lucas (or k-Fibonacci-Lucas) sequence $\{l_{k,n+1}\}_{n\geq 0}$, which is defined by the same recurrence $l_{k,n+1} = kl_{k,n} + l_{k,n-1}$, for $n \geq 1$, and with initial terms $l_{k,0} = 2$ and $l_{k,1} = k$. Thus, the first seven terms are 2, k, k^2+2, k^3+3k, k^4+4k^2+2, k^5+5k^3+5k, $k^6+6k^4+9k^2+2, \ldots$ Applications and properties of k-Fibonacci sequence are explored, for example, in [3, 8, 11], and other works present new results and generalizations, such as the k-Fibonacci generating matrices and an extension of the type (k,t)-Fibonacci numbers, (see [1], [2], [4], [5], [12]).

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For a given sequence $\{a_n\}_{n\geq 0}$, one can define the finite difference operator $\Delta(a_n)$, and, recursively, one can define its iterated i-th, that is, $\Delta(a_n) = a_{n+1} - a_n$ and $\Delta^i(a_n) = \Delta^{i-1}(\Delta(a_n))$ In [6], Falcón introduces the so-called *k-Fibonacci difference sequence*, $\{F^{(j)}_{k,n}\}_{n\geq 0}$, by considering $a_n = f_{k,n}$ and $F^{(j)}_{k,n} = \Delta^j(f_{k,n})$. However, although several properties and identities have been presented, for example, *k-Fibonacci difference sequence* verifies the recurrence relation of the k–Fibonacci sequence. However, no classical identities have been addressed.

In this work, we present some classical identities for *k-Fibonacci difference sequence* $\{F^{(j)}_{k,n}\}_{n\geq 0}$. For this, we use the technique provided by the work [10] in which the authors showed that the pair of k-Fibonacci sequences $\{f_{k,n}\}_{n\geq 0}$ and $\{l_{k,n}\}_{n\geq 0}$ can be used as a framework to obtain a large number of identities for any generalized k-Fibonacci sequence.

In this introduction, we situate our investigation object. In what follows, in Section 2, we regard the k-Fibonacci and k-Lucas sequences and list key results (known or new) with respect to these sequences that will be used in this work. In Section 3, for all real $k > 0$ we exhibit the Binet formula for the k-Fibonacci difference sequence and explore its applications. We show the connection between k-Fibonacci-type sequences and any k-Fibonacci difference sequence; we also extend the k-Fibonacci difference sequence to include negative subscripts, and we establish some generating functions for the k-Fibonacci difference sequence. In Section 4, the most important for our goal, we give some identities for the k-Fibonacci difference sequence. The classical identities are studied for this sequence; for instance, the Convolution identity, the Tagiuri-Vajda identity, and its consequences are presented. In Section 5, the limit of the quotient between two terms of the k-Fibonacci difference sequence is analyzed, and the partial sum involving elements of the k-Fibonacci difference numbers is studied.

2. Background and Auxiliary Results

In this section, we present some identities involving k-Fibonacci and k-Lucas numbers, which will be used to derive several results for the k-Fibonacci difference sequence in the following sections.

The k-Fibonacci type sequence is recognized to be linked to the characteristic equation

$$r^2 = kr + 1\,, \tag{2.1}$$

whose two distinct roots are denoted by $r_{k,1}$ and $r_{k,2}$. These roots are given by $r_{k,1} = (k + \sqrt{k^2+4})/2$ and $r_{k,2} = (k - \sqrt{k^2+4})/2$, which play a central role in obtaining properties of the sequence.

Lemma 2.1 ([1], Proposition 2). *Let $\{f_{k,\,n}\}_{n\geq 0}$ be the k-Fibonacci sequence. Then*

$$f_{k,n} = \frac{r^n_{k,1} - r^n_{k,2}}{r_{k,1} - r_{k,2}}\,, \tag{2.2}$$

where $r_{k,1}$ and $r_{k,2}$ are the distinct roots from Equation (2.1).

Lemma 2.2 ([3], Theorem 2.2). *Let $\{l_{k,\,n}\}_{n\geq 0}$ be the k-Lucas sequence. Then*

$$l_{k,n} = r^n_{k,1} + r^n_{k,2}\,, \tag{2.3}$$

where $r_{k,1}$ and $r_{k,2}$ are the distinct roots of the Equation (2.1).

Equations (2.2) and (2.3), respectively, are the Binet formula for the k-Fibonacci and k-Lucas sequences.

For the purpose of establishing identities involving the k- Fibonacci difference numbers, in this article, we will consider the following identities for the k-Fibonacci and k-Lucas numbers as support.

Lemma 2.3 ([3], Theorem 2.4). *For $n \geq 1$ and any positive real numbers k, we have*

$$l_{k,n} = f_{k,n-1} + f_{k,n+1}\,, \tag{2.4}$$

where $\{f_{k,n}\}_{n\geq 0}$ and $\{l_{k,n}\}_{n\geq 0}$ are, respectively, the k-Fibonacci sequence and the k-Lucas sequence.

Lemma 2.4 ([2], Equation (26)). *For $n \geq 1$ and any positive real numbers k, we have*

$$(k^2+4)f_{k,n} = l_{k,n-1} + l_{k,n+1}\,, \tag{2.5}$$

where $\{f_{k,n}\}_{n\geq 0}$ and $\{l_{k,n}\}_{n\geq 0}$ are the k-Fibonacci sequence and the k-Lucas sequence, respectively.

Lemma 2.5 ([7], Equation (6)). *For $n \geq 1$ and any positive real numbers k, we have*

$$f_{k,n-1}f_{k,n+1} - (f_{k,n})^2 = (-1)^n\,, \tag{2.6}$$

where $\{f_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence.

Lemma 2.6 ([10], Proposition 2.1). *For $n \geq 1$ and any positive real numbers k, we have*

$$(k^2+4)f_{k,n}f_{k,n+1} = l_{k,2n+1} + k(-1)^{n+1}, \tag{2.7}$$

$$(k^2+4)(f_{k,n})^2 = l_{k,2n} + 2(-1)^{n+1}, \tag{2.8}$$

where $\{f_{k,n}\}_n$ is the k-Fibonacci sequence and $\{l_{k,n}\}_n$ is the k-Lucas sequence.

We use $\{g_{k,n}\}_{n\geq 0}$ to denote the generalized k-Fibonacci sequence, or a k-Fibonacci-type sequence, defined by the recurrence

$$g_{k,\,n+1} = kg_{k,n} + g_{k,n-1}, \quad n \geq 1 \tag{2.9}$$

with initial terms $g_{k,0} = a$ and $g_{k,1} = b$, where a and b are fixed integer constants.

According to Theorem 2.1 in [10] and its consequences, we have the next result.

Lemma 2.7. *For non-negative integer values of n and m, and for any generalized k-Fibonacci sequence $\{g_{k,\,n}\}_{n\geq 0}$, we have*

$$g_{k,\,n+m} = f_{k,m-1}\cdot g_{k,\,n} + f_{k,m}\cdot g_{k,\,n+1}\,,$$

$$g_{k,\,-m} = (-1)^m f_{k,m+1}\cdot a + (-1)^{m+1} f_{k,m}\cdot b\,,$$

where $\{f_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence.

Consider $\{g_{k,n}\}$ and $\{h_{k,n}\}$ two k-Fibonacci-type sequences. The next statement shows some connections between them.

Lemma 2.8 ([10], Theorem 2.2). *For any generalized k-Fibonacci sequences $\{g_{k,n}\}_{n\geq 0}$ and $\{h_{k,n}\}_{n\geq 0}$, that is, both satisfy (2.9), the following identity holds*

$$g_{k,\,n+m}h_{k,\,n+l} - g_{k,\,n}h_{k,\,n+m+l} = (-1)^n(g_{k,m}h_{k,l} - g_{k,0}h_{k,\,m+l}). \tag{2.10}$$

3. Binet's formula and applications

In this section, we will first review the Binet formula for the k-Fibonacci difference and explore some applications. We will show the relationship between the k-Fibonacci type sequences and any k-Fibonacci difference sequence. We will also present an extension of the k-Fibonacci difference with negative subscripts. Finally, we will give the generating functions for the k-Fibonacci difference sequence.

3.1. Connection of k-Fibonacci difference sequence with k-Fibonacci numbers.

In this subsection, we will explore the relationship between k-Fibonacci type sequences and any k-Fibonacci difference sequence.

Consider any k-Fibonacci sequence $\{f_{k,n}\}_{n\geq 0}$, and for any positive integer j make $F_{k,n}^{(j)} = \Delta^j(f_{k,n})$, being Δ the difference operator. The first result follows directly from the existence of roots of the characteristic equation (2.1), and using the initial conditions

$$F_{k,0}^{(j)} = a_{k,j} \quad \text{and} \quad F_{k,1}^{(j)} = b_{k,j}\ ,$$

that is, $a_{k,j}$ and $b_{k,j}$ are the initial terms of the sequence $F_{k,n}^{(j)}$ given by the j-th application of the difference operator Δ.

Firstly, [6] proves that the k–Fibonacci difference sequences verify also the initial relation (1.1).

Lemma 3.1 ([6], Lemma 1)**.** *Let* $\{\mathrm{F}_{k,n}^{(j)}\}_{n\geq 0}$ *be the* k*-Fibonacci difference sequence. Then* $\{\mathrm{F}_{k,n}^{(j)}\}_{n\geq 0}$ *verifies the recurrence relation of the* k*–Fibonacci sequence:*

$$F_{k,n+1}^{(j)} = kF_{k,n}^{(j)} + F_{k,n-1}^{(j)} \tag{3.1}$$

From the relation (3.1), in [6] it is deduced the Binet formula for k–Fibonacci difference sequences.

Lemma 3.2 ([6], subsection 2.2)**.** *Let* $\{F_{k,\,n}^{(j)}\}_{n\geq 0}$ *be the* k*-Fibonacci difference sequence. Then*

$$F_{k,n}^{(j)} = \frac{(b_{k,j} - a_{k,j}r_{k,2})\, r_{k,1}^n - (b_{k,j} - a_{k,j}r_{k,1})\, r_{k,2}^n}{r_{k,1} - r_{k,2}}\ , \tag{3.2}$$

where $r_{k,1}$ *and* $r_{k,2}$ *are the distinct roots of the Equation* (2.1)*.*

According to Corollary 2 in [6], it is possible to write the k–Fibonacci difference numbers depending on the classical k–Fibonacci numbers. In a similar way, we present the following result, where also the k–Fibonacci difference numbers depend on the classical k–Fibonacci numbers in the manner indicated.

Theorem 3.3. *Let* $\{F_{k,n}^{(j)}\}_{n\geq 0}$ *be the* k*-Fibonacci difference sequence. Then*

$$F_{k,n}^{(j)} = b_{k,j}f_{k,\,n} + a_{k,j}f_{k,\,n-1}\ , \tag{3.3}$$

for all $n \geq 1$*, where* $\{f_{k,n}\}_{n\geq 0}$ *is the* k*-Fibonacci sequence.*

Proof. In Lemma 2.7 replace $\{g_{k,n}\}_{n\geq 0}$ by $\{F_{k,n}^{(j)}\}_{n\geq 0}$, since $\{\mathrm{F}_{k,n}^{(j)}\}_{n\geq 0}$ being the k-Fibonacci sequence (see Equation (3.1)). As $F_{k,0}^{(j)} = a_{k,j}$ and $F_{k,1}^{(j)} = b_{k,j}$, the end of the proof is established. □

The following result states the k-Fibonacci difference numbers as a linear combination formula involving the k–Fibonacci and k–Lucas numbers.

Theorem 3.4. *Let* $\{F^{(j)}_{k,n}\}_{n\geq 0}$ *be the* k*-Fibonacci difference sequence. Then, for all* $n \geq 0$*,*

$$F^{(j)}_{k,n} = \frac{a_{k,j}}{2} l_{k,n} - \left(\frac{k a_{k,j}}{2} - b_{k,j}\right) f_{k,\,n}\,, \tag{3.4}$$

where $\{f_{k,n}\}_{n\geq 0}$ *is the* k*-Fibonacci sequence and* $\{l_{k,n}\}_{n\geq 0}$ *is the* k*-Lucas sequence.*

Proof. The proof is given by mathematical induction on n. Note that

$$F^{(j)}_{k,\,0} = \frac{a_{k,j}}{2} l_{k,0} - \left(\frac{k a_{k,j}}{2} - b_{k,j}\right) f_{k,\,0} = \frac{a_{k,j}}{2} 2 - \left(\frac{k a_{k,j}}{2} - b_{k,j}\right) 0 = a_{k,j}$$
$$F^{(j)}_{k,\,1} = \frac{a_{k,j}}{2} l_{k,1} - \left(\frac{k a_{k,j}}{2} - b_{k,j}\right) f_{k,\,1} = \frac{a_{k,j}}{2} k - \left(\frac{k a_{k,j}}{2} - b_{k,j}\right) 1 = b_{k,j}\,.$$

Suppose that the result is valid for any positive integer less than or equal to n, for some $n \geq 1$. Then, by Equation (3.1), we have

$$\begin{aligned} F^{(j)}_{k,n+1} &= k F^{(j)}_{k,n} + F^{(j)}_{k,n-1} \\ &= k\left(\frac{a_{k,j}}{2} l_{k,n} - \left(\frac{k a_{k,j}}{2} - b_{k,j}\right) f_{k,\,n}\right) + \frac{a_{k,j}}{2} l_{k,n-1} - \left(\frac{k a_{k,j}}{2} - b_{k,j}\right) f_{k,\,n-1} \\ &= \frac{a_{k,j}}{2} l_{k,n+1} - \left(\frac{k a_{k,j}}{2} - b_{k,j}\right) f_{k,n+1}\,. \end{aligned}$$

So, the result is valid for any non-negative integer n. □

As a consequence, we get the next result.

Theorem 3.5. *Let* $\{F^{(j)}_{k,n}\}_{n\geq 0}$ *be the* k*-Fibonacci difference sequence. Then*

$$F^{(j)}_{k,n+1} + F^{(j)}_{k,n-1} = (k^2+4)\frac{a_{k,j}}{2} f_{k,n} - \left(\frac{k a_{k,j}}{2} - b_{k,j}\right) l_{k,n} \tag{3.5}$$

where $\{f_{k,n}\}_{n\geq 0}$ *is the* k*-Fibonacci sequence and* $\{l_{k,n}\}_{n\geq 0}$ *is the* k*-Lucas sequence.*

Proof. The result required follows from Equations (2.4), (2.5) and (3.4). □

Let $\{F^{(j)}_{k,n}\}_{n\geq 0}$ be the k-Fibonacci difference sequence. Consider the constants $c = b_{k,j} - a_{k,j} r_{k,2}$ and $d = b_{k,j} - a_{k,j} r_{k,1}$, where $r_{k,1}$ and $r_{k,2}$ are the distinct roots of the Equation (2.1). So, observe that

$$\begin{aligned} cd &= [b_{k,j} - a_{k,j} r_{k,2}][b_{k,j} - a_{k,j} r_{k,1}] \\ &= (b_{k,j})^2 - k a_{k,j} b_{k,j} - (a_{k,j})^2\,, \end{aligned}$$

as $r_{k,1} + r_{k,2} = k$ and $r_1 r_2 = -1$.

Following the literature (for instance, see pag. 139 in [9]), we will denote it by letter $\mu_{k,j}$. The product $\mu_{k,j} = -cd$, and let us call it by the j-characteristic of the k-Fibonacci difference sequence.

From the above discussion and notation, we have the following result.

Proposition 1. The j-characteristic $\mu_{k,j}$ of the k-Fibonacci difference sequence $\{F^{(j)}_{k,n}\}_{n\geq 0}$ is given by

$$\mu_{k,j} = (a_{k,j})^2 + k a_{k,j} b_{k,j} - (b_{k,j})^2\,,$$

where $r_{k,1}$ and $r_{k,2}$ represent the distinct roots of Equation (2.1), $a_{k,j}$ and $b_{k,j}$ are the initial terms of $\{F^{(j)}_{k,n}\}$.

According to [9], the constant cd appears in a lot of formulas for Fibonacci-type numbers. It is called the *characteristic* of the Fibonacci sequence. The authors in [10] showed that this remains valid for the k-Fibonacci type sequences.

3.2. **Negative index for the k-Fibonacci difference sequence.** In this subsection, we extend the definition of the k-Fibonacci difference sequence to negative indexes. Using standard techniques for Fibonacci-type sequences, we establish a recurrence relation for negative subscripts and derive explicit formulas consistent with the properties of the sequence.

The Fibonacci and Lucas (or Fibonacci-Lucas) numbers with negative subscripts, according to [9, Equations 5.19 and 5.20], have the relations

$$f_{-n} = \quad (-1)^{n+1} f_n \quad \text{and} \quad l_{-n} = \quad (-1)^n l_n$$

where $\{f_n\}_{n\geq 0}$ is the ordinary Fibonacci sequence and $\{l_n\}_{n\geq 0}$ is the Lucas sequence. In a similar way, it has been obtained for the k-Fibonacci sequence $\{f_{k,n}\}$ and k-Fibonacci-Lucas sequence $\{l_{k,n}\}$, that is,

Lemma 3.6. [10, Proposition 2.2] *For all non-negative integer values of n, we have*

$$f_{k,-n} = (-1)^{n+1} f_{k,n} \tag{3.6}$$

$$l_{k,-n} = (-1)^n l_{k,n} \tag{3.7}$$

where $\{f_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence and $\{l_{k,n}\}_{n\geq 0}$ is the k-Lucas sequence.

We will give the meaning to the numbers $F^{(j)}_{k,-n}$ for every integer number n, and the recurrence will remain valid. To extend the sequence $\{F^{(j)}_{k,n}\}_{n\geq 0}$ to negative subscripts, we use the modified recurrence relation

$$F^{(j)}_{k,\,n-2} = F^{(j)}_{k,\,n} - kF^{(j)}_{k,n-1}.$$

Then, we have the following pattern:

$$\begin{aligned}
F^{(j)}_{k,\,-1} &= F^{(j)}_{k,1} - kF^{(j)}_{k,0} = b_{k,j} - ka_{k,j} = -1(-1\cdot b_{k,j} + k\cdot a_{k,j}) \\
&= -1(-f_{k,1}\cdot b_{k,j} + f_{k,2}\cdot a_{k,j})\ , \\
F^{(j)}_{k,\,-2} &= F^{(j)}_{k,0} - kF^{(j)}_{k,-1} = a_{k,j}(1+k^2) - kb_{k,j} = 1(-f_{k,2}\cdot b_{k,j} + f_{k,3}\cdot a_{k,j})\ , \\
F^{(j)}_{k,\,-3} &= F^{(j)}_{k,-1} - kF^{(j)}_{k,-2} = b_{k,j} - ka_{k,j} - k(a_{k,j}(1+k^2) - kb_{k,j}) \\
&= -1(-f_{k,3}\cdot b_{k,j} + f_{k,4}\cdot a_{k,j}),
\end{aligned}$$

and so on. Thus, we have the following result, which is a corresponding result to the Equation (3.3) for negative subscripts.

Proposition 2. Let $\{F^{(j)}_{k,n}\}_{n\geq 0}$ be the k-Fibonacci difference sequence. For all integers $n \geq 1$, the general term with negative indexes n-th satisfies

$$F^{(j)}_{k,-n} = (-1)^n \left(a_{k,j}\cdot f_{k,n+1} - b_{k,j}\cdot f_{k,n} \right)\ ,$$

where $\{f_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence.

Proof. Let $U_{k,\,-n} = (-1)^n \left(a_{k,j}\cdot f_{k,n+1} - b_{k,j}\cdot f_{k,n} \right)$ for all integers $n \geq 1$. We want to show that $U_{k,n}$ verifies the recurrence relation $U_{k,-n} = U_{k,-(n+2)} + kU_{k,-(n+1)}$.

Indeed,

$$\begin{aligned}&U_{k,\,-(n+2)}+kU_{k,\,-(n+1)}\\&=(-1)^{n+2}\left(a_{k,j}\cdot f_{k,n+3}-b_{k,j}\cdot f_{k,n+2}\right)+(-1)^{n+1}k\left(a_{k,j}\cdot f_{k,n+2}-b_{k,j}\cdot f_{k,n+1}\right)\\&=(-1)^{n}\left(a_{k,j}\cdot f_{k,n+1}-b_{k,j}\cdot f_{k,n}\right)=U_{k,-n}\,.\end{aligned}$$

Moreover, we get $U_{k,-1}=-(a_{k,j}\cdot f_{k,2}-b_{k,j}\cdot f_{k,1})=F_{k,-1}$ and $U_{k,-2}=a_{k,j}\cdot f_{k,3}-b_{k,j}\cdot f_{k,2}=F_{k,-2}$. So, since $U_{k,n}$ satisfies the recurrence that defines $F_{k,n}$ with the same initial condition, we conclude that $U_{k,-n}=F^{(j)}_{k,-n}$. ☐

Another way of expressing the general term of the sequence $\{F^{(j)}_{k,-n}\}_{n\geq 0}$ is the following result:

Proposition 3. Let $\{F^{(j)}_{k,n}\}_{n\geq 0}$ be the k-Fibonacci difference sequence. For each integer $n\geq 1$, then the negative index n-th number $F^{(j)}_{k,n}$ satisfies the following identity:

$$F^{(j)}_{k,-n}=(-1)^{n}\left(F^{(j)}_{k,n}+2\left(\frac{ka_{k,j}}{2}-b_{k,j}\right)f_{k,n}\right)\,, \tag{3.8}$$

where $\{f_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence.

Proof. Combining Equations (3.4), (3.6) and (3.7) we have

$$\begin{aligned}F^{(j)}_{k,-n}&=\frac{a_{k,j}}{2}l_{k,-n}-\left(\frac{ka_{k,j}}{2}-b_{k,j}\right)f_{k,-n}\\&=\frac{a_{k,j}}{2}(-1)^{n}l_{k,n}-(-1)^{n+1}\left(\frac{ka_{k,j}}{2}-b_{k,j}\right)f_{k,n}\\&=(-1)^{n}\left[F^{(j)}_{k,n}+2\left(\frac{ka_{k,j}}{2}-b_{k,j}\right)f_{k,n}\right]\,,\end{aligned}$$

which verifies the result. ☐

3.3. **Generating functions.** In this subsection, we determine the generating functions for the k-Fibonacci difference sequence. More specifically, we present three types of generating functions for the k-Fibonacci difference sequence: the ordinary generating function, the exponential generating function, and the Poisson generating function. We point out that [6] introduced the ordinary generator.

The ordinary generating function for a sequence $\{a_n\}_{n\geq 0}$, denoted as $G_{a_n}(x)$, is defined as a formal series given by:

$$G_{a_n}(x)=\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_nx^n+\cdots\,.$$

The following result provides the explicit form of the generating function for the k-Fibonacci difference sequence $\{F^{(j)}_{k,n}\}_{n\geq 0}$.

Proposition 4 ([6], Equation (12))**.** The generating function for the k-Fibonacci difference sequence $\{F^{(j)}_{k,n}\}_{n\geq 0}$, denoted by $G_{F^{(j)}_{k,n}}(x)$, is given by

$$G_{F^{(j)}_{k,n}}(x)=\frac{a_{k,j}+(b_{k,j}-ka_{k,j})x}{1-kx-x^2}$$

The exponential generating function $E_{a_n}(x)$ for a sequence $\{a_n\}_{n\geq 0}$ is represented as a power series given by:

$$E_{a_n}(x) = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \cdots + \frac{a_n x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}.$$

In the following result, we consider the case where $a_n = F_{k,n}^{(j)}$ and apply the Binet formula for the k-Fibonacci difference sequence. The proof of the next result follows directly from Binet's formula.

Proposition 5. The exponential generating function for the k-Fibonacci difference sequence $\{F_{k,n}^{(j)}\}_{n\geq 0}$ is

$$E_{F_{k,n}^{(j)}}(x) = \sum_{n=0}^{\infty} F_{k,n}^{(j)} \frac{x^n}{n!} = \frac{(b_{k,j} - a_{k,j} r_{k,2}) e^{r_{k,1} x} - (b_{k,j} - a_{k,j} r_{k,1}) e^{r_{k,2} x}}{r_{k,1} - r_{k,2}}$$

where $r_{k,1}$ and $r_{k,2}$ are the distinct roots of the Equation (2.1).

The Poisson generating function $P_{a_n}(x)$ for a sequence $\{a_n\}_{n\geq 0}$ is defined as:

$$P_{a_n}(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} e^{-x} = e^{-x} E_{a_n}(x).$$

This relationship establishes a direct connection between the two generating functions, allowing the Poisson generating function to be derived from its exponential function.

As a particular case, considering the exponential generating function for the k-Fibonacci difference sequence given in Proposition 5 as $E_{F_{k,n}^{(j)}}(x)$, we obtain the following result.

Corollary 3.7. *The Poisson generating function for the k-Fibonacci difference sequence $\{F_{k,n}^{(j)}\}_{n\geq 0}$ is:*

$$P_{F_{k,n}^{(j)}}(x) = \frac{(b_{k,j} - a_{k,j} r_{k,2}) e^{(r_{k,1}-1) x} - (b_{k,j} - a_{k,j} r_{k,1}) e^{(r_{k,2}-1) x}}{r_{k,1} - r_{k,2}}$$

where $r_{k,1}$ and $r_{k,2}$ are the distinct roots of Equation (2.1).

4. Some Properties

In this section, we establish some identities for the k-Fibonacci difference sequence. We also provide the classical Tagiuri-Vajda identity, among others.

4.1. Identities for the k-Fibonacci difference sequence.

The first result is the establishment of the multiplication formula for two consecutive terms of the k-Fibonacci difference sequence.

Proposition 6. For all non-negative integers n and $k > 0$, the k-Fibonacci difference sequence $\{F_{k,n}^{(j)}\}_{n\geq 0}$ satisfies the following identity:

$$(k^2+4) F_{k,n}^{(j)} F_{k,n+1}^{(j)} = (b_{k,j})^2 l_{k,2n+1} + 2 a_{k,j} b_{k,j} l_{k,2n} + (a_{k,j})^2 l_{k,2n-1} + \mu_{k,j} k (-1)^n$$

where $\{l_{k,n}\}$ is the k-Lucas sequence and $\mu_{k,j}$ is the j-characteristic of $\{F_{k,n}^{(j)}\}$.

Proof. By Equation (3.3), we have

$$\begin{aligned}F_{k,n}^{(j)}F_{k,n+1}^{(j)} &= (b_{k,j}f_{k,n}+a_{k,j}f_{k,n-1})(b_{k,j}f_{k,n+1}+a_{k,j}f_{k,n})\\ &= (b_{k,j})^2 f_{k,n}f_{k,n+1}+a_{k,j}b_{k,j}[f_{k,n}^2+f_{k,n-1}f_{k,n+1}]+(a_{k,j})^2 f_{k,n-1}f_{k,n}\,.\end{aligned}$$

So, by equations (2.4), (2.5), (2.6), (2.7) and (2.8), it follows that

$$\begin{aligned}(k^2{+}4)F_{k,n}^{(j)}F_{k,n+1}^{(j)} &= (b_{k,j})^2[l_{k,2n+1}-k(-1)^n]+a_{k,j}b_{k,j}(k^2+4)[(-1)^n+2(f_{k,n})^2]\\ &\qquad + (a_{k,j})^2[l_{k,2n-1}+k(-1)^n]\\ &= (b_{k,j})^2 l_{k,2n+1}-k(b_{k,j})^2(-1)^n+2a_{k,j}b_{k,j}l_{k,2n}+k^2a_{k,j}b_{k,j}(-1)^n\\ &\qquad + (a_{k,j})^2 l_{k,2n-1}+k(a_{k,j})^2(-1)^n\\ &= (b_{k,j})^2 l_{k,2n+1}+2a_{k,j}b_{k,j}l_{k,2n}+(a_{k,j})^2 l_{k,2n-1}\\ &\quad + \left[(a_{k,j})^2+ka_{k,j}b_{k,j}-(b_{k,j})^2\right]k(-1)^n.\end{aligned}$$

Now, using $\mu_{k,j}$ we get the result. □

The next result is one of the most important in this paper and provides a direct connection between the terms of the k-Fibonacci difference sequence with Fibonacci and Lucas numbers.

Theorem 4.1. *For non-negative integer values of n, q, k, and for the k-Fibonacci difference sequence $\{F_{k,n}^{(j)}\}_{n\geq 0}$, the following identity holds:*

$$2F_{k,n+q}^{(j)} = l_{k,q}F_{k,n}^{(j)} + f_{k,q}\left((k^2+4)\frac{a_{k,j}}{2}f_{k,n} - \left(\frac{ka_{k,j}}{2}-b_{k,j}\right)l_{k,n}\right), \tag{4.1}$$

where $\{l_{k,n}\}_{n\geq 0}$ is the k-Lucas sequence and $\{f_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence.

Proof. Take $g_{k,n} = F_{k,n}^{(j)}$, for some j, and change m by $-q$ in Lemma 2.7. Then, by using the Equations (2.4) and (3.6) we have

$$F_{k,n-q}^{(j)} = (-1)^q\left(f_{k,q+1}F_{k,n}^{(j)} - f_{k,q}F_{k,n+1}^{(j)}\right).$$

Then,

$$\begin{aligned}F_{k,n+q}^{(j)}+(-1)^qF_{k,n-q}^{(j)} &= f_{k,\,q-1}F_{k,n}^{(j)}+f_{k,q}F_{k,n+1}^{(j)}+f_{k,q+1}F_{k,n}^{(j)}-f_{k,q}F_{k,n+1}^{(j)}\\ &= (f_{k,q-1}+f_{k,q+1})F_{k,n}^{(j)} = l_{k,q}\cdot F_{k,n}^{(j)}\end{aligned}$$

and

$$\begin{aligned}F_{k,n+q}^{(j)}-(-1)^qF_{k,n-q}^{(j)} &= f_{k,q-1}F_{k,n}^{(j)}+f_{k,q}F_{k,n+1}^{(j)}-f_{k,q+1}F_{k,n}^{(j)}+f_{k,q}F_{k,n+1}^{(j)}\\ &= (f_{k,q-1}-f_{k,q+1})F_{k,n}^{(j)}+2f_{k,q}F_{k,n+1}^{(j)}\\ &= f_{k,q}\left(F_{k,n-1}^{(j)}+F_{k,n+1}^{(j)}\right).\end{aligned}$$

Now, by summing the last two equations, we obtain

$$2F_{k,n+q}^{(j)} = l_{k,q}F_{k,n}^{(j)} + f_{k,q}\left(F_{k,n-1}^{(j)}+F_{k,n+1}^{(j)}\right).$$

The result follows from Equation (3.5). □

4.2. **Some classical identities for k-Fibonacci difference sequence.** In this subsection, we will provide some classical identities for the k-Fibonacci difference sequence by considering the Binet formula and the results of the previous sections.

The convolution identity for k-Fibonacci difference sequence is shown below.

Proposition 7 (Convolution's identity)**.** Let m, n be non-negative integer numbers, then the k-Fibonacci difference sequence satisfies the following identity:

$$F^{(j)}_{k,m-1}F^{(j)}_{k,n} + F^{(j)}_{k,m}F^{(j)}_{k,n+1} = b_{k,j}F^{(j)}_{k,n+m} + a_{k,j}F^{(j)}_{k,n+m-1}\,,$$

where $\{F^{(j)}_{k,n}\}_{n\geq 0}$ is the k-Fibonacci difference sequence .

Proof. By Equation (3.3) we have $F^{(j)}_{k,n} = b_{k,j}f_{k,n} + a_{k,j}f_{k,n-1}$, then

$$\begin{aligned}
&F^{(j)}_{k,m-1}F^{(j)}_{k,n} + F^{(j)}_{k,m}F^{(j)}_{k,n+1}\\
&= (b_{k,j}f_{k,m-1} + a_{k,j}f_{k,m-2})(b_{k,j}f_{k,n} + a_{k,j}f_{k,n-1}) + (b_{k,j}f_{k,m} + a_{k,j}f_{k,m-1})(b_{k,j}f_{k,n+1} + a_{k,j}f_{k,n})\\
&= (b_{k,j})^2 f_{k,m-1}f_{k,n} + a_{k,j}b_{k,j}f_{k,m-1}f_{k,n-1} + a_{k,j}b_{k,j}f_{k,m-2}f_{k,n} + (a_{k,j})^2 f_{k,m-2}f_{k,n-1}\\
&\qquad + (b_{k,j})^2 f_{k,m}f_{k,n+1} + a_{k,j}b_{k,j}f_{k,m}f_{k,n} + a_{k,j}b_{k,j}f_{k,m-1}f_{k,n+1} + (a_{k,j})^2 f_{k,m-1}f_{k,n}\\
&= (b_{k,j})^2(f_{k,m-1}f_{k,n} + f_{k,m}f_{k,n+1}) + a_{k,j}b_{k,j}(f_{k,m-1}f_{k,n-1} + f_{k,m}f_{k,n})\\
&\qquad + a_{k,j}b_{k,j}(f_{k,m-2}f_{k,n} + f_{k,m-1}f_{k,n+1}) + (a_{k,j})^2(f_{k,m-2}f_{k,n-1} + f_{k,m-1}f_{k,n})\\
&= (b_{k,j})^2 f_{k,m+n} + a_{k,j}b_{k,j}f_{k,m+n-1} + a_{k,j}b_{k,j}f_{k,m+n-1} + (a_{k,j})^2 f_{k,m+n-2}\\
&= b_{k,j}(b_{k,j}f_{k,m+n} + a_{k,j}f_{k,m+n-1}) + a_{k,j}(b_{k,j}f_{k,m+n-1} + a_{k,j}f_{k,m+n-2})\\
&= b_{k,j}F^{(j)}_{k,n+m} + a_{k,j}F^{(j)}_{k,n+m-1}\,,
\end{aligned}$$

as required. □

As a direct consequence, then we recover the standard form of the ordinary convolution identity:

Corollary 4.2. *Let j a non-negative integer such that $a_{k,j} = 0$ and $b_{k,j} = 1$, then*

$$F^{(j)}_{k,m-1}F^{(j)}_{k,n} + F^{(j)}_{k,m}F^{(j)}_{k,n+1} = F^{(j)}_{k,n+m}\,,$$

where $\{F^{(j)}_{k,n}\}_{n\geq 0}$ is the k-Fibonacci difference sequence .

In the following, we will introduce the Tagiuri-Vajda identity. It is a key result of this section as it allows us to derive a wide range of identities which we get using Lemma 2.8.

Theorem 4.3 (Tagiuri-Vajda's identity)**.** *Let $\{F^{(j)}_{k,n}\}$ be the k-Fibonacci difference sequence and $\mu_{k,j}$ your j-characteristic, then for all non-negative integers n, m, q, the following identity holds:*

$$F^{(j)}_{k,n+m}F^{(j)}_{k,n+q} - F^{(j)}_{k,n}F^{(j)}_{k,n+m+q} = (-1)^{n+1}\,\mu_{k,j}\cdot f_{k,m}f_{k,q}\,, \tag{4.2}$$

where $\{f_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence.

Proof. Taking $g_{k,i} = h_{k,i} = F^{(j)}_{k,i}$ in Equation (2.10) we have

$$F^{(j)}_{k,n+m}F^{(j)}_{k,n+q} - F^{(j)}_{k,n}F^{(j)}_{k,n+m+q} = (-1)^n\left[F^{(j)}_{k,m}F^{(j)}_{k,q} - F^{(j)}_{k,0}F^{(j)}_{k,m+q}\right]$$

Using the Equations (3.4) and (4.1), we have

$$
\begin{aligned}
&F^{(j)}_{k,n+m}F^{(j)}_{k,n+q}-F^{(j)}_{k,n}F^{(j)}_{k,n+m+q}\\
&=(-1)^n\left[F^{(j)}_{k,m}F^{(j)}_{k,q}-\frac{a_{k,j}}{2}\left(l_{k,q}F^{(j)}_{k,m}+(k^2+4)\frac{a_{k,j}}{2}f_{k,q}f_{k,m}-\left(\frac{ka_{k,j}}{2}-b_{k,j}\right)l_{k,m}f_{k,q}\right)\right],\\
&=(-1)^n\left[-f_{k,q}\left[\left(\frac{ka_{k,j}}{2}-b_{k,j}\right)\left(F^{(j)}_{k,m}-\frac{a_{k,j}}{2}l_{k,m}\right)+(k^2+4)\left(\frac{a_{k,j}}{2}\right)^2 f_{k,m}\right]\right]\\
&=(-1)^n\left[-f_{k,q}\left[-\left(\frac{ka_{k,j}}{2}-b_{k,j}\right)\left(\frac{ka_{k,j}}{2}-b_{k,j}\right)f_{k,m}+(k^2+4)\left(\frac{a_{k,j}}{2}\right)^2 f_{k,m}\right]\right]\\
&=(-1)^{n+1}f_{k,m}f_{k,q}\left[(a_{k,j})^2+ka_{k,j}b_{k,j}-(b_{k,j})^2\right]\ .
\end{aligned}
$$

Since $\mu_{k,j}=(a_{k,j})^2+ka_{k,j}b_{k,j}-(b_{k,j})^2$, we conclude the proof. ☐

As a direct consequence of the Tagiuri-Vajda identity, the following results are also obtained for the d'Ocagne, the Catalan and the Cassini identities for the k-Fibonacci difference sequence.

Proposition 8 (d'Ocagne's identity)**.** Let h,n be non-negative integer numbers with $h\geq n$, then

$$F^{(j)}_{k,h}F^{(j)}_{k,n+1}-F^{(j)}_{k,n}F^{(j)}_{k,h+1}=(-1)^{n+1}\mu_{k,j}f_{k,h-n}\ ,$$

where $\{F^{(j)}_{k,n}\}_{n\geq 0}$ is the k-Fibonacci difference sequence and $\mu_{k,j}$ your j-characteristic and $\{f_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence.

Proof. Consider $m+n=h$ and $q=1$ in Equation (4.2), then

$$
\begin{aligned}
F^{(j)}_{k,h}F^{(j)}_{k,n+1}-F^{(j)}_{k,n}F^{(j)}_{k,h+1}&=(-1)^{n+1}\mu_{k,j}f_{k,h-n}f_{k,1}\\
&=(-1)^{n+1}\mu_{k,j}f_{k,h-n}\ ,
\end{aligned}
$$

as $f_{k,1}=1$ for all integers $k>0$, which proves the result. ☐

Proposition 9 (Catalan's identity)**.** Let n and m be non-negative integer numbers with $m\geq n$, then

$$(F^{(j)}_{k,n})^2-F^{(j)}_{k,n-m}F^{(j)}_{k,n+m}=(-1)^{n+m+1}\mu_{k,j}(f_{k,m})^2\ , \tag{4.3}$$

where $\{F^{(j)}_{k,n}\}_{n\geq 0}$ is the k-Fibonacci difference sequence, $\mu_{k,j}$ your j-characteristic and $\{f_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence.

Proof. Taking $q=-m$ in Equation (4.2), we have

$$F^{(j)}_{k,n+m}F^{(j)}_{k,m-n}-(F^{(j)}_{k,n})^2=(-1)^{n+1}\mu_{k,j}f_{k,m}f_{k,-m}\,.$$

The result follows using Equation (3.6). ☐

A direct consequence of the Catalan identity is the following identity. To obtain the desired result, simply substitute $m=2$ into Equation (4.3).

Corollary 4.4. *For all non-negative integers $n\geq 2$, we have*

$$(F^{(j)}_{k,n})^2-F^{(j)}_{k,n-2}F^{(j)}_{k,n+2}=\mu_{k,j}(-1)^{n+1}(f_{k,2})^2\ , \tag{4.4}$$

where $\{F^{(j)}_{k,n}\}_{n\geq 0}$ is the k-Fibonacci difference sequence, $\{f_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence, and $\mu_{k,j}$ your j-characteristic.

Another consequence of the Catalan identity, as $f_{k,1}=1$ and by doing $m=1$ in Equation (4.3), we have the following result.

Corollary 4.5 (Cassini-Simson's identity). *For all non-negative integers n, we have*

$$(F^{(j)}_{k,n})^2 - F^{(j)}_{k,n-1}F^{(j)}_{k,n+1} = (-1)^n \mu_{k,j}\ , \tag{4.5}$$

where $\{F^{(j)}_{k,n}\}_{n\geq 0}$ is the k-Fibonacci difference sequence and $\mu_{k,j}$ your j-characteristic.

A direct consequence is the Cassini-Simson identity for subscripts even, and this is given above.

Corollary 4.6. *For all non-negative integers n, we have*

$$(F^{(j)}_{k,2n})^2 - F^{(j)}_{k,2n-1}F^{(j)}_{k,2n+1} = \mu_{k,j}\ ,$$

where $\{F^{(j)}_{k,n}\}_{n\geq 0}$ is the k-Fibonacci difference sequence and $\mu_{k,j}$ is your j-characteristic.

Now, we present the Gelin-Cesàro identity for k-Fibonacci difference sequence. Note that, this result also follows from Catalan's identity.

Proposition 10 (Gelin-Cesàro's identity). The k-Fibonacci difference sequence $\{F^{(j)}_{k,n}\}_{n\geq 0}$ satisfies the following identity:

$$F^{(j)}_{k,n+2}F^{(j)}_{k,n+1}F^{(j)}_{k,n-1}F^{(j)}_{k,\,n-2} - (F_{k,n)})^4 = \mu_{k,j}(-1)^n(F^{(j)}_{k,n})^2(k^2-1) + k^2\mu^2_{k,j}\ ,$$

where $\mu_{k,j}$ is the j-characteristic of the k-Fibonacci difference sequence, $\{F^{(j)}_{k,n}\}_{n\geq 0}$ is the k-Fibonacci difference sequence.

Proof. According to Equation (4.4), we have

$$F^{(j)}_{k,n-2}F^{(j)}_{k,n+2} = (F^{(j)}_{k,n})^2 + \mu_{k,j}(-1)^{n+1}(f_{k,2})^2\ , \tag{4.6}$$

By Equation (4.5)

$$F^{(j)}_{k,n-1}F^{(j)}_{k,n+1} = F^{(j)}_{k,n})^2 + (-1)^n\mu_{k,j}\ , \tag{4.7}$$

Combining the Equations (4.6) and (4.7), we get

$$\begin{aligned}
&F^{(j)}_{k,n+2}F^{(j)}_{k,n+1}F^{(j)}_{k,n-1}F^{(j)}_{k,n-2} - (F^{(j)}_{k,n})^4\\
&= \Big[(F^{(j)}_{k,n})^2 + \mu_{k,j}(-1)^{n+1}(f_{k,2})^2\Big]\Big[(F^{(j)}_{k,n})^2 + \mu_{k,j}(-1)^n\Big] - (F^{(j)}_{k,n})^4\\
&= \Big[(F^{(j)}_{k,n})^2 + \mu_{k,j}(-1)^{n+1}k^2\Big]\Big[(F^{(j)}_{k,n})^2 - \mu_{k,j}(-1)^{n+1}\Big] - (F_{k,n})^4\\
&= (F^{(j)}_{k,n})^4 - \mu_{k,j}(-1)^{n+1}(F^{(j)}_{k,n})^2 + k^2\mu_{k,j}(-1)^{n+1}(F^{(j)}_{k,n})^2 - k^2\mu^2_{k,j}(-1)^{2n+1} - (F^{(j)}_{k,n})^4\\
&= \mu_{k,j}(-1)^{n+1}(F^{(j)}_{k,n})^2(k^2-1) + k^2\mu^2_{k,j}\ ,
\end{aligned}$$

as $f_{k,1}=1$ and $f_{k,2}=k$, which verifies the result. □

5. Some Ratio and Partial Sum

The ratio between two terms (successive or not) of a sequence $\{a_n\}_{n\geq 0}$ is defined as $q_{n+t} = \frac{a_{n+t}}{a_n}$, where t is a natural number. In particular, for $t = 1$ we have the quotient between successive terms a_{n+1} and a_n. For instance, in the classical Fibonacci sequence $\{f_n\}_{n\geq 0}$, the ratio is given by $q_{n+1} = \frac{f_{n+1}}{f_n}$. In this case, as n becomes large, the sequence $\{q_{n+1}\}_{n\geq 0}$ converges to the golden ratio, $\alpha = \frac{1+\sqrt{5}}{2}$. In this section we will examine the generalized quotient $q_{n+t} = \frac{a_{n+t}}{a_n}$, for a fixed positive integer t.

The first result shows that the sequence $\{q_{k,\,n+t}\}_{n\geq 0}$ for the $\{F^{(j)}_{k,n}\}_{n\geq 0}$ converges to $r_{k,1}$ as n goes to infinity, and t is a fixed positive integer.

Proposition 11. If $F^{(j)}_{k,n}$ is the n-th term of k-Fibonacci difference sequence, then

$$\lim_{n\to\infty} \frac{F^{(j)}_{k,n+t}}{F^{(j)}_{k,n}} = (r_{k,1})^t\,, \tag{5.1}$$

and

$$\lim_{n\to\infty} \frac{F^{(j)}_{k,-(n+t)}}{F^{(j)}_{k,-n}} = (-r_{k,1})^t\,, \tag{5.2}$$

for any positive integer t, where $r_{k,1}$ is a root of Equation (2.1).

Proof. According to Binet's formula (3.2), we have

$$\frac{F^{(j)}_{k,n+t}}{F^{(j)}_{k,n}} = (r_{k,1})^t\,\frac{(b_{k,j}-a_{k,j}r_{k,2})-(b_{k,j}-a_{k,j}r_{k,1})(\frac{r_{k,2}}{r_{k,1}})^{n+t}}{(b_{k,j}-a_{k,j}r_{k,2})-(b_{k,j}-a_{k,j}r_{k,1})(\frac{r_{k,2}}{r_{k,1}})^{n}}\ .$$

Since $|r_{k,2}/r_{k,1}| < 1$, it follows that $(r_{k,2}/r_{k,1})^n \to 0$ as $n\to\infty$. Thus,

$$\lim_{n\to\infty} \frac{F^{(j)}_{k,n+t}}{F^{(j)}_{k,n}} = (r_{k,1})^t\,\frac{b_{k,j}-a_{k,j}r_{k,2}}{b_{k,j}-b_{k,j}r_{k,2}} = (r_{k,1})^t\ ,$$

and thus (5.1) follows.

Using the Equation (3.8), we can write

$$\frac{F^{(j)}_{k,\,-(n+t)}}{F^{(j)}_{k,\,-n}} = (-1)^t\,\frac{F^{(j)}_{k,\,n+t}+2(\frac{ka_{k,j}}{2}-b_{k,j})f_{k,n+t}}{F^{(j)}_{k,\,n}+2(\frac{ka_{k,j}}{2}-b_{k,j})f_{k,n}}\ .$$

Also follows from Binet's formula that

$$\begin{aligned}&\frac{F^{(j)}_{k,\,n+t}+2(\frac{ka_{k,j}}{2}-b_{k,j})f_{k,n+t}}{F^{(j)}_{k,\,n}+2(\frac{ka_{k,j}}{2}-b_{k,j})f_{k,n}}\\ &= (r_{k,1})^t\,\frac{(b_{k,j}-a_{k,j}r_{k,2})-(b_{k,j}-a_{k,j}r_{k,1})(\frac{r_{k,2}}{r_{k,1}})^{n+t}+2(\frac{ka_{k,j}}{2}-b_{k,j})(1-(\frac{r_{k,2}}{r_{k,1}})^{n+t})}{(b_{k,j}-a_{k,j}r_{k,2})-(b_{k,j}-a_{k,j}r_{k,1})(\frac{r_{k,2}}{r_{k,1}})^{n}+2(\frac{ka_{k,j}}{2}-b_{k,j})(1-(\frac{r_{k,2}}{r_{k,1}})^{n})}\ .\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\left[\frac{F^{(j)}_{k,\,n+t}+2(\frac{ka_{k,j}}{2}-b_{k,j})f_{k,n+t}}{F^{(j)}_{k,n}+2(\frac{ka_{k,j}}{2}-b_{k,j})f_{k,n}}\right]$$

$$=(r_{k,1})^t\ \frac{(b_{k,j}-a_{k,j}r_{k,2})+2(\frac{ka_{k,j}}{2}-b_{k,j})}{(b_{k,j}-a_{k,j}r_{k,2})+2(\frac{ka_{k,j}}{2}-b_{k,j})}=(r_{k,1})^t\,,$$

that concludes the proof. □

In what follows, we can immediately establish the next result using fundamental tools from the calculus of limits, along with (5.1) and (5.2).

Corollary 5.1. *If $F^{(j)}_{k,n}$ are the n-th term of the k-Fibonacci difference sequence, then*

$$\lim_{n\to\infty}\frac{F^{(j)}_{k,n}}{F^{(j)}_{k,n+t}}=\left(\frac{1}{r_{k,1}}\right)^t\,,$$

and

$$\lim_{n\to\infty}\frac{F^{(j)}_{k,\,-n}}{F^{(j)}_{k,\,-(n+t)}}=\left(-\frac{1}{r_{k,1}}\right)^t\,,$$

for any integer t, where $r_{k,1}$ and $r_{k,2}$ are the roots of Equation (2.1).

Note that if $t=1$, then $F^{(j)}_{k,n+1}/F^{(j)}_{k,n}$ converges to $r_{k,1}$ independently of j. It is similar to what happens with both the ordinary Fibonacci and k-Fibonacci sequences; the positive root of the j characteristic equation is the limit of the ratio between successive terms. This relationship is well documented in the mathematical literature.

Now, we investigate the properties and identities associated with the partial sums of the $n+1$ terms of the k-Fibonacci difference sequence, providing insights into their pattern and applications. The sum of the first $n+1$ terms of the sequence is expressed as:

$$\sum_{i=0}^{n}F^{(j)}_{k,i}=F^{(j)}_{k,0}+F^{(j)}_{k,1}+F^{(j)}_{k,2}+\cdots+F^{(j)}_{k,n-1}+F^{(j)}_{k,n}\,.$$

We begin by presenting three key results concerning the partial sums of the k-Fibonacci difference sequence.

First, we present the sum of the first $n+1$ terms of the k-Fibonacci difference sequence.

Theorem 5.2. *For all non-negative integers n, then*

$$\sum_{i=0}^{n}F^{(j)}_{k,i}=\frac{1}{k}[F^{(j)}_{k,n}+F^{(j)}_{k,n+1}+(k-1)a_{k,j}-b_{k,j}]\,,$$

where $\{F^{(j)}_{k,\,n}\}_{n\geq 0}$ is the k-Fibonacci difference sequence.

Proof. According to Equation (2.9) we have the following equations:

$$\begin{aligned} F_{k,0}^{(j)} &= F_{k,2}^{(j)} - kF_{k,1}^{(j)}, \\ F_{k,1}^{(j)} &= F_{k,3}^{(j)} - kF_{k,2}^{(j)}, \\ &\vdots \\ F_{k,n-1}^{(j)} &= F_{k,n+1}^{(j)} - kF_{k,n}^{(j)}. \end{aligned}$$

By adding both sides of these equations, we have

$$\sum_{i=0}^{n-1} F_{k,i}^{(j)} = \sum_{i=2}^{n+1} F_{k,i}^{(j)} - k\sum_{i=1}^{n} F_{k,i}^{(j)},$$

that is,

$$k\sum_{i=1}^{n} F_{k,i}^{(j)} = F_{k,n}^{(j)} + F_{k,n+1}^{(j)} - F_{k,0}^{(j)} - F_{k,1}^{(j)},$$

as $F_{k,0}^{(j)} = a_{k,j}$ and $F_{k,1}^{(j)} = b_{k,j}$, we conclude the result. □

In a similar way to what we have done in Theorem 5.2, we will have similar results in the following, but here we will only present the statement. The sum of the terms at even indexes of the k-Fibonacci difference sequence can be expressed as:

Proposition 12. For all non-negative integers n, then

$$\begin{aligned} \sum_{i=0}^{\frac{n}{2}} F_{k,2i}^{(j)} &= \frac{1}{k}\left[F_{k,n+1}^{(j)} + ka_{k,j} - b_{k,j}\right], \text{ if } n \text{ is even} \\ \sum_{i=0}^{\frac{n-1}{2}} F_{k,2i}^{(j)} &= \frac{1}{k}\left[F_{k,n}^{(j)} + ka_{k,j} - b_{k,j}\right], \text{ if } n \text{ is odd}, \end{aligned}$$

where $\{F_{k,n}^{(j)}\}_{n\geq 0}$ is the k-Fibonacci difference sequence.

Next, the sum of the terms with odd indexes of the k-Fibonacci difference sequence is given by:

Proposition 13. For all non-negative integers n, then

$$\begin{aligned} \sum_{i=0}^{\frac{n}{2}} F_{k,2i-1}^{(j)} &= \frac{1}{k}\left[F_{k,n}^{(j)} - a_{k,j}\right], \text{ if } n \text{ is even} \\ \sum_{i=0}^{\frac{n-1}{2}} F_{k,2i}^{(j)} &= \frac{1}{k}\left[F_{k,n+1}^{(j)} - a_{k,j}\right], \text{ if } n \text{ is odd}. \end{aligned}$$

A direct consequence of the previous results is the result presented below. This follows naturally from the relationships established and reinforces the conclusions drawn from the sentence.

Proposition 14. Let $\{F_{k,n}^{(j)}\}_{n\geq 0}$ be the k-Fibonacci difference sequence. For all non-negative integers m, we have the following formulas:

(a) $\sum_{i=0}^{m}(-1)^i F_{k,i}^{(j)} = \frac{1}{k}[F_{k,m}^{(j)} - F_{k,m+1}^{(j)} + (k-1)a_{k,j} - b_{k,j}]$; if m is odd,

and

(b) $\sum_{i=0}^{m}(-1)^i F_{k,j}^{(j)} = \frac{1}{k}[F_{k,m+1}^{(j)} - F_{k,m}^{(j)} + (k+1)a_{k,j} - b_{k,j}]$; if m is even.

To finalize, we derive an expression for the sum of the squares of the first $n+1$ terms of the k-Fibonacci difference sequence, relating it to the classical Fibonacci sequence.

Proposition 15. The sum of the squares of the first $n+1$ terms of the k-Fibonacci difference sequence is given by:

$$\sum_{i=0}^{n}(F_{k,i}^{(j)})^2 = \frac{1}{k}[F_{k,n}^{(j)} \cdot F_{k,n+1}^{(j)} + k(a_{k,j})^2 - a_{k,j}b_{k,j}]\,,$$

for all non-negative integers n.

Proof. To begin, observe that for $n \geq 2$, the following identity holds:

$$F_{k,n}^{(j)}F_{k,n+1}^{(j)} - F_{k,n-1}^{(j)}F_{k,n}^{(j)} = F_{k,n}^{(j)}(F_{k,n+1}^{(j)} - F_{k,n-1}^{(j)}) = k(F_{k,n}^{(j)})^2\,.$$

Thus, we find:

$$\begin{aligned}
k(F_{k,2}^{(j)})^2 &= F_{k,2}^{(j)}F_{k,3}^{(j)} - F_{k,1}^{(j)}F_{k,2}^{(j)},\\
k(F_{k,3}^{(j)})^2 &= F_{k,3}^{(j)}F_{k,4}^{(j)} - F_{k,2}^{(j)}F_{k,3}^{(j)},\\
&\cdots\\
k(F_{k,n-1}^{(j)})^2 &= F_{k,n-1}^{(j)}F_{k,n}^{(j)} - F_{k,n-2}^{(j)}F_{k,n-1},\\
k(F_{k,n}^{(j)})^2 &= F_{k,n}^{(j)}F_{k,n+1}^{(j)} - F_{k,n-1}^{(j)}F_{k,n}^{(j)}\,.
\end{aligned}$$

Adding both sides of these equations yields:

$$k(F_{k,2}^{(j)})^2 + k(F_{k,3}^{(j)})^2 + \cdots + k(F_{k,n-1}^{(j)})^2 + k(F_{k,n})^2 = F_{k,n}^{(j)}F_{k,n+1}^{(j)} - F_{k,1}^{(j)}F_{k,2}^{(j)}\,.$$

Since $F_{k,0}^{(j)} = a_{k,j}$, $F_{k,1}^{(j)} = b_{k,j}$ and $F_{k,2}^{(j)} = a_{k,j} + kb_{k,j}$, it follows

$$\begin{aligned}
k\sum_{j=0}^{n}(F_{k,j}^{(j)})^2 &= k(F_{k,0}^{(j)})^2 + k(F_{k,1}^{(j)})^2 + F_{k,n}^{(j)}F_{k,n+1}^{(j)} - F_{k,1}^{(j)}F_{k,2}^{(j)}\\
&= k(a_{k,j})^2 - a_{k,j}b_{k,j} + F_{k,n}^{(j)}F_{k,n+1}^{(j)}\,,
\end{aligned}$$

which completes the proof. □

6. CONCLUSION

The present work contributes to the list of known identities for the k-Fibonacci difference sequence with original results, including the classical Tagiuri–Vajda identity and other related results. One of our primary goals was to highlight that this sequence remains a particular case of the k-Fibonacci family, as previously noted by Falcon [6]. This observation has motivated us to explore several classical identities through multiple perspectives. Following a standard methodological framework, we also examined certain consequences derived from Binet's formula, as exhibited by Falcon [6]. In this context, we obtain a characterization for the k-Fibonacci difference in terms of k-Fibonacci and k-Lucas sequences by linear combination, which allows us to extract a wide variety of properties and identities for the k-Fibonacci difference numbers, as well as some expressions for the negative subscripts of the sequence. We explore some properties of a generalized k-Fibonacci sequence, that is, a k-Fibonacci sequence with arbitrary initial terms and apply the results to the k-Fibonacci difference sequence. Also, this allowed us to establish expressions for the limit of the quotient of successive terms and to evaluate some partial sums associated with the sequence. In future work, we intend to apply this approach to other k-difference sequences, for instance k-difference Mersenne, among others.

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(E. Costa) Federal University of Tocantins, Campus Arraias, 77330-000, Arraias, Tocantins, Brazil

Current address: Federal University of Tocantins, Campus Arraias, 77330-000, Arraias, Tocantins, Brazil

Email address, E. Costa: `eudes@uft.edu.br`

(author two) Federal University of Tocantins, Campus Palmas, 77020-021, Palmas, Tocantins, Brazil

Email address, É. Mesquita: `elisgardel@uft.edu.br`

(author three) University of Trás-os-Montes and Alto Douro, Department of Mathematics, 5000-801 Vila Real, Portugal

Email address, P. Catarino: `pcatarin@utad.pt`