Sakarya University Journal of Science, 22 (6), 1799-1803, 2018.

SAKARYA UNIVERSITY
JOURNAL OF SCIENCE

**e-ISSN:** 2147-835X
http://www.saujs.sakarya.edu.tr

Received
03-05-2018
Accepted
26-06-2018

Doi
10.16984/saufenbilder.420612

# Extending property on *EC*-Fully Submodules

Adnan TERCAN[1], Ramazan YAŞAR[*2]

**Abstract**

There are several generalizations of $CS$-modules in literature. One of the generalization is based on fully invariant submodules. Recall that a module $M$ is called $FI$-extending if every fully invariant submodule is essential in a direct summand. We call a module $EFI$-extending if every fully invariant submodule which contains essentially a cyclic submodule is essential in a direct summand. Initially we obtain basic properties in the general module setting. For example, a direct sum of $EFI$-extending modules is $EFI$-extending. Again, like the $FI$-extending property, the $EFI$-extending property is shown to carry over to matrix rings.

**Keywords:** fully invariant, $ec$-fully submodule, $FI$-extending, extending

## 1. INTRODUCTION

In recent years, the theory of extending modules and rings and their generalizations has come to play an important role in the theory of rings and modules. Recall that a module $M$ is called an *extending* (or $CS$) module if every submodule of $M$ is essential in a direct summand of $M$ (see [4], [9] or [10]).

One of the extremely useful generalization of $CS$ concept is $FI$-extending property (see [1] or [2]). Recall a module $M$ is called $FI$*-extending* if every fully invariant submodule of $M$ is essential in a direct summand. Following [3] and [5], by an $ec$-*fully submodule* $N$ of a module $M$, we mean a fully invariant submodule $N$ which contains essentially a cyclic submodule i.e., there exists an element $x$ in $N$ such that $xR$ is essential in $N$.

In this paper, we are concerned with the study of modules $M$ that every $ec$-fully submodule is essential in a direct summand of $M$. We call such a module as $EFI$*-extending*. Moreover, a ring $R$ is called right $EFI$-extending ring if $R_R$ is an $EFI$-extending module. Clearly the notion of an $EFI$-extending module generalizes that of a $FI$-extending module by requiring that only every $ec$-fully submodule is essential in a direct summand rather than every fully invariant submodule.

In Section 2, we provide basic properties of $ec$-fully submodules. After defining $EFI$-extending modules, in Section 3 we prove basic results and properties of $EFI$-extending modules. It is shown that any direct sum of $EFI$-extending modules is $EFI$-extending and that the $EFI$-extending property of a ring $R$ carries over to the full matrix ring $M_n(R)$, $n \geq 1$.

Throughout this paper, all rings are associative with unity and $R$ denotes such a ring. All modules are unital right $R$-modules.

Recall that a submodule $X$ of $M$ is called *fully invariant* if for every $\alpha \in End_R(M)$, $\alpha(X) \subseteq X$. If $M$ is an $R$-module and $A \subseteq M$, then we use $A \leq M$, $A \leq_e M$, $A \trianglelefteq M$, $A \trianglelefteq_{ec} M$, and $E(M)$ to denote that $A$ is a submodule, essential submodule, fully

[1] Hacettepe University, Faculty of Science, Department of Mathematics, Beytepe, Ankara, Turkey
[*] Corresponding Author
[2] Hacettepe University, Faculty of Science, Department of Mathematics, Beytepe, Ankara, Turkey

invariant submodule, $ec$-fully submodule, and the injective hull of $M$, respectively.

Moreover $M_n(R)$ denotes the full ring of $n$-by-$n$ matrices over $R$. For other terminology and notation, we refer to [2], [4], [7] and [10].

## 2. *EC-FULLY* SUBMODULES

Since $ec$-fully submodules are building bricks to the establishment of $EFI$-extending notion; first, we deal with this kind of submodules. To this end, we begin this section by recording some basic facts about them.

### 2.1. Lemma.

Let $M$ be a module.

(i) If $X \trianglelefteq_{ec} Y$ and $Y \trianglelefteq_{ec} M$ then $X \trianglelefteq_{ec} M$.

(ii) If $M = \bigoplus_{i\in\wedge} X_i$ and $S \trianglelefteq_{ec} M$, then $S = \bigoplus_{i\in\wedge} \pi_i(S) = \bigoplus_{i\in\wedge} (S \cap X_i)$, where $\pi_i$ is the $i^{th}$-projection homomorphism of $M$.

*Proof.* The proof is routine.

The class of $ec$-fully submodules is properly contained in the class of fully invariant submodules. Next example provides a fully invariant submodule which is not $ec$-fully submodule. For details on this example, we refer to [8] or [10].

### 2.2. Example.

Let $\mathbb{R}$ be the real field and $S$ the polynomial ring $\mathbb{R}[x,y,z]$. Then the ring $R = {}^{S}/_{Ss}$ , where $s = x^2 + y^2 + z^2 - 1$, is a commutative Noetherian domain. The free $R$-module $M = R \oplus R \oplus R$ contains an indecomposable submodule $X_R$ of uniform dimension 2.

Now, let us build up the trivial extension of $R$ with $X_R$ i.e., let

$$T = \begin{bmatrix} R & X \\ 0 & R \end{bmatrix} = \left\{ \begin{bmatrix} r & x \\ 0 & r \end{bmatrix} : r \in R, x \in X \right\}.$$

Then $N = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \trianglelefteq T_T$ but $N$ is not $ec$-fully submodule of $T_T$.

*Proof.* It is easy to check that $R$ is a commutative Noetherian domain. Let $\phi: M \to R$ be the homomorphism defined by $\phi(a + Ss, b + Ss, c + Ss) = ax + by + cz + Ss$ for all $a, b, c \in S$. Clearly, $\phi$ is an epimorphism, and hence, its kernel $X$ is a direct summand of $M$, i.e., $M = X \oplus X'$ for some submodule $X' \cong R$. Observe that $R$ is uniform i.e., $X'$ has uniform dimension 1 and hence $X_R$ has uniform dimension 2.

Note that $X$ is the $R$-module of regular sections of the tangent bundle of the 2-sphere $S^2$. Furthermore, a celebrated result in differential geometry yields that $X_R$ is an indecomposable module. Now the trivial extension of $R$ with $X_R$ i.e., $T = \begin{bmatrix} R & X \\ 0 & R \end{bmatrix}$ is a commutative ring and hence $N = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$ is a fully invariant submodule of $T$. Assume that $N$ contains essentially a cyclic submodule, say $\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} T$, where $x \in X$. Thus $xR$ is essential in $X_R$. It follows that $xR$ has uniform dimension 2. However, this is impossible, because the mapping $\alpha: xR \to R$, defined by $\alpha(xr) = r$, where $r \in R$, is an $R$-isomorphism. Thus $xR$ has uniform dimension 1. Therefore $N$ does not contain essentially a cyclic submodule. Hence $N$ is not $ec$-fully submodule of $T_T$.

Notice that the rank of free $R$-module $M$ in the previous example can be replaced by any odd integer $n > 3$ (see [8]). There are more examples in this trend. We refer reader to look at [6] for the construction of this kind of examples. Following easy lemma shows that certain fully invariant submodules are $ec$-fully submodules.

### 2.3. Lemma.

Let $M$ be a module which contains essentially a cyclic submodule. If $K$ is a fully invariant direct summand of $M$, then $K$ is an $ec$-fully submodule of $M$.

*Proof.* Suppose $Y = xR$ is an essential submodule of $M$, where $x \in M$. Let $\pi: M \to K$ be the canonical projection map. Then $xR \cap K = Y \cap K \leq \pi(Y) = \pi(x)R \leq K$. Since $xR$ is essential in $M$ then $xR \cap K$ is essential in $K$. It follows that $\pi(x)R$ is essential in $K$. Hence $K$ is an $ec$-fully submodule of $M$.

It is natural to think of which modules (even rings) have the property that every $ec$-fully submodule is a direct summand. Next result provides a class of rings which satisfy the aforementioned property. First, recall the following module condition:

$C_2$: If $X \leq M$ is isomorphic to a direct summand of $M$, then $X$ is a direct summand of $M$ (see [4] or [10]).

It is well-known that (von Neumann) regular rings satisfy the $C_2$ condition (see, for example [7]).

**2.4. Proposition.**

Let $R$ be a (von Neumann) regular ring. Then an $ec$-fully submodule of $R$-module $R$ is a direct summand.

*Proof.* Let $I$ be an $ec$-fully submodule of $R_R$. Then there exists $x \in I$ such that $xR$ is essential in $I$. By assumption, $xR$ is a direct summand of $R_R$. Thus $R_R = xR \oplus L$ for some $L \leq R_R$. Now $xR \cap L$ is essential in $I \cap L$ which yields that $I \cap L = 0$. Therefore $R = xR \oplus L = I \oplus L$. It follows that $I \cong xR$. Since $R_R$ has $C_2$ condition, $I$ is a direct summand of $R_R$ as required.

## 3. *EFI*-EXTENDING MODULES

In this section, we define and obtain basic properties of $EFI$-extending modules. Let us start by mentioning the definition of this new class of modules.

**3.1. Definition**

A module $M$ is called $EFI$*-extending* if every $ec$-fully submodule of $M$ is essential in a direct summand of $M$.

Obviously $FI$-extending modules (and hence extending modules) are $EFI$-extending modules. Moreover, (von Neumann) regular rings enjoy with the $EFI$-extending property. On the other hand, the ring of integers is an $EFI$-extending ring which is not regular. One might expect that whether $EFI$-extending property implies $FI$-extending or not? However, the following examples show that the class of $FI$-extending modules are properly contained in the class of $EFI$-extending modules.

**3.2. Example**

Let $F$ be any field and let $F_i = F$, $i \in \wedge$, where $\wedge$ is infinite. Define $R = \bigoplus_{i\in\wedge} F_i + F1$, which is an $F$-subalgebra of $\Pi_{i\in\wedge} F_i$, where 1 is the identity of $\Pi_{i\in\wedge} F_i$. It is known that $R$ is a regular (and hence $EFI$-extending ring by Proposition 2.4) ring which is not $FI$-extending (see [2, Ex. 2.3.32]).

**3.3. Example [7, Ex. 7.54]**

Let $F$ be a field, and let $A = F \times F \times \cdots$. So this ring is commutative. Now, let $R$ be the subring of $A$ consisting of sequences $(a_1, a_2, \ldots) \in A$ that are eventually constant. For any $(a_1, a_2, \ldots) \in R$, define $x = (x_1, x_2, \ldots)$ by; $x_n = a_n^{-1}$ if $a_n \neq 0$, and $x_n = 0$ if $a_n = 0$. Then $x \in R$ and $a = axa$. Therefore, $R$ is (von Neumann) regular. By Proposition 2.4, $R$ is $EFI$-extending. Note that $R$ is not a Baer ring. Hence $R$ is not an $FI$-extending ring by [1, Theorem 4.7(iii)].

It is an open problem to determine if a direct summand of an $FI$-extending (or, also $EFI$-extending) module is always $FI$-extending ($EFI$-extending) (see [1]). The following result is in related with the $EFI$-extending version of the aforementioned problem.

**3.4. Proposition**

Let $M$ be a module and $X \trianglelefteq_{ec} M$. If $M$ is $EFI$-extending, then $X$ is $EFI$-extending.

*Proof.* Assume $M$ is $EFI$-extending module. Let $S \trianglelefteq_{ec} X$. By Lemma 2.1 (i), $S \trianglelefteq_{ec} M$. Hence there exists a direct summand $D$ of $M$ such that $S \leq_e D$. Let $\pi: M \to D$ be the canonical projection endomorphism. Then $S = \pi(S) \leq \pi(X) \cap D = \pi(X)$. Hence $S \leq_e \pi(X)$ and $\pi(X)$ is a direct summand of $X$.

Next result deals with characterization of $EFI$-extending modules in terms of endomorphisms of injective hulls of the modules and complements of $ec$-fully submodules. To this end, the proof of the next theorem is based on the proof of the corresponding result for $FI$-extending modules (see [2, Proposition 2.3.2]).

### 3.5. Theorem

Let $M$ be a module. Then the following are equivalent:

(i) $M$ is $EFI$-extending

(ii) For $X \trianglelefteq_{ec} M$, there is $e^2 = e \in End(E(M))$ such that $X \leq_e eE(M)$ and $eM \leq M$.

(iii) Each $X \trianglelefteq_{ec} M$ has a complement which is a direct summand.

*Proof.* $(i) \Rightarrow (ii)$. Assume that $X \trianglelefteq_{ec} M$. Then there is $f^2 = f \in End(M)$ such that $X \leq_e fM$. Let $e: E(M) \rightarrow E(fM)$ be the canonical projection. Then we see that $X \leq_e eE(M)$ and $eM = fM \leq M$.

$(ii) \Rightarrow (iii)$. Let $X \trianglelefteq_{ec} M$. Then there exists $e^2 = e \in End(E(M))$ such that $X \leq_e eE(M)$ and $eM \leq M$. Now, let us put $c = (1-e)|_M$ . Then $c^2 = c \in End(M)$. We show that $cM$ is a complement of $X$. For this, first note that $cM \cap X = 0$ as $cM = (1-e)M$. Say $K \leq M$ such that $cM = (1-e)M \leq K$ and $K \cap X = 0$. From $M = (1-e)M \oplus eM$, $K = (1-e)M \oplus (K \cap eM)$ by the modular law. As $K \cap X = 0$ and $X \leq_e eE(M)$, $K \cap eE(M) = 0$ and so $K \cap eM = 0$. Thus, we get that $K = (1-e)M$, then $K = cM$. Therefore $cM$ is a complement of $X$.

$(iii) \Rightarrow (i)$. Let $X \trianglelefteq_{ec} M$. There exists $g^2 = g \in End(M)$ so that $gM$ is a complement of $X$. As $X \trianglelefteq_{ec} M$, $gX \leq X \cap gM = 0$. Hence $X = (1-g)X$. To show that $M$ is $EFI$-extending, we claim that $X \leq_e (1-g)M$. For this, assume that $K \leq (1-g)M$ such that $X \cap K = 0$. Then note that $gM \cap K = 0$. Take $gm + k = n \in (gM \oplus K) \cap X$ with $m \in M$, $k \in K$, and $n \in X$. Then $(1-g)gm + (1-g)k = (1-g)n$, so $k = n \in X \cap K$ because $K \leq (1-g)M$ and $X = (1-g)X$. Now as $X \cap K = 0$, $k = n = 0$. Thus, $(gM \oplus K) \cap X = 0$. Since $gM$ is a complement of $X$, $gM \oplus K = gM$ and so $K = 0$. Therefore, $X \leq_e (1-g)M$. It follows that $M$ is $EFI$-extending.

It is well-known that a direct sum of $FI$-extending modules is also $FI$-extending module. Now, we intend to have the corresponding result for $EFI$-extending modules.

### 3.6. Theorem

Let $M = \bigoplus_{i\in\Lambda} N_i$. If each $N_i$ is an $EFI$-extending module, then $M$ is an $EFI$-extending module.

*Proof.* Let $S \trianglelefteq_{ec} M$. By Lemma 2.1(ii), $S = \bigoplus_{i\in\Lambda} (S \cap N_i)$, and $S \cap N_i \trianglelefteq N_i$ for each $i \in \Lambda$. Assume $S$ contains essentially the cyclic submodule $xR$, where $x \in S$. Let $\pi: S \rightarrow S \cap N_i$ be the projection map. Then $xR \cap (S \cap N_i) \leq \pi(xR) = \pi(x)R \leq S \cap N_i$. Since $xR \leq_e S$ then $xR \cap (S \cap N_i) \leq_e S \cap N_i$. It follows that $\pi(x)R \leq_e S \cap N_i$. Hence $S \cap N_i \trianglelefteq_{ec} N_i$ for each $i \in \Lambda$. As $N_i$ is $EFI$-extending, there is a direct summand $D_i$ of $N_i$ with $S \cap N_i \leq_e D_i$ for every $i \in \Lambda$. Thus $S = \bigoplus_{i\in\Lambda} (S \cap N_i) \leq_e \bigoplus_{i\in\Lambda} D_i$. Since $\bigoplus_{i\in\Lambda} D_i$ is a direct summand of $M$ we have that $M$ is an $EFI$-extending module.

### 3.7. Corollary

If $M$ is a direct sum of $FI$-extending (e.g., extending) modules, then $M$ is $EFI$-extending.

*Proof.* Immediate by Theorem 3.6.

Applying Theorem 3.6 to Abelian groups (i.e., $\mathbb{Z}$-modules) we obtain the following corollary.

### 3.8. Corollary

Let $M$ be a $\mathbb{Z}$-module. If $M$ satisfies any of the following conditions, then $M$ is an $EFI$-extending $\mathbb{Z}$-module.

(i) $M$ is finitely generated

(ii) $M$ is of bounded order (i.e., $nM = 0$ for some positive integer $n$)

(iii) $M$ is divisible.

*Proof.* (i) and (ii) $M$ is a direct sum of uniform submodules. Then the result follows from Theorem 3.6.

(iii) $M$ is extending and hence $FI$-extending. Thus $M$ is $EFI$-extending.

Observe that an easy modification yields that the Corollary 3.8 above remains true when the ring of integers replaced with a Dedekind domain.

One more application of the Theorem 3.6 gives an affirmative answer for the direct summand problem of $EFI$-extending Abelian groups which as follows.

**3.9. Theorem**

Let $M$ be a direct sum of uniform $\mathbb{Z}$-modules. Then any direct summand of $M$ is an $EFI$-extending module.

*Proof.* Let $N$ be a direct summand of $M$. Then $N$ is also a direct sum of uniform modules by [9, Theorem 4.45] (see, also [10]). Now, Theorem 3.6 gives that $N$ is an $EFI$-extending module.

Our next objective is to carry over $EFI$-extending property to full matrix ring. First of all, we give an example of $EFI$-extending ring which shows that $EFI$-extending property is not left-right symmetric.

**3.10. Example**

Let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{bmatrix}$ . Then the ring $R$ is right $EFI$-extending, but it is not left $EFI$-extending.

*Proof.* Note that $R$ is right $FI$-extending by [2, Example 2.3.14]. Hence $R$ is right $EFI$-extending ring. On the other hand, let $I = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix} \trianglelefteq_{ec} {}_R R$. It is easy to check that $I$ is not essential in a direct summand of ${}_R R$. It follows that $R$ is not left $EFI$-extending ring.

**3.11. Theorem**

Let $R$ be a right $EFI$-extending ring. Then $M_n(R)$ is a right $EFI$-extending ring for all positive integer $n$.

*Proof.* Let $N \trianglelefteq_{ec} M_n(R)$. Then it is easy to see that $N = M_n(I)$ for some $I \trianglelefteq_{ec} R$. As $R$ is right $EFI$-extending, there exists $e^2 = e \in R$ such that $I_R \leq_e eR_R$. This yields that as a right ideal of $M_n(R)$, $N$ is essential in a direct $(e\boldsymbol{I})M_n(R)$ of $M_n(R)$, where $\boldsymbol{I}$ is the identity matrix of $M_n(R)$. Thus $M_n(R)$ is right $EFI$-extending, as required.

## REFERENCES


[1] Birkenmeier, G. F., Müller, B. J. and Rizvi S. T., “Modules in which every fully invariant submodule is Essential in a Direct Summand,” *Communications in Algebra,* vol. 30, no.3, pp. 1395-1415, 2007.

[2] Birkenmeier, G. F., Park, J. K. and Rizvi, S. T., *Extensions of rings and modules*, New York: Birkhäuser, 2013.

[3] Celep, C. and Tercan A., "Modules whose ec-closed submodules are direct summand," *Taiwanese Journal of Mathematics,* vol.13, no.4, pp. 1247-1256, 2009.

[4] Dung, N. V., Huynh, D. V., Smith, P. F., and Wisbauer, R., *Extending Modules*, Pitman Research Notes in Mathematics Series, 313, Longmon, New York, 1994.

[5] Kamal, M. A., and Elmnophy, O. A., "On P-extending modules," *Acta Math. Univ. Comenianae,* vol. 74, no. 2, pp. 279-286, 2005.

[6] Kara, Y., Tercan, A. and Yaşar R., "*PI*-extending modules via nontrivial complex bundles and Abelian endomorphism rings," *Bulletin of the Iranian Mathematical Society,* vol. 43, no. 1, pp. 121-129, 2017.

[7] Lam, T.Y., *Lectures on modules and rings*, vol. 189. Springer Science and Business Media, 2012.

[8] Smith, P. F., and Tercan, A., "Direct summands of modules which satisfy (C-11)," *Algebra Colloquium*, vol. 11, pp. 231-237, 2004.

[9] Smith, P. F., and Tercan, A., “Generalizations of CS-modules,” *Communications in Algebra,* vol. 21, no.6, pp.1809-1847, 1993.

[10] Tercan, A. and Yücel, C. C., *Module theory, extending modules and generalizations*, Basel: Birkhäuser, 2016.