A generalization of amenability for topological semigroups and semigroup algebras

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Abstract

In this paper for two topological semigroups $S$ and $T$, and a continuous homomorphism $\varphi$ from $S$ into $T$, we introduce and study the concept of $(\varphi, T)$-derivations on $S$ and $\varphi$-amenability of $T$ and investigate the relations between these two concepts. For two Banach algebras $A$ and $B$ and a continuous homomorphism $\varphi$ from $A$ into $B$ we also introduce the notion of $(\varphi, B)$-amenability of $A$ and show that if a foundation semigroup $T$ with identity is $\varphi$-amenable whenever the Banach algebra $M_a(S)$ is $(\tilde{\varphi}, M_a(T))$-amenable, where $\tilde{\varphi} : M(S) \to M(T)$ denotes the unique extension of $\varphi$. An example is given to show that the converse is not true.

Keywords: Continuous homomorphism, semigroup, Banach algebra, $(\varphi, T)$-derivation, $\varphi$-amenable.


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1. Introduction

The concept of amenability for Banach algebras was initiated by Johnson in [9]. He showed that a locally compact Hausdorff group $G$ is amenable if and only if the Banach algebra $L^1(G)$ is amenable. This fails to be true for discrete semigroups. Duncan and Nomiloka [4] proved that if $l^1(S)$ is amenable then $S$ is amenable and $l^1(S)$ fails to be amenable if $E_S$ is infinite. Johnson proved that $H^1(L^1(G), X^*) = \{0\}$ if and only if every $G$-derivation into $X^*$ is inner, where $X$ is a neo-unital Banach $L^1(G)$-bimodule (see [2] and [9]).

Recently, Kaniuth, Lau and Pym introduced $\varphi$-amenability of a Banach algebra $A$ where $\varphi$ is a homomorphism from $A$ to $\mathbb{C}$ [10]. Here for two Banach algebras $A$ and $B$ we study the concept of $(\varphi, B)$-amenability of $A$ where $\varphi : A \to B$ is a continuous homomorphism. In the case where $A = B$, $A$ is called $\varphi$-amenable. Several authors have

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studied \(\varphi\)-derivations, and \(\varphi\)-amenability of a Banach algebra \(A\) (see [7], [8], [15] and [16]).

Authors in [7], introduced the notion of \(\varphi\)-amenability for a locally compact group \(G\), where \(\varphi\) is a continuous homomorphism on \(G\). They proved that if the group algebra \(L^1(G)\) is \(\hat{\varphi}\)-amenable, then \(G\) is \(\varphi\)-amenable and when \(\varphi\) is an isomorphism on \(G\), the converse is valid. Here \(\hat{\varphi}\) is the unique extension of \(\varphi\) to \(M(G)\).

In this paper for two Banach algebras \(A\) and \(B\) and a continuous homomorphism \(\varphi: A \to B\), we first introduce the notion of \((\varphi, B)\)-amenability of \(A\). This concept reduces to that of \(\varphi\)-amenability introduced by Kaniuth, Lau, and Pym, when \(B = \mathbb{C}\). Also for two topological semigroups \(S\) and \(T\), and a continuous homomorphism \(\varphi\) from \(S\) into \(T\), we introduce and study the concept of \((\varphi, T)\)-derivations on \(S\) and \(\varphi\)-amenability of \(T\) and investigate the relation between these two concepts. Then we apply our results to the case where \(S\) and \(T\) are foundation semigroups and prove that \(M_a(S)\) is \((\hat{\varphi}, M_a(T))\)-amenable if and only if every \((\varphi, T)\)-derivation on \(S\) is \(\varphi\)-inner, where \(\hat{\varphi}: M_a(S) \to M_a(T)\) denotes the extension of \(\varphi\). This extends a known result due to Johonson for groups to foundation semigroups. Finally, we show that \((\hat{\varphi}, M_a(T))\)-amenability of \(M_a(S)\) implies \(\varphi\)-amenability of \(T\), and present an example to show that the converse is not true.

2. Preliminaries

Let \(A\) be a Banach algebra, and let \(X\) be an \(A\)-bimodule. Then \(X\) is a Banach \(A\)-bimodule if \(X\) is a Banach space and there is a constant \(k > 0\) such that

\[
\|a.x\| \leq k\|a\|\|x\|, \quad \|x.a\| \leq k\|a\|\|x\| \quad (a \in A, x \in X).
\]

By renorming, we can suppose that \(k = 1\). For example, \(A\) itself is a Banach \(A\)-bimodule, and \(X^*\), the dual space of a Banach \(A\)-bimodule \(X\), is a Banach \(A\)-bimodule if for every \(a \in A\) and \(f \in X^*\) we define

\[
\langle x, af \rangle = \langle x, a.f \rangle, \quad \langle x, f.a \rangle = \langle a.x, f \rangle \quad (x \in X).
\]

We say that \(X^*\) is the dual module of \(X\).

Suppose that \(A\) is a Banach algebra and \(X\) is a Banach \(A\)-bimodule. A derivation from \(A\) into \(X\) is a linear operator \(D: A \to X\) satisfying

\[
D(ab) = D(a)b + aD(b) \quad (a, b \in A).
\]

A derivation \(D\) is inner if there is \(x_0 \in X\) such that \(D(a) = a.x_0 - x_0.a\) for \(a \in A\) and \(x_0 \in X\).

A derivation \(D\) is amenable if for any Banach \(A\)-bimodule \(X\), every continuous derivation \(D: A \to X^*\) is inner.

Let \(A\) and \(B\) be two Banach algebras. The set of continuous homomorphisms from \(A\) into \(B\) is denoted by \(\text{Hom}(A, B)\). We denote the set \(\text{Hom}(A, A)\) by \(\text{Hom}(A)\).

Suppose that \(\varphi: A \to B\) is a continuous homomorphism. A Banach space \(X\) over \(\mathbb{C}\) is a Banach \(\varphi(B)\)-bimodule if it is two-sided \(\varphi(B)\)-module and there is a positive real number \(K\) such that

\[
\|\varphi(a).x\| \leq K\|\varphi(a)\|\|a\|\|x\|, \quad \|x.\varphi(a)\| \leq K\|x\|\|\varphi(a)\|,\]}

for all \(a \in A\) and \(x \in X\).

Let \(X\) be a Banach \((\varphi, B)\)-bimodule and \(\varphi \in \text{Hom}(A, B)\), a linear operator \(D: A \to X\) is called a \((\varphi, B)\)-derivation if

\[
D(ab) = D(a).\varphi(b) + \varphi(a).D(b) \quad (a, b \in A).
\]

A \((\varphi, B)\)-derivation \(D\) is called \((\varphi, B)\)-inner if there is \(x \in X\) such that \(D(a) = \varphi(a)x - x.\varphi(a)\) (\(a \in A\)). A Banach algebra \(A\) is called \((\varphi, B)\)-amenable if for any Banach \((\varphi, B)\)-bimodule \(X\), every continuous \((\varphi, B)\)-derivation \(D: A \to X^*\) is \((\varphi, B)\)-inner. In the case that \(A = B\), \(D\) is called \(\varphi\)-derivation and \(A\) is called \(\varphi\)-amenable.
3. \(\varphi\text{-derivation and }\varphi\text{-amenable semigroups}

We commence this section with the following definition:

**3.1. Definition.** Let \(S\) and \(T\) be two topological semigroups and \(\varphi : S \rightarrow T\) be a continuous homomorphism of \(S\) into \(T\). We say that the complex Banach space \(X\) is a left Banach \(\varphi\)-module if there exists a mapping

\[
\varphi(S) \times X \rightarrow X, \quad (\varphi(s), x) \mapsto \varphi(s).x,
\]

having the following properties:

(i) \(\varphi(s_1 + s_2).x = \varphi(s_1).x + \varphi(s_2).x\), \(\varphi(\lambda s).x = \lambda \varphi(s).x\) for all \(s, s_1, s_2 \in S, x_1, x_2 \in X\) and \(\lambda \in \mathbb{C}\),

(ii) if \(s_n \rightarrow s\) in \(S\) and \(x \in X\), then \(\varphi(s_n).x \rightarrow \varphi(s).x\) in the norm topology, and

(iii) there is \(M > 0\) such that for every \(x \in X\) and \(s \in S\), we have

\[
\|\varphi(s).x\| \leq M\|x\|.
\]

In the same way, one defines a right Banach \(\varphi\)-module. The (two sided) Banach \(\varphi\)-module \(X\) is a left and right Banach \(\varphi\)-module such that

\[
(\varphi(s_1).x), \varphi(s_2) = \varphi(s_1), (x, \varphi(s_2)) \quad (s_1, s_2 \in S, x \in X).
\]

Note that if \(X\) is a Banach \(\varphi\)-module, then \(X^*\), the dual space of \(X\), is also an \(\varphi\)-module through the following actions:

\[
\langle \varphi(s).x^*, x \rangle = \langle x^*, \varphi(s) \rangle, \quad (x^*, \varphi(s), x) = \langle x^*, \varphi(s), x \rangle \quad (s \in S, x \in X, x^* \in X^*).
\]

A left (resp. right) action of \(\varphi(S)\) on \(X\) is trivial if \(\varphi(s).x = x\) (\(s \in S, x \in X\)) (resp. \(x, \varphi(s) = x\) (\(s \in S, x \in X\))).

**3.2. Definition.** Let \(S\) and \(T\) be two topological semigroups and \(\varphi : S \rightarrow T\) be a continuous homomorphism of \(S\) into \(T\). Let \(X\) be a Banach \(\varphi\)-module. A weak*-continuous map \(D : S \rightarrow X^*\) is called a \((\varphi, T)\)-derivation (\(\varphi\)-derivation in the case that \(S = T\)) if

(i) \(D(s_1 s_2) = \varphi(s_1).D(s_2) + D(s_1).\varphi(s_2)\) (\(s_1, s_2 \in S\));

(ii) \(\sup_{s \in S} \|D(s)\| < \infty\).

Furthermore, \(D\) is called \(\varphi\)-inner if there exists \(x^* \in X^*\) such that

\[
D(s) = \varphi(s).x^* - x^*.\varphi(s) \quad (s \in S).
\]

For a topological semigroup \(S\) let \(C_b(S)\) be the set of all bounded continuous complex valued functions on \(S\) and let \(\text{LUC}(S) = \{ f \in C_b(S) \mid x \rightarrow l_x f \text{ is norm continuous} \}\) (resp. \(\text{RUC}(S) = \{ f \in C_b(S) \mid x \rightarrow r_x f \text{ is norm continuous} \}\), where \(l_x f(y) = f(xy) \quad (x, y \in S)\) (resp. \(r_x f(y) = f(yx) \quad (x, y \in S)\)).

Recall that a linear functional \(m \in \text{LUC}(S)^*\) is called a mean if \(\|m\| = 1\); \(m\) is called a left invariant mean, if \(m(l_x f) = m(f)\) for all \(s \in S\) and \(f \in \text{LUC}(S)\). A topological semigroup \(S\) is called left amenable if \(\text{LUC}(S)\) has a left invariant mean (see [13] and [17]). Right amenability of a topological semigroup may be defined similarly. A topological semigroup which is both left and right amenable is called amenable.

**3.3. Definition.** Let \(S\) and \(T\) be two topological semigroups, and \(\varphi : S \rightarrow T\) be a continuous homomorphism. A \(\varphi\)-left invariant mean (resp. \(\varphi\)-right invariant mean) on \(\text{LUC}(T)\) (resp. on \(\text{RUC}(T)\)) is a functional \(m \in \text{LUC}(T)^*\) (resp. \(m \in \text{RUC}(T)^*\)) such that

(i) \(m(1) = \|m\| = 1\) and \(m(l_x f) = m(f)\) (\(f \in \text{LUC}(T), s \in S\)) (resp. \(m(1) = \|m\| = 1\) and \(m(r_x f) = m(f)\) (\(f \in \text{RUC}(T), s \in S\)).
3.4. **Definition.** For two topological semigroups $S$ and $T$, and continuous homomorphism $\varphi : S \to T$, $T$ is called $\varphi$-left amenable (resp. $\varphi$-right amenable) if there is a $\varphi$-left invariant mean (resp. $\varphi$-right invariant mean) on $\text{LUC}(T)$ (resp. on $\text{RUC}(T)$). A semigroup $T$ is called $\varphi$-amenable if it is both $\varphi$-left and $\varphi$-right amenable.

The proof of the following lemma is straightforward.

3.5. **Lemma.** Let $S$ and $T$ be two topological semigroups, and $\varphi : S \to T$ a continuous homomorphism. If $T$ is amenable then $T$ is $\varphi$-amenable. The converse is true if $\varphi(S)$ is dense in $T$.

3.6. **Proposition.** Let $S$ and $T$ be two topological semigroups, and $\varphi : S \to T$ a continuous homomorphism. If for every Banach $\varphi$-module $X$, any $(\varphi,T)$-derivation $D : S \to X^*$ is $\varphi$-inner, then $T$ is $\varphi$-amenable.

**Proof.** We first note that $\text{LUC}(T)$ is a Banach $\varphi$-module through the following actions given by

$$\varphi(s)f = f, \quad (f, \varphi(s))(t') = f(\varphi(s)t') \quad (s \in S, t' \in T, f \in \text{LUC}(T)).$$

Let $n \in \text{LUC}(T)^*$ such that $(1,n) = 1$. Define $d : S \to \text{LUC}(T)^*$ by $d(s) = \varphi(s)n - n$. It is easy to see that $C_1T$ is a closed submodule of $\text{LUC}(T)$. Let $X = \frac{\text{LUC}(T)^*}{C_1T}$. Since for each $s \in S$, $(1,d(s)) = 0$, there exists a $(\varphi,T)$-derivation $D : S \to X^*$ such that $\pi^* \circ D(s) = d(s)(s \in S)$, where $\pi$ is the canonical map from $\text{LUC}(T)$ onto $X$. Thus there exists $g \in X^*$ such that $D(s) = \varphi(s)g - g \ (s \in S)$. Hence

$$\pi^* \circ D(s) = \pi^*(\varphi(s)g) - \pi^*g = \varphi(s)n - n.$$  

So

$$\varphi(s)n - \pi^*(\varphi(s)g) = n - \pi^*g.$$  

Let $\tilde{n} = n - \pi^*g$. Then $\tilde{n} \in \text{LUC}(T)^*$. From (3.1), it follows that

$$\varphi(s)\tilde{n} = \varphi(s)n - \varphi(s)(\pi^*g) = \varphi(s)n - \pi^*(\varphi(s)g) = n - \pi^*g = \tilde{n} \ (s \in S).$$

Since $\text{LUC}(T)$ is a commutative $C^*$-algebra with identity, there exists a compact Hausdorff space $\Delta$ such that $C(\Delta)$ and $\text{LUC}(T)$ are isometrically $*$-isomorphic $C^*$-algebras. Thus we can consider $\tilde{n}$ as a $\varphi$-left invariant complex Borel regular measure on $\Delta$. Let $m = \frac{\tilde{n}}{||\tilde{n}||}$, then $\varphi(s)m = m$. Therefore for every $f \in \text{LUC}(T)$ and $s \in S$

$$\langle l_{\varphi(s)}f, m \rangle = \langle f, \varphi(s)m \rangle = \langle f, \varphi(s)m \rangle = \langle f, m \rangle.$$

Hence $T$ is $\varphi$-left amenable. Similarly we can show that $T$ is $\varphi$-right amenable. Therefore $T$ is $\varphi$-amenable.$\square$

The following proposition provides a converse for Proposition 3.6 in a special case.

3.7. **Proposition.** Let $S$ be a topological semigroup, and $\varphi : S \to S$ be a continuous homomorphism such that $\varphi(S)$ is dense in $S$. If $S$ is $\varphi$-left amenable, then for every Banach $\varphi$-module $X$ with trivial left action, any $\varphi$-derivation $D : S \to X^*$ is $\varphi$-inner.

**Proof.** Suppose $S$ is $\varphi$-left amenable and $X$ is a Banach $\varphi(S)$-module with trivial left action, and $D : S \to X^*$ is a $\varphi$-derivation. For every $x \in X$ we define $f_x : S \to \mathbb{C}$ by $f_x(s) = \langle x, D(s) \rangle$ ($s \in S$). Thus

$$||f_x||_\infty = \sup_{s \in S} |f_x(s)| \leq \sup_{s \in S} ||D(s)|| ||x|| \leq M||x||,$$
where \( M > 0 \) is a uniform bound for \( D \), clearly, \( f_x \) is continuous. We claim that 
\( f_x \in \text{LUC}(S) \). To see this let \( s_n \rightarrow s \) in \( S \), then 
\[
\| l_{s_n} f_x - l_s f_x \| = \sup_{s' \in S} |f_x(s_n s') - f_x(s s')| \\
= \sup_{s' \in S} |x, D(s_n s') - x, D(s s')| \\
\leq \sup_{s' \in S} |x, D(s_n s') - D(s), \varphi(s s')| \\
+ \sup_{s' \in S} |x, \varphi(s_n s') - D(s) s', \varphi(s')| \\
= |x, D(s_n) - D(s)| + \sup_{s' \in S} |x, \varphi(s_n s') - x, \varphi(s), D(s')|.
\]
Since \( D \) is weak*-continuous, we infer that 
\[
|\langle x, D(s_n) - D(s) \rangle| \rightarrow 0. \quad \text{Also}
\]
\[
\sup_{s' \in S} |\langle x, \varphi(s_n s') - x, \varphi(s), D(s') \rangle| \leq M|\langle x, \varphi(s_n) - x, \varphi(s) \rangle|,
\]
and by definition 3.1 (ii), 
\[
\| x, \varphi(s_n) - x, \varphi(s) \| \rightarrow 0. \quad \text{Thus} \quad \| l_{s_n} f_x - l_s f_x \| \rightarrow 0.
\]
So \( f_x \in \text{LUC}(S) \). Let \( m \in \text{LUC}(S)^* \) be such that \( \langle 1, m \rangle = 1 \) and \( m(l_{\varphi(s_n)} f) = m(f) \) \( (s \in S, f \in \text{LUC}(S)) \), and define a linear functional \( f \) on \( X \) by \( \langle x, f \rangle = \langle f_x, m \rangle \) \( (x \in X) \). For every \( x \in X \) and \( s, s' \in S \), we have
\[
f_{x, \varphi(s)(s')} = \langle x, \varphi(s), D(s') \rangle = \langle x, \varphi(s), D(s') \rangle \\
= \langle x, D(s s') - x, D(s) \varphi(s') \rangle \\
= \langle x, D(s s') - x, D(s) \rangle \\
= f_x(s s') - f_x - (x, D(s)) 1_S(s').
\]
Therefore \( f_{x, \varphi(s)} = l_s f_x - \langle x, D(s) \rangle 1_S \). Since \( \varphi(s) \) is dense in \( S \) it follows that there exists a net \( \{ s_n \} \) in \( S \) such that \( \lim_n \varphi(s_n) = s \), and \( \lim_n l_{\varphi(s_n)} f_x = l_s f_x \) by the definition of \( \text{LUC}(S) \). Thus
\[
\langle x, f - \varphi(s), f \rangle = \langle x, f \rangle - \langle x, \varphi(s), f \rangle = \langle f_x - f_{x, \varphi(s)}, m \rangle \\
= \langle f_x - l_s f_x + \langle x, D(s) \rangle 1_S, m \rangle \\
= \langle f_x - \lim_n l_{\varphi(s_n)} f_x + \langle x, D(s) \rangle 1_S, m \rangle \\
= \lim_n \langle f_x - l_{\varphi(s_n)} f_x + \langle x, D(s) \rangle 1_S, m \rangle \\
= \langle x, D(s) \rangle.
\]
Hence \( D(s) = f - \varphi(s), f \) \( (s \in S) \). Therefore \( D \) is \( \varphi \)-inner.

By a similar argument one can prove the following proposition:

3.8. Proposition. Let \( S \) be topological semigroup, and \( \varphi : S \rightarrow S \) be a continuous homomorphism such that \( \varphi(S) \) is dense in \( S \). If \( S \) is \( \varphi \)-right amenable, then for every Banach \( \varphi \)-module \( X \) with trivial right action, any \( \varphi \)-derivation \( D : S \rightarrow X^* \) is \( \varphi \)-inner.

4. \( \varphi \)-amenability of \( M_u(S) \)

We start this section with the following.

For a topological semigroup \( S \) let \( M(S) \) denote the space of all bounded, regular, complex Borel measures on \( S \). This space with the convolution product and norm \( \| \mu \| = \| \mu \| \) is a Banach algebra. The space of all measures \( \mu \in M(S) \) for which the mappings \( x \rightarrow |\mu| * \delta_x \) and \( x \rightarrow \delta_x * |\mu| \) from \( S \) into \( M(S) \) are weakly continuous is denote by \( M_u(S) \) (\( L(S) \), as in [1]). A Hausdorff locally compact topological semigroup \( S \) is called a foundation semigroup if \( S \) coincides with the closure of \( \bigcup \{ \text{supp} \mu : \mu \in M_u(S) \} \). Note
that in the case where $S$ is a foundation semigroup with identity, for every $\mu \in M_a(S)$ both mappings $x \to |\mu| \ast \delta_x$, and $x \to \delta_x \ast |\mu|$ from $S$ into $M(S)$ are norm continuous and $M_a(S)$ has a bounded approximate identity (see [6]). Finally, for any topological semigroup $S$, $\mu \in M(S)$ and $f \in C_b(S)$, we define the complex valued functions $\mu \circ f$ and $f \circ \mu$ on $S$ by

\[ (4.1) \quad \mu \circ f(x) = \int_S f(y|x) d\mu(y), \quad f \circ \mu(x) = \int_S f(xy) d\mu(y). \]

Lemma 1.3.4 of [6] shows that $\mu \circ f$ and $f \circ \mu$ are in $C_b(S)$. Also for every $\mu, \nu \in M_a(S)$ and $f \in C_b(S)$ $\langle \mu \ast \nu, f \rangle = \langle \mu, f \circ \nu \rangle = \langle \nu, \mu \circ f \rangle$.

4.1. Definition. If a Banach algebra $A$ is contained in a Banach algebra $B$ as a closed ideal, then the strict topology or strong operator topology (so) on $B$ with respect to $A$ is defined through the family of seminorms $(p_a)_{a \in A}$, where

\[ p_a(b) := ||ba|| + ||ab|| \quad (b \in B). \]

For a topological semigroup $S$ the strict topology on $M(S)$ with respect to $M_a(S)$ is simply called so topology or the strict topology on $M(S)$.

4.2. Lemma. Let $S$ and $T$ be two foundation semigroups with identity, and let $\varphi : S \rightarrow T$ be a continuous homomorphism. Define $\tilde{\varphi} : M(S) \rightarrow M(T)$ by

\[ \langle \tilde{\varphi}(\mu), f \rangle = \int_S f(\varphi(x)) d\mu(x) \quad (f \in C_0(T)). \]

Then $\tilde{\varphi}$ is a continuous homomorphism (with respect to the strict topology on $M(S)$) that extends $\varphi$ uniquely and $\tilde{\varphi}(M_a(S)) \subseteq M_a(T)$.

Proof. It is easy to see that $\tilde{\varphi}$ is continuous. By using (4.1), for every $f \in C_0(T)$ and $\mu_1, \mu_2 \in M(S)$, we have

\[ \langle \tilde{\varphi}(\mu_1) \ast \tilde{\varphi}(\mu_2), f \rangle = \langle \tilde{\varphi}(\mu_1), f \circ \tilde{\varphi}(\mu_2) \rangle \]

\[ = \int_S f \circ \tilde{\varphi}(\mu_2)(\varphi(x)) d\mu_1(x) \]

\[ = \int_S \int_S f(\varphi(x)y) d\tilde{\varphi}(\mu_2)(y) d\mu_1(x) \]

\[ = \int_S \int_S f(\varphi(x)\varphi(y)) d\mu_2(y) d\mu_1(x) \]

\[ = \int_S \int_S f(\varphi(x)\varphi(y)) d\mu_1(x) d\mu_2(y) \]

\[ = \int_S f(\varphi(x)) d\mu_1 \ast \mu_2(x) \]

\[ = \langle \tilde{\varphi}(\mu_1 \ast \mu_2), f \rangle. \]

Therefore $\tilde{\varphi}$ is a continuous homomorphism. Let $\overline{\varphi}$ be another extension of $\varphi$ and let $\mu \in M(S)$. By Theorem 3.3 of [14], $\mu$ is the s-lim (strict-limit) of a net $(\mu_i)$ such that each $\mu_i$ is a combination of point masses. So,

\[ \tilde{\varphi}(\mu) = \tilde{\varphi}(s - \lim_i \mu_i) = \lim_i \tilde{\varphi}(\mu_i) = \lim_i \overline{\varphi}(\mu_i) = \overline{\varphi}(s - \lim_i \mu_i) = \overline{\varphi}(\mu). \]

Thus $\tilde{\varphi} = \overline{\varphi}$.

To complete the proof, let $(\mu_\alpha)$ be a bounded approximate identity for $M_a(T)$, then as in Lemma 2.1 of [11],

\[ ||\mu_\alpha \circ f - f||_\infty \rightarrow 0 \quad (f \in C_0(T)). \]
Now for every \( \mu \in M_a(T) \), we obtain
\[
\|\tilde{\varphi}(\mu) * \mu_\alpha - \tilde{\varphi}(\mu)\| = \sup_{f \in C_0(\tau), \|f\| \leq 1} \|\langle \tilde{\varphi}(\mu) * \mu_\alpha, f \rangle - \langle \tilde{\varphi}(\mu), f \rangle\|
\]
\[
= \sup_{f \in C_0(\tau), \|f\| \leq 1} \|\langle \tilde{\varphi}(\mu), \mu_\alpha \circ f \rangle - \langle \tilde{\varphi}(\mu), f \rangle\|
\]
\[
= \sup_{f \in C_0(\tau), \|f\| \leq 1} \|\langle \tilde{\varphi}(\mu), \mu_\alpha \circ f - f \rangle\|
\]
\[
\leq \sup_{f \in C_0(\tau), \|f\| \leq 1} \|\tilde{\varphi}(\mu)\| \|\mu_\alpha \circ f - f\|_\infty \to 0.
\]
This means that \( \tilde{\varphi}(\mu) * \mu_\alpha \to \tilde{\varphi}(\mu) \) in norm. So \( \tilde{\varphi}(\mu) \in M_a(T) \). Therefore \( \tilde{\varphi}(M_a(S)) \subseteq M_a(T) \). □

4.3. Definition. Let \( A \) and \( B \) be two Banach algebras and \( \varphi : A \to B \) be a continuous homomorphism. A Banach \((\varphi, B)\)-bimodule \( X \) is called \((\varphi, B)\)-pseudo-unital if
\[
X = \{\varphi(a_1).x.\varphi(a_2) : a_1, a_2 \in A, x \in X\}.
\]

The proof of the following proposition is omitted, since it can be proved in the same direction of Proposition 2.1.3 of [18].

4.4. Proposition. Let \( A \) and \( B \) be two Banach algebras which \( A \) has a bounded right approximate identity, and \( \varphi : A \to B \) be a continuous homomorphism. Let \( X \) be a Banach \((\varphi, B)\)-bimodule such that \( \varphi(A).X = \{0\} \). Then every \((\varphi, B)\)-derivation on \( A \) is \((\varphi, B)\)-inner.

Similarly, we can prove above proposition for a Banach algebra \( A \) with a bounded left approximate identity, where the module action from the right is trivial.

By using above proposition and similar argument as in the proof of the Proposition 2.1.5 of [18], we can prove following proposition.

4.5. Proposition. Let \( A \) and \( B \) be two Banach algebras with bounded approximate identity, and \( \varphi : A \to B \) be a continuous homomorphism. Then the following two condition are equivalent:

(i) For each Banach \((\varphi, B)\)-bimodule \( X \), any continuous \((\varphi, B)\)-derivation on \( A \) is \((\varphi, B)\)-inner.

(ii) For each \((\varphi, B)\)-pseudounital Banach \((\varphi, B)\)-bimodule \( X \), any continuous \((\varphi, B)\)-derivation on \( A \) is \((\varphi, B)\)-inner.

The following proposition generalizes Proposition 2.1.6 of [18].

4.6. Proposition. Let \( A_1 \) and \( A_2 \) be two Banach algebras with bounded approximate identity which are closed ideals of Banach algebras \( B_1 \) and \( B_2 \), respectively. Let \( \varphi : A_1 \to A_2 \) be a continuous homomorphism and \( X \) be a \((\varphi, A_2)\)-bimodule, and \( \hat{\varphi} : B_1 \to B_2 \) be a continuous homomorphism such that \( \hat{\varphi} |_{A_1} = \varphi \). Let \( D : A_1 \to X^* \) be a \((\varphi, A_2)\)-derivation, then \( X \) is a Banach \((\hat{\varphi}, B_2)\)-bimodule and there is a unique \((\hat{\varphi}, B_2)\)-derivation \( \hat{D} : B_1 \to X^* \) satisfying the following:

(i) \( \hat{D} |_{A_1} = D \);

(ii) \( \hat{D} \) is continuous with respect to the strict topology on \( B_1 \) and the \( w^* \)-topology on \( X^* \).

Proof. For \( x \in X \), let \( \varphi(a_1) \in A_2 \) and \( y \in X \) be such that \( x = \varphi(a_1).y \). For \( b_1 \in B_1 \), define \( \tilde{\varphi}(b_1).x := \varphi(b_1a_1).y \). We claim that \( \tilde{\varphi}(b_1).x \) is well define. Let \( \varphi(a'_1) \in A_2 \) and
$y' \in X$ be such that $x = \varphi(a'_1).y'$, and let $(f_\beta)_\beta$ be a bounded approximate identity for $A_2$. Then

$$\tilde{\varphi}(b_1a_1).y = \lim_\beta \tilde{\varphi}(b_1)f_\beta\varphi(a_1).y = \lim_\beta \tilde{\varphi}(b_1)f_\beta\varphi(a'_1).y' = \tilde{\varphi}(b_1a_1).y' \ (b_1 \in B_1).$$

It is obvious that this operation of $\tilde{\varphi}(B_1)$ on $X$ turns $X$ into a left $\tilde{\varphi}(B_1)$-bimodule. Similarly, one defines a right Banach $\tilde{\varphi}(B_1)$-module structure on $X$, so that $X$ becomes a Banach $(\tilde{\varphi}, B_2)$-bimodule. Now we define $\tilde{D} : B_1 \rightarrow X^*$ by

$$\tilde{D}(b_1) = w^* - \lim_\alpha \left( D(b_1e_\alpha) - \tilde{\varphi}(b_1).D(e_\alpha) \right),$$

where $(e_\alpha)_\alpha$ is a bounded approximate identity for $A_1$. By the similar argument as in the proof of Proposition 3.1 of [7], one can show that $\tilde{D}$ is a $\varphi$-derivation on $B_1$ where $\tilde{D}|_{A_1} = D$ and $\tilde{D}$ is continuous with respect to the strict topology on $B_1$ and the $w^*$-topology on $X^*$.

4.7. Theorem. Let $S$ and $T$ be two foundation semigroups with identity, and $\varphi : S \rightarrow T$ be a continuous homomorphism, and $\tilde{\varphi}$ be as in Lemma 4.2. Then $M_\alpha(S)$ is $(\tilde{\varphi}, M_\alpha(T))$-amenable if and only if every $(\varphi, T)$-derivation on $S$ is $\varphi$-inner.

Proof. Suppose $M_\alpha(S)$ is $(\tilde{\varphi}, M_\alpha(T))$-amenable, and $D : S \rightarrow X^*$ is a $(\varphi, T)$-derivation on $S$ for some $\varphi$-module $X$. For every $\mu \in M_\alpha(S)$ and $x \in X$ we define

$$\tilde{\varphi}(\mu).x = \int_{\varphi(S)}t.xd\tilde{\varphi}(\mu)(t), \quad x.\tilde{\varphi}(\mu) = \int_{\varphi(S)}x.td\tilde{\varphi}(\mu)(t).$$

So for some $k > 0$,

$$\int_{\varphi(S)} \|t.x\|d|\tilde{\varphi}(\mu)|(t) \leq k\|x\||\tilde{\varphi}(\mu)|| < \infty.$$

Therefore $\tilde{\varphi}(\mu).x$ is well defined with $\|\tilde{\varphi}(\mu).x\| \leq k\|\tilde{\varphi}(\mu)||x||$. Similarly one can define $x.\tilde{\varphi}(\mu)$ satisfying $\|x.\tilde{\varphi}(\mu)\| \leq k\|x\||\tilde{\varphi}(\mu)||$. Thus $X$ defines a Banach $(\tilde{\varphi}, M_\alpha(T))$-bimodule. Using (4.3) and the definition of $\tilde{\varphi}$ we obtain

$$\tilde{\varphi}(\mu).x = \int_S \varphi(s).xd\mu(s), \quad x.\tilde{\varphi}(\mu) = \int_S x.\varphi(s)d\mu(s),$$

for all $x \in X, \mu \in M_\alpha(S)$. Define $\tilde{D} : M_\alpha(S) \rightarrow X^*$ by

$$\langle x, \tilde{D}(\mu) \rangle = \int_S \langle x, D(s) \rangle d\mu(s) \quad (\mu \in M_\alpha(S), x \in X).$$
Clearly, $\tilde{D}$ is continuous. By using (4.4), for every $\mu_1, \mu_2 \in M_a(S)$ and $x \in X$, we have

$$
\langle x, \tilde{D}(\mu_1 \ast \mu_2) \rangle = \int_S \langle x, D(s) \rangle d\mu_1 \ast \mu_2(s)
$$

$$
= \int_S \left( \int_S \langle x, D(s_1 s_2) \rangle d\mu_1(s_1) \right) d\mu_2(s_2)
$$

$$
= \int_S \left( \int_S \langle x, D(s_1) \varphi(s_2) + \varphi(s_1)D(s_2) \rangle d\mu_1(s_1) \right) d\mu_2(s_2)
$$

$$
= \int_S \int_S \langle \varphi(s_2)x, D(s_1) \rangle d\mu_1(s_1) d\mu_2(s_2)
$$

$$
+ \int_S \int_S \langle x, \varphi(s_1), D(s_2) \rangle d\mu_1(s_1) d\mu_2(s_2)
$$

$$
= \int_S \varphi(s_2)xd\mu_2(s_2), D(s_1))d\mu_1(s_1)
$$

$$
+ \int_S \langle \varphi(s_1), D(s_2) \rangle d\mu_1(s_1) d\mu_2(s_2)
$$

$$
= \int_S \langle \tilde{D}(\mu_2)x, D(s_1) \rangle d\mu_1(s_1) + \int_S \langle x, \tilde{D}(\mu_1)D(s_2) \rangle d\mu_2(s_2)
$$

$$
= \langle \tilde{D}(\mu_2)x, \tilde{D}(\mu_1) \rangle + \langle x, \tilde{D}(\mu_1) \rangle
$$

$$
= \langle x, \tilde{D}(\mu_1) \rangle + \tilde{\varphi}(\mu), \tilde{D}(\mu_2)
$$

That is $\tilde{D}(\mu_1 \ast \mu_2) = \tilde{D}(\mu_1) \circ \tilde{\varphi}(\mu) + \tilde{\varphi}(\mu_1) \circ \tilde{D}(\mu_2)$. Hence $\tilde{D}$ is a $(\tilde{\varphi}, M_a(T))$-derivation. From the $(\tilde{\varphi}, M_a(T))$-amenability of $M_a(S)$ it follows that there exists $x^* \in X^*$ such that $\tilde{D}(\mu) = \tilde{\varphi}(\mu)x^* - x^* \tilde{\varphi}(\mu)$ $(\mu \in M_a(S))$. Moreover, for every $x \in X$ and $\mu \in M_a(S)$, we have

$$
\int_S \langle x, D(s) \rangle d\mu(s) = \langle x, \tilde{D}(\mu) \rangle
$$

$$
= \langle x, \tilde{\varphi}(\mu)x^* - x^* \tilde{\varphi}(\mu) \rangle
$$

$$
= \langle x, \tilde{\varphi}(\mu)x^* \rangle - \langle x, x^* \tilde{\varphi}(\mu) \rangle
$$

$$
= \langle x, \int_S \varphi(s)x^* d\mu(s) \rangle - \langle x, \int_S x^* \varphi(s) d\mu(s) \rangle
$$

$$
= \int_S \langle x, \varphi(s)x^* - x^* \varphi(s) \rangle d\mu(s).
$$

So $\langle x, D(s) \rangle = \langle x, \varphi(s)x^* - x^* \varphi(s) \rangle$ $(x \in X)$, by Lemma 2.2 of [12]. Hence

$$
D(s) = \varphi(s)x^* - x^* \varphi(s) (s \in S).
$$

Therefore $D$ is $\varphi$-inner.

Conversely, suppose every $(\varphi, T)$-derivation on $S$ is $\varphi$-inner. Let $D : M_a(S) \rightarrow X^*$ be a $(\tilde{\varphi}, M_a(T))$-derivation for some Banach $(\tilde{\varphi}, M_a(T))$-bimodule $X$. By Proposition 4.5, there is no loss of generality if we suppose that $X$ is $\tilde{\varphi}$-pseudo-unital. So by Proposition 4.6, $X$ is a Banach $(\tilde{\varphi}, M(T))$-bimodule and there is a unique $(\tilde{\varphi}, M(T))$-derivation $\tilde{D} : M(S) \rightarrow X^*$ that extends $D$ and is continuous with respect to the strict topology on $M(S)$ and the $w^*$-topology on $X^*$. We consider the following module actions $\varphi(S)$ on $X$ by

$$
\varphi(s)x := \delta_{\varphi(s)}x, \quad x\varphi(s) := x\delta_{\varphi(s)} \quad (s \in S, x \in X),
$$

and define $D_S : S \rightarrow X^*$ by $D_S(s) = \tilde{D}(\delta_s)$ $(s \in S)$. It is easy to see that $D_S$ defines a $(\varphi, T)$-derivation. So there exists $x^* \in X^*$ such that

$$
D_S(s) = \varphi(s)x^* - x^* \varphi(s) \quad (s \in S).
$$
Consequently, for every \( s \in S \)
\[
\tilde{D}(\delta_s) = D_S(s) = \varphi(s).x^* - x^*.\varphi(s) = \tilde{\varphi}(\delta_s).x^* - x^*.\tilde{\varphi}(\delta_s).
\]
Since every measure \( \mu \) in \( M(S) \) is the \( s \)-lim of a net \( \langle \mu_i \rangle \) such that each \( \mu_i \) is a combination of point masses (see Theorem 3.3 of [14]), from the definition of the strict topology it follows that \( \nu * \mu_i \rightarrow \nu * \mu \) \((\nu \in M_0(S))\) and \( \mu_i * \nu \rightarrow \mu * \nu \) \((\nu \in M_0(S))\) in the norm topology. Let \( x \in X \), and \( \tilde{\varphi}(\nu) \in M_0(T) \) and \( y \in X \) be such that \( x = y.\tilde{\varphi}(\nu) \). Hence
\[
|\langle x, \tilde{\varphi}(\mu_i).x^* \rangle - \langle x, \tilde{\varphi}(\mu).x^* \rangle| = |\langle y, \tilde{\varphi}(\nu * \mu_i) - y, \tilde{\varphi}(\nu * \mu).x^* \rangle |
\]
\[
= |\langle y, \tilde{\varphi}(\nu * \mu_i) - y, \tilde{\varphi}(\nu * \mu).x^* \rangle |
\]
\[
\leq k\|y\|\|\tilde{\varphi}(\nu * \mu_i) - \tilde{\varphi}(\nu * \mu)\|\|x^*\| \rightarrow 0.
\]
This means that \( w^*-\lim \tilde{\varphi}(\mu_i).x^* = \tilde{\varphi}(\mu).x^* \). Similarly, \( w^*-\lim x^*.\tilde{\varphi}(\mu_i) = x^*.\tilde{\varphi}(\mu) \).

Now for every \( \mu \in M(S) \), we obtain
\[
\tilde{D}(\mu) = \tilde{D}(s - \lim_i \mu_i) = w^* - \lim_i \tilde{D}(\mu_i)
\]
\[
= w^* - \lim_i \left( \tilde{\varphi}(\mu_i).x^* - x^*.\tilde{\varphi}(\mu_i) \right)
\]
\[
= \tilde{\varphi}(\mu).x^* - x^*.\tilde{\varphi}(\mu).
\]
Thus \( \tilde{D} \) is a \((\tilde{\varphi}, M(T))\)-inner derivation and so \( D \) is \((\varphi, M_0(T))\)-inner derivation. Therefore \( M_0(S) \) is \((\tilde{\varphi}, M_0(T))\)-amenable. \( \square \)

A combination of Proposition 3.6 and Theorem 4.7, gives the following result.

4.8. Theorem. Let \( S \) and \( T \) be two foundation semigroups with identity, and let \( \varphi : S \rightarrow T \) be a continuous homomorphism, and \( \tilde{\varphi} \) be as in Lemma 4.2. If \( M_0(S) \) is \((\tilde{\varphi}, M_0(T))\)-amenable, then \( T \) is \( \varphi \)-amenable.

Before turning the next result, we first need to prove the following proposition.

4.9. Proposition. Let \( A \) and \( B \) be two Banach algebras and let \( \varphi : A \rightarrow B \) be a continuous homomorphism. If \( \varphi(A) \) is dense in \( B \) and \( A \) is \((\varphi,B)\)-amenable, then \( B \) is amenable.

Proof. Let \( D : B \rightarrow X^* \) be a continuous derivation for a Banach \( B \)-bimodule \( X \), and \( \tilde{D} = D \circ \varphi \). Obviously \( \tilde{D} \) defines a continuous \((\varphi,B)\)-derivation from \( A \) into \( X^* \). By \((\varphi,B)\)-amenability of \( A \) there exists \( f \in X^* \) such that
\[
\tilde{D}(a) = \varphi(a).f - f.\varphi(a) \quad (a \in A).
\]
Let \( b \in B \), since \( \varphi(A) \) is dense in \( B \), there exists a net \( \{a_\alpha\} \subset A \) such that \( \lim_\alpha \varphi(a_\alpha) = b \). Hence
\[
D(b) = \lim \varphi(a_\alpha) = \lim \tilde{D}(a_\alpha) = \lim (\varphi(a_\alpha).f - f.\varphi(a_\alpha)) = b.f - f.b.
\]
Thus \( D \) is an inner derivation. This completes the proof. \( \square \)

Note that if \( S \) is a discrete semigroup then \( S \) is a foundation semigroup with \( LUC(S) = l^\infty(S) \), and \( M(S) = M_0(S) = l^1(S) \).

The next example shown that the converses of the Theorem 4.8 and Proposition 3.6 are not true.

4.10. Example. Let \( S \) be the set \( \mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\} \), with the product
\[
(m, n) \rightarrow m \lor n = \max\{m, n\}, \quad \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}.
\]
Define \( \varphi : S \rightarrow S \) by \( \varphi(s) = s + 1 \). Then it is easy to check that \( \varphi \) is a homomorphism on \( S \). Since \( S \) is an abelian semigroup it follows \( S \) is amenable (see [3]) and so by Lemma 3.5, \( S \) is \( \varphi \)-amenable. Since \( E(S) = S \), from Corollary 1 of [5], the
convolution semigroup algebra $l^1(S)$ is not amenable. Also since $\varphi$ is an epimorphism on $S$ it follows that $\tilde{\varphi}$ is an epimorphism on $l^1(S)$. Therefore by Proposition 4.9, $l^1(S)$ is not $\tilde{\varphi}$-amenable. So the converse of the Theorem 4.8 is not valid. Also by Theorem 4.7, we conclude that the converse of the Proposition 3.6 is not true.

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References
