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AN EFFICIENT FAMILY OF RATIO-CUM-PRODUCT ESTIMATORS FOR FINITE POPULATION MEAN IN SAMPLE SURVEYS

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Abstract: This paper considers the problem of estimating the finite population mean $\bar{Y}$ of the study variable using information on two auxiliary variables (x,z) . A family of ratio-cum-product estimators for population mean $\bar{Y}$ has been suggested. It has been shown that the usual unbiased estimator $\bar{y}$, ratio estimator, product estimator, dual to ratio estimator and dual to product estimator due to Srivenkatramana (1980) and Bandyopadhyaya (1980), Singh et al's (2005, 2011) estimator, Tailor et al's (2012) estimator, Vishwakarma et al's (2014) estimator and Vishwakarma and Kumar (2015) estimator are members of the suggested family of estimators. In addition to these estimators, various unknown estimators are shown to be the member of the suggested family of estimators. The bias and mean squared error of the proposed family are obtained under large sample approximation. Efficiency comparisons are made to demonstrate the performance of the suggested family over other existing estimators. An empirical study is carried out in support of the present study.

Key words: Auxiliary variables, Study variable, Ratio-cum-Product method of estimation, Bias, Mean Squared Error.
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1. Introduction

The use of auxiliary information has been widely discussed in the literature in order to improve the precision of estimators of population parameters. Out of many ratio, product and regression methods of estimation are good examples in this context. A large number of estimators for population mean using information on single auxiliary variable is available in the literature for instance see Singh, H.P. (1986) and Singh, S. (2003) and the references cited there in. Some times it may possible that information on the auxiliary variables are readily available. In such situations it is applicable to use information on two auxiliary variables at the estimation stage for estimating the population mean of the study variable, for instance, see Olkin (1958), Singh, M.P. (1967), Yasmeen et al. (2015), Vishwawkarma and Kumar (2015) among others. While estimating the population mean $\bar{Y}$ of the study character y , we can use the parameters such as coefficients of variation (C_x, C_z), coefficients of skewness $(\beta_1(x), \beta_1(z))$, coefficients of kurtosis $(\beta_2(x), \beta_2(z))$, standard deviations (S_x, S_z), population means $(\bar{X}, \bar{Z})$, associated with the auxiliary variables (x,z) respectively and the correlation coefficients ρ_{yx}(between auxiliary variable x and auxiliary variable z), ρ_{yz} (between study variable y and auxiliary variable z) and ρ_{xz} (between study variables x and auxiliary variable z),for instance see Upadhyaya and Singh (1999), Singh and Tailor (2003), Kadilar and Cingi (2004, 2006) etc.

Consider the finite population $U = (U_1, U_2, ..., U_N)$ of N units. Let y denote the study variable and (x,z) denote the auxiliary variables. Let $(\bar{Y}, \bar{X}, \bar{Z})$ be the population means of the study variable y and auxiliary variables (x,z) respectively. It is assumed that the population means $(\bar{X}, \bar{Z})$ of

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the auxiliary variables (x, z) respectively are known. It is desired to estimate the population mean $\bar{Y}$ based on the information available on two auxiliary variables (x, z). For estimating population mean $\bar{Y}$, a simple random sample (SRS) of size n is drawn without replacement (WOR) from the population U. Let $(\bar{y}, \bar{x}, \bar{z})$ be the sample means of the variables (y, x, z) respectively based on sample observations of size n.
When no auxiliary information is available, the usual unbiased estimator for the population mean $\bar{Y}$ is given by

$$T_1 = \bar{y} \tag{1.1}$$

When the population mean $\bar{X}$ of the auxiliary variable x is known, the classical ratio estimator for the population mean $\bar{Y}$ is defined by

$$T_2 = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right) \tag{1.2}$$

With known population mean $\bar{Z}$ of the auxiliary variable z, the classical product estimator for the population mean $\bar{Y}$ is given by

$$T_3 = \bar{y}\left(\frac{\bar{z}}{\bar{Z}}\right) \tag{1.3}$$

In the situation, where the study variable y is positively correlated with the auxiliary variable x and negatively correlated with the auxiliary variable z , Singh (1967) suggested a ratio-cum-product estimator for the population mean $\bar{Y}$ as

$$T_4 = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)\left(\frac{\bar{z}}{\bar{Z}}\right) \tag{1.4}$$

When the population correlation coefficient ρ_{xz} between the auxiliary variables x, z , is known , Singh and Tailor (2005) defined a ratio-cum-product estimator for the population mean $\bar{Y}$ as

$$T_5 = \bar{y}\left(\frac{\bar{X}+\rho_{xz}}{\bar{x}+\rho_{xz}}\right)\left(\frac{\bar{z}+\rho_{xz}}{\bar{Z}+\rho_{xz}}\right) \tag{1.5}$$

Using transformation $x_i^* = (1+g)\bar{X} - gx_i$ and $z_i^* = (1+g)\bar{Z} - gz_i,\ i = 1, 2, ...N;$ with $g = \frac{n}{(N-n)}$ Srivenkataramna (1980) and Bandyopadhyaya (1980) suggested duals to ratio and product estimators, respectively, for population mean $\bar{Y}$ as

$$T_6 = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right) \tag{1.6}$$

and

$$T_7 = \bar{y}\left(\frac{\bar{z}^*}{\bar{Z}}\right) \tag{1.7}$$

where $\bar{x}^* = \frac{1}{n}\sum_{i=1}^{n} x_i^* = (1+g)\bar{X} - g\bar{x}$, and $\bar{z}^* = \frac{1}{n}\sum_{i=1}^{n} z_i^* = (1+g)\bar{Z} - g\bar{z}$. Singh et al (2005) suggested a dual to ratio-cum-product estimator for the population mean $\bar{Y}$ as

$$T_8 = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)\left(\frac{\bar{Z}}{\bar{z}^*}\right) \tag{1.8}$$

Singh et al (2011) suggested a generalized version of the estimator T_8 for the population mean$\bar{Y}$ as

$$T_9 = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\delta_1}\left(\frac{\bar{Z}}{\bar{z}^*}\right)^{\delta_2} \tag{1.9}$$

where (δ_1, δ_2) are suitably chosen constants.
A dual to Singh and Tailor (2005) estimatorT_5 due to Tailor et al (2012) is given by

$$T_{10} = \bar{y}\left(\frac{\bar{x}^* + \rho_{xz}}{\bar{X} + \rho_{xz}}\right)\left(\frac{\bar{Z} + \rho_{xz}}{\bar{z}^* + \rho_{xz}}\right) \tag{1.10}$$

A generalized version of the estimator T_{10} due to Vishwakarma et al (2014) is given by

$$T_{11} = \bar{y}\left(\frac{\bar{x}^* + \rho_{xz}}{\bar{X} + \rho_{xz}}\right)^{\delta_1}\left(\frac{\bar{Z} + \rho_{xz}}{\bar{z}^* + \rho_{xz}}\right)^{\delta_2} \tag{1.11}$$

where (δ_1, δ_2) are suitably chosen constants.
Vishwakarma and Kumar (2015) suggested a family of dual to ratio-cum-product estimators for population mean$\bar{Y}$ as

$$T_{12} = \bar{y}\left(\frac{a\bar{x}^* + b}{a\bar{X} + b}\right)^{\delta_1}\left(\frac{a\bar{Z} + b}{a\bar{z}^* + b}\right)^{\delta_2} \tag{1.12}$$

where $a(\neq 0)$ and b are either real numbers or functions of some known parameters of auxiliary variates x and z such as the correlation coefficient, coefficient of variation etc. and (δ_1, δ_2) are suitably chosen constants.
In this paper we have suggested a class of estimators for finite population mean $\bar{Y}$ of the study variable y using information on two auxiliary variables(x, z). Expressions of bias and mean squared error(MSE) of the suggested class of estimators are obtained under large sample approximation. The minimum MSE of the suggested class of estimators is obtained. It has been shown that the proposed class of estimators is more efficient than the one recently proposed family of estimators due to Singh et al (2011), Tailor et al (2012), Vishwakarma et.al.(2014) and Vishwakarma and Kumar (2015). An empirical study is carried one in support of the present study.

2. Suggested Class of Estimators

Keeping the form of the estimators $T_j = (j = 1to12)$ and motivated by Searls (1964) and Upadhayaya et al (1985) we define a class of estimators for population mean$\bar{Y}$ as

$$T = \left[W_1\bar{y}\left(\frac{a\bar{X} + b}{a\bar{x} + b}\right)^{\alpha_1}\left(\frac{c\bar{z} + d}{c\bar{Z} + d}\right)^{\alpha_2} + W_2\bar{y}\left(\frac{a\bar{x}^* + b}{a\bar{X} + b}\right)^{\delta_1}\left(\frac{c\bar{Z} + d}{c\bar{z}^* + d}\right)^{\delta_2}\right] \tag{2.1}$$

where $(a \neq 0, b, c \neq 0, d)$ being real numbers and also may take the values of parameters associated with either study variable y or auxiliary variable x or both variables (x, y);$(\alpha_1, \alpha_2, \delta_1, \delta_2)$ are scalars which help in designing the estimators,(W_1, W_2) are suitably chosen scalars whose sum need not be unity and $\bar{x}^* = \{,(1+g)\bar{X} - g\bar{x}\}$, $\bar{z}^* = \{(1+g)\bar{Z} - g\bar{z}\}$ are unbiased estimators of population means $\bar{X}$ and $\bar{Z}$ respectively, $g = \frac{n}{(N-n)} = \frac{f}{(1-f)}$ and $f = \frac{n}{N}$.
We note that the class of estimators T reduces to a large number of known and unknown estimators of the population mean $\bar{Y}$ of the study variable y. Table 1 presents the set of known estimators of the population mean $\bar{Y}$. In Table 2 we have given some unknown members of the suggested class of estimators T.

TABLE 1. Some known members of the class of estimators T.

		Values of scalars									
S.No.	Estimator	W_1	W_2	α_1	α_2	δ_1	δ_2	a	b	c	d
1.	$T_1 = \bar{y}$	1	0	0	0	-	-	-	-	-	-
2.	$T_2 = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)$	1	0	1	0	-	-	1	0	-	-
3.	$T_3 = \bar{y}\left(\frac{\bar{z}}{\bar{Z}}\right)$	1	0	0	1	-	-	-	-	1	0
4.	$T_4 = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)\left(\frac{\bar{z}}{\bar{Z}}\right)$ $Singh(1967)$	1	0	1	1	-	-	1	0	1	0
5.	$T_5 = \bar{y}\left(\frac{\bar{X}+\rho_{xz}}{\bar{x}+\rho_{xz}}\right)\left(\frac{\bar{z}+\rho_{xz}}{\bar{Z}+\rho_{xz}}\right)$ $SinghandTailor(2005)$	1	0	1	1	-	-	1	ρ_{xz}	1	ρ_{xz}
6.	$T_6 = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)$ $Srivenkataramana(1980)$ $andBandyopadhyaya(1980)$	0	1	-	-	1	0	1	0	-	-
7.	$T_7 = \bar{y}\left(\frac{\bar{Z}}{\bar{z}^*}\right)$ $Srivenkataramana(1980)$ $andBandyopadhyaya(1980)$	0	1	-	-	0	1	-	-	1	0
8.	$T_8 = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)\left(\frac{\bar{Z}}{\bar{z}^*}\right)$ $Srivenkataramana(1980)$ $andBandyopadhyaya(1980)$	0	1	-	-	1	1	1	0	1	0
9.	$T_9 = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\delta_1}\left(\frac{\bar{Z}}{\bar{z}^*}\right)^{\delta_2}$ Srivenkataramana (1980)and Bandyopadhyaya (1980)	0	1	-	-	δ_1	δ_2	1	0	1	0
10.	$T_{10} = \bar{y}\left(\frac{\bar{x}^*+\rho_{xz}}{\bar{X}+\rho_{xz}}\right)\left(\frac{\bar{Z}+\rho_{xz}}{\bar{z}^*+\rho_{xz}}\right)$ $Tailoretal(2012)$	0	1	-	-	1	1	1	ρ_{xz}	1	ρ_{xz}
11.	$T_{11} = \bar{y}\left(\frac{\bar{x}^*+\rho_{xz}}{\bar{X}+\rho_{xz}}\right)^{\delta_1}\left(\frac{\bar{Z}+\rho_{xz}}{\bar{z}^*+\rho_{xz}}\right)^{\delta_2}$ $Vishwakarmaetal(2014)$	0	1	-	-	δ_1	δ_2	1	ρ_{xz}	1	ρ_{xz}
12.	$T_{12} = \bar{y}\left(\frac{a\bar{x}^*+b}{a\bar{X}+b}\right)^{\delta_1}\left(\frac{a\bar{Z}+b}{\bar{z}^*+b}\right)^{\delta_2}$ $Vishwakarmaetal(2014)$	0	1	-	-	δ_1	δ_2	a	b	a	b

TABLE 2. Some unknown members of the proposed class of estimators T.

	Values of scalars							
Estimator	α_1	α_2	δ_1	δ_2	a	b	c	d
$T_1^* = W\bar{y} with W = W_1 + W_2$	1	0	1	0	-	-	-	-
$T_2^* = W_1\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right) + W_2\bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)$	1	0	1	0	1	0	1	0
$T_3^* = W_1\bar{y}\left(\frac{\bar{z}}{\bar{Z}}\right) + W_2\bar{y}\left(\frac{\bar{Z}}{\bar{z}^*}\right)$	0	1	0	1	-	-	1	0
$T_4^* = W_1\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)\left(\frac{\bar{z}}{\bar{Z}}\right) + W_2\bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)\left(\frac{\bar{Z}}{\bar{z}^*}\right)$	1	1	1	1	1	0	1	0
$T_5^* = W_1\bar{y}\left(\frac{(N+1)\bar{X}}{\bar{x}+N\bar{X}}\right)\left(\frac{\bar{z}+N\bar{Z}}{(N+1)\bar{Z}}\right) + W_2\bar{y}\left(\frac{\bar{x}^*+N\bar{X}}{(N+1)\bar{X}}\right)\left(\frac{(N+1)\bar{Z}}{\bar{z}^*+N\bar{Z}}\right)$	1	1	1	1	1	$N\bar{X}$	1	$N\bar{X}$
$T_6^* = W_1\bar{y}\left(\frac{\bar{X}+C_x}{\bar{x}+C_x}\right)\left(\frac{\bar{z}+C_z}{\bar{Z}+C_z}\right) + W_2\bar{y}\left(\frac{\bar{x}^*+C_x}{\bar{X}+C_x}\right)\left(\frac{\bar{Z}+C_z}{\bar{z}^*+C_z}\right)$	1	1	1	1	1	C_x	1	C_x
$T_7^* = W_1\bar{y}\left(\frac{\bar{X}+\rho_{xz}}{\bar{x}+\rho_{xz}}\right)\left(\frac{\bar{z}+\rho_{xz}}{\bar{Z}+\rho_{xz}}\right) + W_2\bar{y}\left(\frac{\bar{x}^*+\rho_{xz}}{\bar{X}+\rho_{xz}}\right)\left(\frac{\bar{Z}+\rho_{xz}}{\bar{z}^*+\rho_{xz}}\right)$	1	1	1	1	1	ρ_{xz}	1	ρ_{xz}
$T_8^* = W_1\bar{y}\left(\frac{\bar{X}C_x+\rho_{xz}}{\bar{x}C_x+\rho_{xz}}\right)\left(\frac{\bar{z}C_z+\rho_{xz}}{\bar{Z}C_z+\rho_{xz}}\right) + W_2\bar{y}\left(\frac{\bar{x}^*C_x+\rho_{xz}}{\bar{X}C_x+\rho_{xz}}\right)\left(\frac{\bar{Z}C_z+\rho_{xz}}{\bar{z}^*C_z+\rho_{xz}}\right)$	1	1	1	1	C_x	ρ_{xz}	C_z	ρ_{xz}
$T_9^* = W_1\bar{y}\left(\frac{\bar{X}S_x+C_x}{\bar{x}S_x+C_x}\right)\left(\frac{\bar{z}S_x+C_z}{\bar{Z}S_x+C_z}\right) + W_2\bar{y}\left(\frac{\bar{x}^*S_x+C_x}{\bar{X}S_x+C_x}\right)\left(\frac{\bar{Z}C_z+S_x}{\bar{z}^*C_z+S_x}\right)$	1	1	1	1	S_x	C_x	S_z	C_z
$T_{10}^* = W_1\bar{y}\left(\frac{\bar{X}+S_x}{\bar{x}+S_x}\right)\left(\frac{\bar{z}+S_z}{\bar{Z}+S_z}\right) + W_2\bar{y}\left(\frac{\bar{x}^*+S_x}{\bar{X}+S_x}\right)\left(\frac{\bar{Z}+S_z}{\bar{z}^*+S_z}\right)$	1	1	1	1	1	S_x	1	S_z
$T_{11}^* = W_1\bar{y}\left(\frac{\bar{X}S_x+\rho_{xz}}{\bar{x}S_x+\rho_{xz}}\right)\left(\frac{\bar{z}S_z+\rho_{xz}}{\bar{Z}S_z+\rho_{xz}}\right) + W_2\bar{y}\left(\frac{\bar{x}^*S_x+\rho_{xz}}{\bar{X}S_x+\rho_{xz}}\right)\left(\frac{\bar{Z}S_z+\rho_{xz}}{\bar{z}^*S_z+\rho_{xz}}\right)$	1	1	1	1	s_x	ρ_{xz}	S_z	ρ_{xz}
$T_{12}^* = W_1\bar{y}\left(\frac{\bar{X}\rho_{xz}+S_x}{\bar{x}\rho_{xz}+S_x}\right)\left(\frac{\bar{z}\rho_{xz}+S_z}{\bar{Z}\rho_{xz}+S_z}\right) + W_2\bar{y}\left(\frac{\bar{x}^*\rho_{xz}+S_x}{\bar{X}\rho_{xz}+S_x}\right)\left(\frac{\bar{Z}\rho_{xz}+S_z}{\bar{z}^*\rho_{xz}+S_z}\right)$	1	1	1	1	ρ_{xz}	s_x	ρ_{xz}	S_z
$T_{13}^* = W_1\bar{y}\left(\frac{\bar{X}+C_x}{\bar{x}+C_x}\right)^{\alpha_{1opt}}\left(\frac{\bar{z}+C_z}{\bar{Z}+C_z}\right)^{\alpha_{2opt}} + W_2\bar{y}\left(\frac{\bar{x}^*+C_x}{\bar{X}+C_x}\right)^{\delta_{1opt}}\left(\frac{\bar{Z}+C_z}{\bar{z}^*+C_z}\right)^{\delta_{2opt}}$	α_{1opt}	α_{2opt}	δ_{1opt}	δ_{2opt}	1	C_x	1	C_z

To obtain the bias andMSE of the class of estimators T, we write

$$\bar{y} = \bar{Y}(1+e_0), \bar{x} = \bar{X}(1+e_1), \bar{z} = \bar{Z}(1+e_2)$$

such that

$$E(e_0) = E(e_1) = E(e_2) = 0$$

and

$$E(e_0^2) = \frac{(1-f)}{n}C_y^2 \qquad E(e_1^2) = \frac{(1-f)}{n}C_x^2$$

$$E(e_2^2) = \frac{(1-f)}{n}C_z^2 \qquad E(e_0e_1) = \frac{(1-f)}{n}\rho_{yx}C_yC_x$$

$$E(e_0e_2) = \frac{(1-f)}{n}\rho_{yz}C_yC_z \qquad E(e_1e_2) = \frac{(1-f)}{n}\rho_{xz}C_xC_z$$

Expressing T at 2.1 in terms of $e's$ we have

$$T=[W_1\bar{Y}(1+e_0)(1+\tau_1e_1)^{-\alpha_1}(1+\tau_2e_2)^{-\alpha_2}+W_2\bar{y}(1+e_0)(1+g\tau_1e_1)^{\delta_1}(1-g\tau_2e_2)^{-\delta_2}] \quad (2.2)$$

where $\tau_1=\frac{a\bar{X}}{a\bar{X}+b}$ and $\tau_2=\frac{c\bar{Z}}{c\bar{Z}+d}$.

We assume that$|\tau_ie_i|<1$ and $|g\tau_ie_i|<1$ so that $(1+\tau_1e_1)^{-\alpha_1}$, $(1+\tau_2e_2)^{\alpha_2}$, $(1-g\tau_1e_1)^{\delta_1}$ and $(1-g\tau_2e_2)^{-\delta_2}$ are expandable. Now expanding the right side of (2.2) we have

$$\begin{aligned}T=\bar{Y}(1+e_0)\Bigg[&W_1\{1-\alpha_1\tau_1e_1+\frac{\alpha_1(\alpha_1+1)}{2}\tau_1^2e_1^2-...\}\{1+\alpha_2\tau_2e_2+\frac{\alpha_2(\alpha_2-1)}{2}\tau_2^2e_2^2-...\}\\ &+W_2\{1-\delta_1g\tau_1e_1+\frac{\delta_1(\delta_1-1)}{2}g^2\tau_1^2e_1^2-...\}\{1+\delta_2g\tau_2e_2+\frac{\delta_2(\delta_2+1)}{2}g^2\tau_2^2e_2^2+...\}\Bigg]\end{aligned} \quad (2.3)$$

Multiplying out terms of right hand side of (2.3) and neglecting terms of $e's$ having power greater than two we have

$$\begin{aligned}T\cong\bar{Y}\Bigg[&W_1\{1+e_0-\alpha_1\tau_1e_1+\alpha_2\tau_2e_2-\alpha_1\tau_1e_0e_1+\alpha_2\tau_2e_0e_2-\alpha_1\alpha_2\tau_1\tau_2e_1e_2+\\ &\frac{\alpha_1(\alpha_1+1)}{2}\tau_1^2e_1^2+\frac{\alpha_2(\alpha_2-1)}{2}\tau_2^2e_2^2\}\\ &+W_2\{1+e_0-\delta_1g\tau_1e_1+\delta_2g\tau_2e_2-\delta_1g\tau_1e_0e_1+\delta_2g\tau_2e_0e_2-\delta_1\delta_2g^2\tau_1\tau_2e_1e_2+\\ &\frac{\delta_1(\delta_1-1)}{2}g^2\tau_1^2e_1^2+\frac{\delta_2(\delta_2+1)}{2}g^2\tau_2^2e_2^2\}]\end{aligned}$$

or

$$\begin{aligned}(T-\bar{Y})=\bar{Y}\Bigg[&W_1\{1+e_0-\alpha_1\tau_1e_1+\alpha_2\tau_2e_2-\alpha_1\tau_1e_0e_1+\alpha_2\tau_2e_0e_2-\alpha_1\alpha_2\tau_1\tau_2e_1e_2+\\ &\frac{\alpha_1(\alpha_1+1)}{2}\tau_1^2e_1^2+\frac{\alpha_2(\alpha_2-1)}{2}\tau_2^2e_2^2\}\\ &+W_2\{1+e_0-\delta_1g\tau_1e_1+\delta_2g\tau_2e_2-\delta_1g\tau_1e_0e_1+\delta_2g\tau_2e_0e_2-\delta_1\delta_2g^2\tau_1\tau_2e_1e_2+\\ &\frac{\delta_1(\delta_1-1)}{2}g^2\tau_1^2e_1^2+\frac{\delta_2(\delta_2+1)}{2}g^2\tau_2^2e_2^2\}\Bigg]\end{aligned} \quad (2.4)$$

Taking expectation of both sides of (2.4) we get the bias of the estimator T to the first degree of approximation as

$$\begin{aligned}B(T)=\bar{Y}\Bigg[&W_1\Bigg\{1+\frac{(1-f)}{n}\Bigg[\frac{\alpha_1(\alpha_1+1)}{2}\tau_1^2C_1^2+\frac{\alpha_2(\alpha_2-1)}{2}\tau_2^2C_2^2\\ &-\alpha_1\tau_1\rho_{01}C_0C_1+\alpha_2\tau_2\rho_{02}C_0C_2-\alpha_1\alpha_2\tau_1\tau_2\rho_{12}C_1C_2\Bigg]\Bigg\}\\ &+W_2\Bigg\{1+\frac{(1-f)}{n}\Bigg[\frac{\delta_1(\delta_1-1)}{2}g^2\tau_1^2C_1^2+\frac{\delta_2(\delta_2+1)}{2}g^2\tau_2^2C_2^2\\ &-g\delta_1\tau_1\rho_{01}C_0C_1+g\delta_2\tau_2\rho_{02}C_0C_2-\delta_1\delta_2g^2\tau_1\tau_2\rho_{12}C_1C_2\Bigg]\Bigg\}-1\Bigg]\end{aligned} \quad (2.5)$$

Squaring both sides of (2.4) and neglecting terms of $e's$ having power greater than two we have

$$\begin{aligned}(T-\bar{Y})^2 = \bar{Y}^2\Bigg[1 + W_1^2\Bigg\{&1+2e_0-2\alpha_1\tau_1e_1+2\alpha_2\tau_2e_2+e_0^2+\alpha_1(2\alpha_1+1)\tau_1^2e_1^2\\ &+\alpha_2(2\alpha_2-1)\tau_2^2e_2^2-4\alpha_1\tau_1e_0e_1+4\alpha_2\tau_2e_0e_2-4\alpha_1\alpha_2\tau_1\tau_2e_1e_2\Bigg\}\\ &+W_2^2\Bigg\{1+2e_0-2g\delta_1\tau_1e_1+2g\delta_2\tau_2e_2+e_0^2+g^2\delta_1(2\delta_1-1)\tau_1^2e_1^2\\ &+g^2\delta_2(2\delta_2+1)\tau_2^2e_2^2-4g\delta_1\tau_1e_0e_1+4g\delta_2\tau_2e_0e_2-4g^2\delta_1\delta_2\tau_1\tau_2e_1e_2\Bigg\}\\ &+2W_1W_2\Bigg\{1+2e_0-(\alpha_1+g\delta_1)\tau_1e_1+(\alpha_2+g\delta_2)\tau_2e_2-2(\alpha_1+g\delta_1)\tau_1e_0e_1\\ &+2(\alpha_2+g\delta_2)\tau_2e_0e_2-\tau_1\tau_2(\alpha_1+g\delta_1)(\alpha_2+g\tau_2)e_1e_2+e_0^2\\ &+\big((\alpha_1+g\delta_1)^2+(\alpha_1-g^2\delta_1)\big)\frac{\tau_1^2e_1^2}{2}+\big((\alpha_2+g\delta_2)^2+(g^2\delta_2-\alpha_2)\big)\frac{\tau_2^2e_2^2}{2}\Bigg\}\\ &-2W_1\Bigg\{1+e_0-\alpha_1\tau_1e_1+\alpha_2\tau_2e_2-\alpha_1\tau_1e_0e_1+\alpha_2\tau_2e_0e_2\\ &-\alpha_1\alpha_2\tau_1\tau_2e_1e_2+\frac{\alpha_1(\alpha_1+1)}{2}\tau_1^2e_1^2+\frac{\alpha_2(\alpha_2-1)}{2}\tau_2^2e_2^2\Bigg\}\\ &-2W_2\Bigg\{1+e_0-g\delta_1\tau_1e_1+g\delta_2\tau_2e_2-g\delta_1\tau_1e_0e_1+g\delta_2\tau_2e_0e_2\\ &-\delta_1\delta_2g^2\tau_1\tau_2e_1e_2+\frac{\delta_1(\delta_1-1)}{2}g^2\tau_1^2e_1^2+\frac{\delta_2(\delta_2+1)}{2}g^2\tau_2^2e_2^2\Bigg\}\Bigg]\end{aligned} \tag{2.6}$$

Taking expectation of both sides of (2.6) we get the mean squared error (MSE) of the proposed class of estimators T to the first degree of approximation as

$$MSE(T) = \bar{Y}^2[1+W_1^2A_1+W_2^2A_2+2W_1W_2A_3-2W_1A_4-2W_2A_5], \tag{2.7}$$

where

$$\begin{aligned}A_1 = \Bigg[1+\frac{1-f}{n}\Bigg\{&C_0^2+\alpha_1(2\alpha_1+1)\tau_1^2C_1^2+\alpha_2(2\alpha_2-1)\tau_2^2C_2^2\\ &-4\alpha_1\tau_1\rho_{01}C_0C_1+4\alpha_2\tau_2\rho_{02}C_0C_2-4\alpha_1\alpha_2\tau_1\tau_2\rho_{12}C_1C_2\Bigg\}\Bigg],\end{aligned} \tag{2.8}$$

$$\begin{aligned}A_2 = \Bigg[1+\frac{1-f}{n}\Bigg\{&C_0^2+g^2\delta_1(2\delta_1-1)\tau_1^2C_1^2+g^2\delta_2(2\delta_2+1)\tau_2^2C_2^2\\ &-4g\delta_1\tau_1\rho_{01}C_0C_1+4g\delta_2\tau_2\rho_{02}C_0C_2-4g^2\delta_1\delta_2\tau_1\tau_2\rho_{12}C_1C_2\Bigg\}\Bigg],\end{aligned} \tag{2.9}$$

$$\begin{aligned}A_3 = \Bigg[1+\frac{1-f}{n}\Bigg\{&C_0^2+\big((\alpha_1+g\delta_1)^2+(\alpha_1-g^2\delta_1)\big)\frac{\tau_1^2C_1^2}{2}\\ &+\big((\alpha_2+g\delta_2)^2+(g^2\delta_2-\alpha_2)\big)\frac{\tau_2^2C_2^2}{2}-2(\alpha_1+g\delta_1)\tau_1\rho_{01}C_0C_1\\ &+2(\alpha_2+g\delta_2)\tau_2\rho_{02}C_0C_2-\tau_1\tau_2(\alpha_1+g\delta_1)(\alpha_2+g\delta_2)\rho_{12}C_1C_2\Bigg\}\Bigg],\end{aligned} \tag{2.10}$$

$$\begin{aligned}A_4 = \Bigg[1+\frac{1-f}{n}\{&\frac{\alpha_1(\alpha_1+1)}{2}\tau_1^2C_1^2+\frac{\alpha_2(\alpha_2-1)}{2}\tau_2^2C_2^2-\alpha_1\alpha_2\tau_1\tau_2\rho_{12}C_1C_2\\ &-\alpha_1\tau_1\rho_{01}C_0C_1+\alpha_2\tau_2\rho_{02}C_0C_2\Bigg\}\Bigg],\end{aligned} \tag{2.11}$$

$$A_5 = \left[1 + \frac{1-f}{n}\left\{\frac{\delta_1(\delta_1-1)}{2}g^2\tau_1^2C_1^2 - \delta_1\delta_2 g^2\tau_1\tau_2\rho_{12}C_1C_2 + \frac{\delta_2(\delta_2+1)}{2}g^2\tau_2^2C_2^2 - g\delta_1\tau_1\rho_{01}C_0C_1 + g\delta_2\tau_2\rho_{02}C_0C_2\right\}\right], \quad (2.12)$$

The MSE of T at (2.7) is minimized for

$$\left.\begin{aligned} W_1 &= \frac{(A_2A_4 - A_3A_5)}{(A_1A_2 - A_3^2)} = W_{10}\ (say) \\ W_2 &= \frac{(A_1A_5 - A_3A_4)}{(A_1A_2 - A_3^2)} = W_{20}\ (say) \end{aligned}\right\} \quad (2.13)$$

Substitution of (2.13) in (2.7) yields the resulting minimum MSE of T as

$$MSE_{min}(T) = \bar{Y}^2\left[1 - \frac{(A_2A_4^2 - 2A_3A_4A_5 + A_1A_5^2)}{(A_1A_2 - A_3^2)}\right] \quad (2.14)$$

Thus we established the following theorem.

THEOREM 1. *To the first degree of approximation,*

$$MSE(T) \geq \bar{Y}^2\left[1 - \frac{(A_2A_4^2 - 2A_3A_4A_5 + A_1A_5^2)}{(A_1A_2 - A_3^2)}\right]$$

with equality holding

$$\left.\begin{aligned} W_1 &= W_{10} \\ W_2 &= W_{20} \end{aligned}\right\},$$

where W_{10} and W_{20} are given by (2.13).

3. Some Special Cases

Case I : If we set $W_2 = 0$ in (2.1) then the class of estimators T reduces to a class of estimators for $\bar{Y}$ as

$$T_{(1)} = W_1\bar{y}\left(\frac{a\bar{X}+b}{a\bar{x}+b}\right)^{\alpha_1}\left(\frac{C\bar{z}+d}{C\bar{Z}+d}\right)^{\alpha_2} \quad (3.1)$$

Putting $W_2 = 0$ in (2.5) and (2.7) we get the bias and MSE of the class of estimators $T_{(1)}$ to the first degree of approximation respectively as

$$B(T_{(1)}) = \bar{Y}[W_1A_4 - 1] = -\bar{Y}(1 - W_1A_4) \quad (3.2)$$

$$MSE(T_{(1)}) = \bar{Y}^2[1 + W_1^2A_1 - 2W_1A_4] \quad (3.3)$$

where (A_1, A_4) are respectively defined in ((2.8), (2.11)).
The$MSE(T_{(1)})$ at (3.3) is minimized for

$$W_1 = \frac{A_4}{A_1} = W_{10}^{(1)} \quad (say) \quad (3.4)$$

Thus the resulting bias and minimum MSE of $T_{(1)}$ are respectively given by

$$B_0(T_{(1)}) = -\bar{Y}[1 - W_{10}^{(1)}A_4] = -\bar{Y}\left(1 - \frac{A_4^2}{A_1}\right) \quad (3.5)$$

$$MSE_{min}(T_{(1)}) = \bar{Y}^2\left(1 - \frac{A_4^2}{A_1}\right) \tag{3.6}$$

From (3.5) and (3.6) we note that

$$ARB_0(T_{(1)}) = \left|\frac{B_0(T_{(1)})}{\bar{Y}}\right| = \left(1 - \frac{A_4^2}{A_1}\right) \tag{3.7}$$

$$RMSE_{min.}(T_{(1)}) = \frac{MSE_{min}(T_{(1)})}{\bar{Y}^2} = \left(1 - \frac{A_4^2}{A1}\right) \tag{3.8}$$

where $ARB_0(T_{(1)})$ and $RMSE_{min}(T_{(1)})$ stand for absolute relative resulting bias of $T_{(1)}$ and relative minimumMSE of $T_{(1)}$.
It follows from (3.7) and (3.8) that

$$ARB_0(T_{(1)}) = RMSE_{min}(T_{(1)}) \tag{3.9}$$

Now, we state the following theorem.

THEOREM 2. *To the first degree of approximation,*

$$MSE(T_{(1)}) \geq \bar{Y}^2\left(1 - \frac{A_4^2}{A_1}\right)$$

with equality holding if

$$W_1 = W_{10}^{(1)}.$$

If we set $(W_1, W_2) = (0, 1)$ in (2.1) we get the estimator for $\bar{Y}$ as

$$T_{(1)}^{(1)} = \bar{y}\left(\frac{a\bar{X} + b}{a\bar{x} + b}\right)^{\alpha_1}\left(\frac{C\bar{z} + d}{C\bar{Z} + d}\right)^{\alpha_2} \tag{3.10}$$

Putting $(W_1, W_2) = (0, 1)$ in (2.5) and (2.7) we get the bias and MSE of the estimators $T_{(1)}^{(1)}$ to the first degree of approximation respectively as

$$B(T_{(1)}^{(1)}) = \bar{Y}(A_4 - 1) \tag{3.11}$$

$$MSE(T_{(1)}^{(1)}) = \bar{Y}^2(1 + A_1 - 2A_4) \tag{3.12}$$

from (3.3) and (3.12) we have

$$MSE(T_{(1)}^{(1)}) - MSE_{min.}(T_{(1)}) = \frac{\bar{Y}^2(A_1 - A_4)^2}{A_1} \geq 0 \tag{3.13}$$

It follows that $T_{(1)}$ - class of estimators is more efficient than$(T_{(1)}^{(1)})$ class of estimators. We note that the estimators $T_1 = \bar{y}, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8$ and T_9 are members of the suggested class of estimators $(T_{(1)}^{(1)})$ and the class of estimators $(T_{(1)}^{(1)})$ is the member of class of estimator $T_{(1)}$. Thus the proposed class of estimator $T_{(1)}$ is more efficient than the estimators $T_{(1)} = \bar{y}$ to T_9 (listed in Table 1) and the class of estimators $T_{(1)}$.
Some unknown members of the class of estimators $T_{(1)}$ are given in Table 3.

TABLE 3. Some Unknown members of the class of estimators T_1.

	Values of scalars					
Estimator	α_1	α_2	a	b	c	d
$T_{(1)1} = W\bar{y} with W = W_1$	0	0	-	-	-	-
$T_{(1)2} = W_1\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)$	1	0	1	0	-	-
$T_{(1)3} = W_1\bar{y}\left(\frac{\bar{z}}{\bar{Z}}\right)$	0	1	-	-	1	0
$T_{(1)4} = W_1\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)\left(\frac{\bar{z}}{\bar{Z}}\right)$	1	1	1	0	1	0
$T_{(1)5} = W_1\bar{y}\left(\frac{(N+1)\bar{X}}{\bar{x}+N\bar{X}}\right)\left(\frac{\bar{z}+N\bar{Z}}{(N+1)\bar{Z}}\right)$	1	1	1	$N\bar{X}$	1	$N\bar{Z}$
$T_{(1)6} = W_1\bar{y}\left(\frac{\bar{X}+C_x}{\bar{x}+C_x}\right)\left(\frac{\bar{z}+C_z}{\bar{Z}+C_z}\right)$	1	1	1	C_x	1	C_x
$T_{(1)7} = W_1\bar{y}\left(\frac{\bar{X}+\rho_{xz}}{\bar{x}+\rho_{xz}}\right)\left(\frac{\bar{z}+\rho_{xz}}{\bar{Z}+\rho_{xz}}\right)$	1	1	1	ρ_{xz}	1	ρ_{xz}
$T_{(1)8} = W_1\bar{y}\left(\frac{\bar{X}C_x+\rho_{xz}}{\bar{x}C_x+\rho_{xz}}\right)\left(\frac{\bar{z}C_z+\rho_{xz}}{\bar{Z}C_z+\rho_{xz}}\right)$	1	1	C_x	ρ_{xz}	C_z	ρ_{xz}
$T_{(1)9} = W_1\bar{y}\left(\frac{\bar{X}S_x+C_x}{\bar{x}S_x+C_x}\right)\left(\frac{\bar{z}S_x+C_z}{\bar{Z}S_x+C_z}\right)$	1	1	S_x	C_x	S_z	C_z
$T_{(1)10} = W_1\bar{y}\left(\frac{\bar{X}+S_x}{\bar{x}+S_x}\right)\left(\frac{\bar{z}+S_z}{\bar{Z}+S_z}\right)$	1	1	1	S_x	1	S_z
$T_{(1)11} = W_1\bar{y}\left(\frac{\bar{X}S_x+\rho_{xz}}{\bar{x}S_x+\rho_{xz}}\right)\left(\frac{\bar{z}S_z+\rho_{xz}}{\bar{Z}S_z+\rho_{xz}}\right)$	1	1	s_x	ρ_{xz}	S_z	ρ_{xz}
$T_{(1)12} = W_1\bar{y}\left(\frac{\bar{X}\rho_{xz}+S_x}{\bar{x}\rho_{xz}+S_x}\right)\left(\frac{\bar{z}\rho_{xz}+S_z}{\bar{Z}\rho_{xz}+S_z}\right)$	1	1	ρ_{xz}	s_x	ρ_{xz}	S_z
$T_{(1)13} = W_1\bar{y}\left(\frac{\bar{X}+C_x}{\bar{x}+C_x}\right)^{\alpha_{1opt}}\left(\frac{\bar{z}+C_z}{\bar{Z}+C_z}\right)^{\alpha_{2opt}}$	α_{1opt}	α_{2opt}	1	C_x	1	C_z

Case II : Inserting $W_1 = 0$ in (2.1) we get another class of estimators for the population mean $\bar{Y}$ as

$$T_{(2)} = W_2\bar{y}\left(\frac{a\bar{x}^* + b}{a\bar{X} + b}\right)^{\delta_1}\left(\frac{C\bar{Z} + d}{C\bar{z}^* + d}\right)^{\delta_2} \tag{3.14}$$

Putting $W_1 = 0$ in (2.5) and (2.7) we get the bias and MSE of the class of estimators $T_{(2)}$ to the first degree of approximation, respectively, as

$$B(T_{(2)}) = \bar{Y}[W_2A_5 - 1] = -\bar{Y}(1 - W_2A_5) \tag{3.15}$$

$$MSE(T_{(2)}) = \bar{Y}^2[1 + W_2^2A_2 - 2W_2A_5] \tag{3.16}$$

The $MSE(T_{(2)})$ at (3.16) is minimized for

$$W_2 = \frac{A_5}{A_2} = W_{20}^{(1)} \quad (say) \tag{3.17}$$

Thus the resulting bias and minimum MSE of $T_{(2)}$ are respectively given by

$$B_0(T_{(2)}) = -\bar{Y}[1 - W_{20}^{(1)}A_5] = -\bar{Y}\left(1 - \frac{A_5^2}{A_2}\right) \tag{3.18}$$

$$MSE_{min.}(T_{(2)}) = \bar{Y}^2\left(1 - \frac{A_5^2}{A_2}\right) \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\left|\frac{B_0(T_{(2)})}{\bar{Y}}\right| = RMSE_{min.}(T_{(2)}) = \left(1 - \frac{A_5^2}{A_2}\right) \tag{3.20}$$

Now, we state the following theorem.

Theorem 3. *To the first degree of approximation,*

$$MSE_{min.}(T_{(2)}) \geq \bar{Y}^2\left(1-\frac{A_5^2}{A_2}\right)$$

with equality holding if

$$W_2 = W_{20}.$$

Putting $(W_1, W_2) = (0,1)$ in (3.1) or $W_2 = 1$ in (3.14) we have the estimator for population mean $\bar{Y}$ as

$$T_{(2)}^{(1)} = \bar{y}\left(\frac{a\bar{x}^* + b}{a\bar{X} + b}\right)^{\delta_1}\left(\frac{C\bar{Z} + d}{C\bar{z}^* + d}\right)^{\delta_2} \tag{3.21}$$

Putting $W_2 = 1$ in (3.15) and (3.16) we have the bias and MSE of $T_{(2)}^{(1)}$ respectively to the first degree of approximation,

$$B(T_{(2)}^{(1)}) = \bar{Y}(A_5 - 1) \tag{3.22}$$

$$MSE(T_{(2)}^{(1)}) = \bar{Y}^2(1 + A_2 - 2A_5) \tag{3.23}$$

from (3.19) and (3.23) we have

$$MSE(T_{(2)}^{(1)}) - MSE_{min.}(T_{(2)}) = \frac{\bar{Y}^2(A_2 - A_5)^2}{A_2} \geq 0 \tag{3.24}$$

It follows from (3.24) that the $T_{(2)}$ class of estimators is better than $T_{(2)}^{(1)}$ class of estimators. We note that the estimators $T_1 = \bar{y}$ and T_6 to T_{12} (listed in Table 1) are members of the $T_{(2)}^{(1)}$ class of estimators and also $T_{(2)}^{(1)}$ is the member of $T_{(2)}$ is also more efficient than the estimators $T_1 = \bar{y}$ and T_6 to T_{12} (listed in Table 1).
From (2.14), (3.8) and (3.19) we have

$$[MSE_{min.}(T_{(1)}) - MSE_{min.}(T)] = \frac{\bar{Y}^2(A_1A_5 - A_3A_4)^2}{A_1(A_1A_2 - A_3^2)} \geq 0, \tag{3.25}$$

$$[MSE_{min.}(T_{(2)}) - MSE_{min.}(T)] = \frac{\bar{Y}^2(A_2A_4 - A_3A_5)^2}{(A_1A_2 - A_3^2)} \geq 0, \tag{3.26}$$

It follows from (3.25) and (3.26) that the proposed class of estimators T is more efficient than the classes of estimators $T_{(1)}$ and $T_{(2)}$ and hence the classes of estimators $T_1^{(1)}$ and $T_2^{(1)}$. Thus the proposed class of estimator T is better than the estimators $T_1 = \bar{y}$ to T_{12} (listed in Table 5).
Some unknown members of the suggested class of estimators $T_{(2)}$ are shown in Table 4.

Table 4. Some unknown members of the proposed class of estimators T.

Estimator	Values of scalars					
	δ_1	δ_2	a	b	c	d
$T_{(2)1}=W_2\bar{y}$	0	0	-	-	-	-
$T_{(2)2}=W_2\bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)$	1	0	-	-	1	0
$T_{(2)3}=W_2\bar{y}\left(\frac{\bar{Z}}{\bar{z}^*}\right)$	0	1	-	-	1	0
$T_{(2)4}=W_2\bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)\left(\frac{\bar{Z}}{\bar{z}^*}\right)$	1	1	1	0	1	0
$T_{(2)5}=W_2\bar{y}\left(\frac{\bar{x}^*+N\bar{X}}{(N+1)\bar{X}}\right)\left(\frac{(N+1)\bar{Z}}{\bar{z}^*+N\bar{Z}}\right)$	1	1	1	$N\bar{X}$	1	$N\bar{Z}$
$T_{(2)6}=W_2\bar{y}\left(\frac{\bar{x}^*+C_x}{\bar{X}+C_x}\right)\left(\frac{\bar{Z}+C_z}{\bar{z}^*+C_z}\right)$	1	1	1	C_x	1	C_3
$T_{(2)7}=W_2\bar{y}\left(\frac{\bar{x}^*+\rho_{xz}}{\bar{X}+\rho_{xz}}\right)\left(\frac{\bar{Z}+\rho_{xz}}{\bar{z}^*+\rho_{xz}}\right)$	1	1	1	ρ_{xz}	1	ρ_{xz}
$T_{(2)8}=W_2\bar{y}\left(\frac{\bar{x}^*C_x+\rho_{xz}}{\bar{X}C_x+\rho_{xz}}\right)\left(\frac{\bar{Z}C_z+\rho_{xz}}{\bar{z}^*C_z+\rho_{xz}}\right)$	1	1	C_x	ρ_{xz}	C_z	ρ_{xz}
$T_{(2)9}=W_2\bar{y}\left(\frac{\bar{x}^*S_x+C_x}{\bar{X}S_x+C_x}\right)\left(\frac{\bar{Z}S_z+C_z}{\bar{z}^*S_z+C_z}\right)$	1	1	S_x	C_x	S_z	C_z
$T_{(2)10}=W_2\bar{y}\left(\frac{\bar{x}^*+S_x}{\bar{X}+S_x}\right)\left(\frac{\bar{Z}+S_z}{\bar{z}^*+S_z}\right)$	1	1	1	S_x	1	S_z
$T_{(2)11}=W_2\bar{y}\left(\frac{\bar{x}^*S_x+\rho_{xz}}{\bar{X}S_x+\rho_{xz}}\right)\left(\frac{\bar{Z}S_z+\rho_{xz}}{\bar{z}^*S_z+\rho_{xz}}\right)$	1	1	S_x	ρ_{xz}	S_z	ρ_{xz}
$T_{(2)12}=W_2\bar{y}\left(\frac{\bar{x}^*\rho_{xz}+S_x}{\bar{X}\rho_{xz}+S_x}\right)\left(\frac{\bar{Z}\rho_{xz}+S_z}{\bar{z}^*\rho_{xz}+S_z}\right)$	1	1	ρ_{xz}	s_x	ρ_{xz}	S_z
$T_{(2)13}=W_2\bar{y}\left(\frac{\bar{x}^*+C_x}{\bar{X}+C_x}\right)^{\delta_{1opt}}\left(\frac{\bar{Z}+C_z}{\bar{z}^*+C_z}\right)^{\delta_{2opt}}$	δ_{1opt}	δ_{2opt}	1	C_x	1	C_z

4. Empirical Study

To see the performance of the members of the suggested class of estimators over other existing estimators, we have considered three natural population data sets earlier used by Vishwakarma and Kumar (2015). The description of the population and the values of the required parameters are given below.

Population I :[Source: Steel and Torrie (1960)]
Y: Log of leaf burn in sec,
X: Potassium percentage,
Z: Chlorine Percentage,
$N=30,\ n=6,\ \bar{Y}=0.6860,\ \bar{X}=4.6537,\ \bar{Z}=0.8077, \rho_{yx}=0.1794,$
$\rho_{yz}=-0.4996,\ \rho_{xz}=0.4074,\ C_y^2=0.4803,\ C_x^2=0.2295,\ C_z^2=0.7493$.

Population II :[Source: Singh (1969)]
Y: Number of females employed,
X: Number of females in service,
Z: Number of educated females,
$N=61,\ n=20,\ \bar{Y}=7.46,\ \bar{X}=5.31,\ \bar{Z}=179.00, \rho_{yx}=0.7737,$
$\rho_{yz}=-0.2070,\ \rho_{xz}=-0.0033,\ C_y^2=0.5046,\ C_x^2=0.5737,\ C_z^2=0.0633$.

Population IIII :[Source: Jhonston (1972)]
Y: Percentage of high affected by disease,
X: Mean January temperature,
Z: Date of flowering of a particular summer species(number of days from January 1),
$N=10,\ n=4,\ \bar{Y}=52,\ \bar{X}=42,\ \bar{Z}=200, \rho_{yx}=0.80,$
$\rho_{yz}=-0.94,\ \rho_{xz}=-0.73,\ C_y^2=0.0244,\ C_x^2=0.0170,\ C_z^2=0.0021$.

We have computed the percent relative efficiency of the members of the suggested class of estimators T and the existing estimators with respect to usual unbiased estimator by using following formulae:

$$PRE(T_j,\bar{y}) = \frac{MSE(\bar{y})}{MSE(T_j)} \times 100$$

$j = 1,2,3,4,5,6,7,8,10.$

$$= \frac{\{(1-f)/n\}C_y^2}{[1+W_1^2A_1+W_2^2A_2+2W_1W_2A_3-2W_1A_4-2W_2A_5]} \times 100 \quad (4.1)$$

$$PRE(T_l,\bar{y}) = (1-\rho_{yxz}^2)^{-1} \times 100 \quad (4.2)$$

$l = 9,11,12$

$$PRE(T_j^*,\bar{y}) = \frac{MSE(\bar{y})}{MSE(T_j)} \times 100 \quad j = 1to13,$$

$$= \frac{\{(1-f)/n\}C_y^2}{\left[1-\frac{(A_2A_4^2-2A_3A_4A_5^2)}{(A_1A_2-A_3^2)}\right]} \times 100, \quad (4.3)$$

$$PRE(T_{(1)j},\bar{y}) = \frac{MSE(\bar{y})}{MSE(T_{(1)j})} \times 100, \quad j = 1to13,$$

$$= \frac{\{(1-f)/n\}C_y^2}{\left[1-\frac{A_4^2}{A_1}\right]} \times 100, \quad (4.4)$$

$$PRE(T_{(2)j},\bar{y}) = \frac{MSE(\bar{y})}{MSE_{min}(T_{(2)j})} \times 100, \quad j = 1to13,$$

$$= \frac{\{(1-f)/n\}C_y^2}{\left[1-\frac{A_5^2}{A_2}\right]} \times 100, \quad (4.5)$$

Findings are displayed in Tables 5, 6, 7 and 8.

Tables 5, 6, 7 and 8 show that the estimators $T_{(1)13}$ (in Table 6), $T_{(2)13}$ (in Table 7), T^* (in Table 8) have $PREs$ larger than the estimators $T_{(9)}$, $T_{(11)}$ and $T_{(12)}$ (in Table 5) proposed by Singh et al (2011), Vishwakarma et al (2014) and Vishwakarma and Kumar (2015) respectively. We note from Table 5 that the estimators T_9, T_{11} and T_{12} have the same and largestPRE (i.e. $PRE(T,\bar{y}) =$ 174.04 (in population I), $PRE(T_j,\bar{y}) = 278.09$ (in population II) and $PRE(T_j,\bar{y}) = 1127.72$ (i.e. in population III) ,$(j = 9,11,12)$ in populations I, II and III; among the estimators considered in Table 5 . Largest gain in efficiency is obtained by using the estimatorT_{13}^* (in Table 8) over other existing estimators. Thus we conclude from the results of Tables 5, 6, 7 and 8 that there is enough scope of selecting the values of scalars (involved in the classes of estimators $T_{(1)}$,$T_{(2)}$ and T) in obtaining estimators better than conventional estimators listed in Table 5($i.e.Table1$). Hence the proposal of the class of estimators T and subclasses of estimators $T_{(1)}$ and $T_{(2)}$ are justified.

TABLE 5. PRE for different estimators (listed in Table 1) of the population mean $\bar{Y}$ with respect to the usual unbiased eastimator $\bar{y}$

(τ_1, τ_2)	Estimator	$PRE(., \bar{y})$		
		Population		
		I	II	III
$-$	$T_1 = \bar{y}$	100	100.00	100.00
$\tau_1 = 1,$	T_2	94.62	205.33	276.85
$\tau_2 = 1,$	T_3	53.33	102.17	187.12
$\tau_1 = 1, \tau_2 = 1,$	T_4	75.50	213.55	394.82
$\tau_1 = \frac{X}{\bar{X}+\rho_{xz}}, \tau_2 = \frac{Z}{\bar{Z}+\rho_{xz}}$	T_5	142.18	213.36	383.49
$\tau_1 = 1,$	T_6	102.94	214.77	238.59
$\tau_2 = 1,$	T_7	131.16	104.35	149.13
$\tau_1 = 1, \tau_2 = 1,$	T_8	143.71	235.49	401.98
$\tau_1 = 1, \tau_2 = 1,$	T_9	174.04	278.00	1127.72
$\tau_1 = \frac{X}{\bar{X}+\rho_{xz}}, \tau_2 = \frac{Z}{\bar{Z}+\rho_{xz}}$	T_{10}	131.99	235.61	405.83
$\tau_1 = \frac{X}{\bar{X}+\rho_{xz}}, \tau_2 = \frac{Z}{\bar{Z}+\rho_{xz}}$	T_{11}	174.04	278.09	1127.72
$\tau_1 = \frac{aX}{a\bar{X}+b}, \tau_2 = \frac{cZ}{c\bar{Z}+d}$	T_{12}	174.04	278.09	1127.72

TABLE 6. PREs of the members of the class of estimators T_1 (listed in Table 3) with respect to the usual unbiased estimator $\bar{y}$

(τ_1, τ_2)	Estimator	$PRE(., \bar{y})$		
		Population		
		I	II	III
$(-, -)$	$T_{(1)1}$	103.08	101.70	100.20
$(1, -)$	$T_{(1)2}$	103.08	209.92	277.09
$(-, 1)$	$T_{(1)3}$	127.65	103.68	186.47
$(1, 1)$	$T_{(1)4}$	174.46	217.43	394.97
$\left(\frac{1}{N+1}\right), \left(\frac{1}{N+1}\right)$	$T_{(1)5}$	108.57	104.60	119.52
$\left(\frac{X}{\bar{X}+C_x}\right), \left(\frac{Z}{\bar{Z}+C_z}\right)$	$T_{(1)6}$	169.72	247.69	396.96
$\left(\frac{X}{\bar{X}+\rho_{xz}}\right), \left(\frac{Z}{\bar{Z}+\rho_{xz}}\right)$	$T_{(1)7}$	142.26	217.26	383.62
$\left(\frac{XC_x}{\bar{X}C_x+\rho_{xz}}\right), \left(\frac{ZC_z}{\bar{Z}C_z+\rho_{xz}}\right)$	$T_{(1)8}$	149.27	217.20	296.92
$\left(\frac{XS_x}{\bar{X}S_x+C_x}\right), \left(\frac{ZS_z}{\bar{Z}S_z+C_z}\right)$	$T_{(1)9}$	173.32	226.59	395.34
$\left(\frac{X}{\bar{X}+S_x}\right), \left(\frac{Z}{\bar{Z}+S_z}\right)$	$T_{(1)10}$	158.00	254.84	459.71
$\left(\frac{XS_x}{\bar{X}S_x+\rho_{xz}}\right), \left(\frac{ZS_z}{\bar{Z}S_z+\rho_{xz}}\right)$	$T_{(1)11}$	165.68	217.38	392.93
$\left(\frac{X\rho_{xz}}{\bar{X}\rho_{xz}+S_x}\right), \left(\frac{Z\rho_{xz}}{\bar{Z}\rho_{xz}+S_z}\right))$	$T_{(1)12}$	167.47	100.80	263.11
$\left(\frac{X}{\bar{X}+C_x}\right), \left(\frac{Z}{\bar{Z}+C_z}\right)$	$T_{(1)13}$	174.38	279.83	1131.42

TABLE 7. PREs of the members of the class of estimators T_2 (listed in Table 4) with respect to the usual unbiased estimator $\bar{y}$

(τ_1, τ_2)	Estimator	$PRE(.,\bar{y})$		
		Population		
		I	II	III
$(-,-)$	$T_{(2)1}$	103.0758	101.6958	100.366
$(1,-)$	$T_{(2)2}$	105.594	194.6444	234.8634
$(-,1)$	$T_{(2)3}$	133.9042	106.0239	149.353
$(1,1)$	$T_{(2)4}$	146.0934	235.5185	402.3912
$\left(\frac{1}{N+1}\right),\left(\frac{1}{N+1}\right)$	$T_{(2)5}$	104.4347	103.0973	112.7033
$\left(\frac{X}{\bar{X}+C_x}\right),\left(\frac{Z}{\bar{Z}+C_z}\right)$	$T_{(2)6}$	128.6612	216.6834	401.7714
$\left(\frac{X}{\bar{X}+\rho_{xz}}\right),\left(\frac{Z}{\bar{Z}+\rho_{xz}}\right)$	$T_{(2)7}$	134.356	235.6092	406.2788
$\left(\frac{XC_x}{\bar{X}C_x+\rho_{xz}}\right),\left(\frac{ZC_z}{\bar{Z}C_z+\rho_{xz}}\right)$	$T_{(2)8}$	130.4054	235.6386	435.9655
$\left(\frac{XS_x}{\bar{X}S_x+C_x}\right),\left(\frac{ZS_z}{\bar{Z}S_z+C_z}\right)$	$T_{(2)9}$	123.337	230.4651	402.2819
$\left(\frac{X}{\bar{X}+S_x}\right),\left(\frac{Z}{\bar{Z}+S_z}\right)$	$T_{(2)10}$	129.9988	169.5826	367.1668
$\left(\frac{XS_x}{\bar{X}S_x+\rho_{xz}}\right),\left(\frac{ZS_z}{\bar{Z}S_z+\rho_{xz}}\right)$	$T_{(2)11}$	129.6144	235.541	403.0553
$\left(\frac{X\rho_{xz}}{\bar{X}\rho_{xz}+S_x}\right),\left(\frac{Z\rho_{xz}}{\bar{Z}\rho_{xz}+S_z}\right))$	$T_{(2)12}$	120.0486	101.2571	430.7065
$\left(\frac{X}{\bar{X}+C_x}\right),\left(\frac{Z}{\bar{Z}+C_z}\right)$	$T_{(2)13}$	175.5731	278.2825	1131.42

TABLE 8. PREs of the members of the class of estimators $T_(1)$ with respect to the usual unbiased estimator $\bar{y}$

(τ_1, τ_2)	Estimator	$PRE(.,\bar{y})$		
		Population		
		I	II	III
$(-,-)$	T_1^*	106.4789	250.7743	278.1218
$(1,-)$	T_2^*	106.4598	250.3207	278.0883
$(-,1)$	T_3^*	135.519	106.111	888.608
$(1,-)$	T_4^*	158.2896	276.9373	456.0533
$\left(\frac{1}{N+1}\right),\left(\frac{1}{N+1}\right)$	T_5^*	157.8524	277.0454	457.2462
$\left(\frac{X}{\bar{X}+C_x}\right),\left(\frac{Z}{\bar{Z}+C_z}\right)$	T_6^*	172.9148	273.8411	456.6116
$\left(\frac{X}{\bar{X}+\rho_{xz}}\right),\left(\frac{Z}{\bar{Z}+\rho_{xz}}\right)$	T_7^*	166.351	276.9466	453.3834
$\left(\frac{XC_x}{\bar{X}C_x+\rho_{xz}}\right),\left(\frac{ZC_z}{\bar{Z}C_z+\rho_{xz}}\right)$	T_8^*	162.7439	276.9487	445.0668
$\left(\frac{XS_x}{\bar{X}S_x+C_x}\right),\left(\frac{ZS_z}{\bar{Z}S_z+C_z}\right)$	T_9^*	169.4218	276.3239	456.1586
$\left(\frac{X}{\bar{X}+S_x}\right),\left(\frac{Z}{\bar{Z}+S_z}\right)$	T_{10}^*	167.052	264.1889	471.8228
$\left(\frac{XS_x}{\bar{X}S_x+\rho_{xz}}\right),\left(\frac{ZS_z}{\bar{Z}S_z+\rho_{xz}}\right)$	T_{11}^*	171.3515	276.9397	455.5144
$\left(\frac{X\rho_{xz}}{\bar{X}\rho_{xz}+S_x}\right),\left(\frac{Z\rho_{xz}}{\bar{Z}\rho_{xz}+S_z}\right))$	T_{12}^*	172.6356	191.1617	432.6855
$\left(\frac{X}{\bar{X}+C_x}\right),\left(\frac{Z}{\bar{Z}+C_z}\right)$	T_{13}^*	175.6382	279.7561	1134.405

5. Conclusion

This article discusses the problem of estimating the population mean $\bar{Y}$ of the study variable y using information on the parameters associated with two auxiliary variables x and z. We have made an effort to unify the several results based on various estimators through defining the class of estimators T. In addition to many the proposed class of estimators T includes the estimators envisaged by Singh (1967), Singh and Tailor (2005), Srivenkatramana (1980) and Bandyopadhyaya (1980), Singh et al (2005, 2011), Tailor (2012), Vishwakarma et al (2014) and Vishwakarma and Kumar (2015). The bias and MSE of the suggested class of estimators T are obtained upto first order of approximation. Asymptotic optimum conditions are obtained in which the suggested class of estimators T has minimum MSE. The biases and mean squared errors of different estimators belonging to the suggested class of estimators T can be obtained for suitable values of the scalars in the proposed class of estimators T .
The theoretical and empirical results show the superiority of the envisaged class of estimators T over other known estimators. Hence, the suggested class of estimators deserves for special attention in sample surveys dealing with estimation and inferential purposes.

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