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Research Article

Isometry Group of Truncated Truncated Cube and Truncated Truncated Octahedron Space

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Abstract
Polyhedra have been extensively studied and examined by scientists, especially geometers and also mathematicians over the years, primarily due to their rich symmetry properties. In particular, there exist notable connections between certain metrics and polyhedral shapes. For instance, it has been demonstrated that the unit spheres corresponding to the maximum metric, taxicab metric, and Chinese Checkers metric are represented, respectively, by the cube, the octahedron, and the deltoidal icositetrahedron.

In the present work, we give two novel metrics and prove that the unit spheres in the associated three-dimensional analytic spaces take the form of the *truncated truncated cube* and the *truncated truncated octahedron*, respectively. Furthermore, we examine several fundamental properties of these metrics. In addition, we prove that the isometry groups of the three-dimensional spaces equipped with the TTC-metric and the TTO-metric are isomorphic to the semidirect product $O_h \ltimes T(3)$, where O_h denotes the octahedral group—the Euclidean symmetry group of the octahedron—and $T(3)$ denotes the translation group consisting of all translations in $\mathbb{R}^3$.

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1. Introduction

Polyhedra are fundamental three-dimensional geometric constructs composed of polygonal faces, linear edges, and distinct vertices. These solids occupy a central role in various branches of mathematics and natural sciences due to their rich structural properties and symmetry. The term polyhedron originates from the Greek words poly (many) and hedron (face or base), emphasizing their multi-faceted nature. Depending on the arrangement of their faces and vertices, polyhedra may be categorized as convex or non-convex. While non-convex polyhedra exhibit more intricate and often self-intersecting structures, convex polyhedra—where any line segment joining two interior points lies entirely within the solid—are of particular mathematical significance and have been the subject of extensive historical and modern inquiry.

Convex polyhedra are traditionally classified into three principal families: Platonic solids, Archimedean solids, and Catalan solids. The Platonic solids, recognized since ancient times and thoroughly examined by the Greeks of old, are noted for their remarkable uniformity: every face comprises a matching regular polygon, and an identical number of faces converge at each vertex. There exist only five such solids—the tetrahedron, cube (or hexahedron), octahedron, dodecahedron, and icosahedron—each possessing complete symmetry under both rotations and reflections. In classical antiquity, these solids were famously linked by Plato to the fundamental elements of nature, thereby imbuing them with both philosophical significance and symbolic meaning in ancient cosmology.

The Archimedean solids, named after the ancient Greek mathematician Archimedes (although the first formal accounts are found in the works of Pappus of Alexandria), are semi-regular polyhedra consisting of two or more types of regular polygons, arranged identically around each vertex. There are precisely thirteen such solids, not including their mirror images. These include well-known forms such as the truncated icosahedron (famous for its resemblance to a soccer ball) and the cuboctahedron. Their rediscovery during the Renaissance, spurred by the emergence of perspective in art and the geometric curiosity of artists such as Leonardo da Vinci and Piero della Francesca, marked a significant milestone in the fusion of mathematical theory and visual representation.

The Catalan solids form the dual set of the Archimedean solids. While all are convex like their duals, their faces are not regular polygons, and they are not vertex-transitive, but rather face-transitive. Introduced by Eugène Charles Catalan in the 19th century, these thirteen solids deepen the understanding of duality and symmetry in polyhedral geometry and are essential in many applied fields, including crystallography and materials science (For more detail see [1–3]).

Beyond Euclidean geometry, polyhedra emerge naturally in the context of Minkowski geometry, a class of normed vector spaces in finite dimensions where the standard Euclidean distance is replaced by a more general metric. In these spaces, while the notions of linearity (points, lines, planes) and angle measurement remain unchanged, the concept of distance—and consequently, the shape of the unit sphere—depends on the specific norm employed. Unlike the perfect roundness of the Euclidean sphere, the unit ball in Minkowski geometry can assume various convex shapes, often corresponding to polyhedral forms.

This connection between metrics and polyhedral unit spheres has led to a rich line of inquiry in metric geometry. For example, in three-dimensional space, the unit ball under the maximum (Chebyshev) metric is a cube, and under the taxicab (Manhattan) metric, it is an octahedron—both of which are Platonic solids. Other metrics yield unit spheres that correspond to Archimedean or Catalan solids. For instance, the unit sphere associated with the Chinese Checkers metric is the deltoidal icositetrahedron, a Catalan solid. Such findings prompt an intriguing inverse problem in geometry: given a convex polyhedron, can one construct a metric space whose unit ball is precisely that polyhedron? Addressing this question not only enriches the understanding of polyhedral geometry but also contributes to the broader theory of normed spaces. (See [4–23]).

In this study, we extend the known catalog of such correspondences by constructing new metrics whose unit spheres are specific convex polyhedra not previously associated with known metrics. Specifically, we present metrics for which the unit balls are the truncated truncated cube and the truncated truncated octahedron. These polyhedra are derived by systematically truncating certain vertices and edges of known Archimedian solids, yielding new convex structures with unique geometric and symmetrical properties. Furthermore, we investigate the isometry groups of the resulting metric spaces and demonstrate that these groups can be expressed as semidirect products of classical polyhedral symmetry groups (e.g., the octahedral or icosahedral group) with the full translation group $T(3)$ of $\mathbb{R}^3$.

Ultimately, this research contributes to the theoretical framework linking polyhedral geometry and metric space theory and may offer new insights into the geometric modeling of natural phenomena, optimization problems in computational geometry, and the design of spatial structures in architecture and engineering.

2. Truncated truncated cube metric and some properties

As discussed in [3], the class of regular polyhedra admits numerous systematic modifications that preserve, adapt, or extend their inherent symmetries. Of particular importance is a distinguished set of eleven solids generated through the truncation of vertices—and, in certain instances, the edges—of regular polyhedra, carried out in such a way that all faces of the resulting structures remain regular polygons. These polyhedra, first described by Archimedes, are collectively known as *Archimedean solids*. A hallmark of this family is the congruence of their vertices, coupled with the presence of two or more distinct types of regular polygons among their faces. This distinctive combination of vertex-uniformity and polygonal variety justifies their alternative designation as *semiregular polyhedra*, setting them apart from the Platonic solids, whose faces are uniformly identical.

Archimedes further demonstrated that, beyond these eleven canonical solids, there exist two additional semiregular polyhedra: the *snub cube* and the *snub dodecahedron*. These solids share the vertex-transitive property of the Archimedean family while exhibiting a more intricate spatial arrangement of faces. A notable subset of the Archimedean solids can be derived from the Platonic solids through a truncation procedure that removes less than half of each vertex [24]. Among these are the *truncated cube* and the *truncated octahedron*, both of which retain a high degree of symmetry while offering a greater variety of face types.

The truncated cube and truncated octahedron also serve as natural starting points for further iterative truncation processes, giving rise to more complex polyhedral forms. In particular, the application of successive truncations yields solids such as the *truncated truncated cube* and the *truncated truncated octahedron*, which not only preserve the essential vertex-uniform property but also display a richer diversity of polygonal faces and enhanced visual complexity. These higher-order truncations occupy an important position in the broader study of polyhedral geometry, providing fertile ground for investigating symmetry groups, metric properties, and geometric transformations.

An illustrative example of a polyhedron obtained via successive truncations is the truncated truncated cube. This solid features six quadratically symmetric hexadecagonal faces, eight regular hexagonal faces, and twenty-four isosceles triangular faces, with a total of seventy-two vertices and one hundred eight edges. The truncated truncated cube is generated by applying two consecutive truncations to a cube: the first truncation produces the truncated cube, followed by a second truncation yielding the truncated truncated cube. Figure 1 sequentially depicts the cube, truncated cube, truncated truncated cube, and a transparent rendering of the truncated truncated cube.

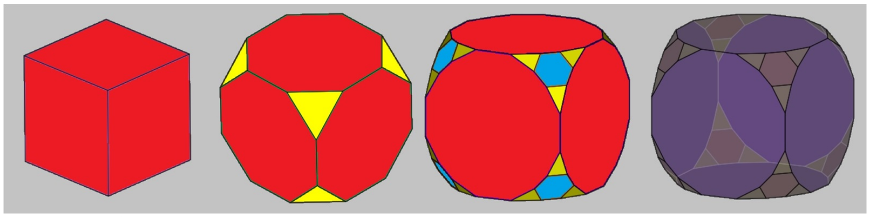

Fig. 1. The cube, the truncated cube and the truncated truncated cube

The metric for which the unit sphere corresponds to the truncated truncated cube is defined as follows:

Definition 1. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. Denote by U, V, and W the maximum, median, and minimum, respectively, of the set $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$. The truncated truncated cube distance function*

$$d_{TTC} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$$

between P_1 and P_2 is then defined by

$$d_{TTC}(P_1, P_2) = \max\{U,\ k_1(U+V) + k_2 W,\ k_3(U+V+W)\},$$

where the constants are given by

$$k_1 = \frac{20+2\sqrt{2}}{49}, \quad k_2 = \frac{-1+2\sqrt{2}}{7}, \quad \text{and} \quad k_3 = \sqrt{2} - 1.$$

With respect to the truncated truncated cube metric, there exist three distinct types of paths connecting points P_1 and P_2:
Type I: A single line segment parallel to one of the coordinate axes.
Type II: A concatenation of three line segments, each parallel to a coordinate axis.
Type III: A concatenation of three line segments, two of which are parallel to coordinate axes, while the third forms an angle

$$\arctan\left(\frac{30+2\sqrt{2}}{56}\right)$$

with the remaining coordinate axis.

Accordingly, the truncated truncated cube distance between P_1 and P_2 is defined as follows:

1. For *Type I*, it equals the Euclidean length of the line segment.
2. For *Type II*, it is $(\sqrt{2} - 1)$ times the sum of the Euclidean lengths of the three segments.
3. For *Type III*, it is

   $$\frac{20+2\sqrt{2}}{49}$$

   times the sum of the Euclidean lengths of the three segments.

In the case where

$$|y_1 - y_2| \geq |x_1 - x_2| \geq |z_1 - z_2|,$$

Figure 2 illustrates examples of truncated truncated cube paths between P_1 and P_2.

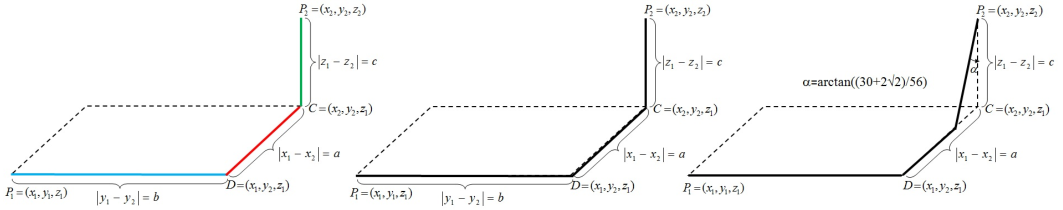

Fig. 2. Some TTC way from P_1 to P_2

It is a well-established fact that, in the three-dimensional analytic space $(\mathbb{R}^3, d_\infty)$ equipped with the maximum metric

$$d_\infty(P_1,P_2) = \max\{|x_1-x_2|, |y_1-y_2|, |z_1-z_2|\},$$

the unit sphere is geometrically represented by a cube whose edges are aligned with the coordinate axes. This follows from the nature of the L_∞-norm, which measures distance solely by the largest coordinate difference between two points.

In [16], the authors introduced a novel metric that modifies this classical geometry, yielding unit spheres in $\mathbb{R}^3$ that take the form of *truncated cubes*. This construction serves as a polyhedral analogue to the maximum metric, altering its symmetry by selectively truncating vertices while preserving convexity.

Let $P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$ be two arbitrary points in $\mathbb{R}^3$. Here, U, V, and W represent, respectively, the largest, intermediate, and smallest coordinate differences between P_1 and P_2.

The two metrics defined in [16] are given by:

$$d_M(P_1,P_2) = U, \quad d_{TC}(P_1,P_2) = \max\left\{U, (\sqrt{2}-1)\big(U+V+W\big)\right\}.$$

The metric d_M coincides with the classical maximum metric, producing cubic unit spheres. In contrast, the metric d_{TC} incorporates an additional term proportional to the sum of all coordinate differences, scaled by the factor $(\sqrt{2}-1)$. This modification geometrically corresponds to truncating the cube's vertices in a symmetric manner, thereby generating the truncated cube as the shape of its unit sphere.

Lemma 2. *Let $P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$ be two distinct points in $\mathbb{R}^3$. Denote by U_{12}, V_{12}, and W_{12} the maximum, the median, and the minimum, respectively, of the set*

$$\{|x_1-x_2|, |y_1-y_2|, |z_1-z_2|\}.$$

Then the following inequalities hold:

$$\begin{aligned}
&d_{TTC}(P_1,P_2) \geq U_{12},\\
&d_{TTC}(P_1,P_2) \geq k_1\,(U_{12}+V_{12}) + k_2 V_{12},\\
&d_{TTC}(P_1,P_2) \geq k_3\,(U_{12}+V_{12}+W_{12}),
\end{aligned}$$

where

$$k_1 = \frac{20+2\sqrt{2}}{49}, \quad k_2 = \frac{-1+2\sqrt{2}}{7}, \quad k_3 = \sqrt{2}-1.$$

Proof. The proof follows directly from the definition of the maximum function. ■

Theorem 3. *The distance function d_{TTC} defines a metric on $\mathbb{R}^3$. Furthermore, with respect to d_{TTC}, the unit sphere corresponds to a truncated truncated cube in $\mathbb{R}^3$.*

Proof. Let

$$d_{TTC} : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow [0,\infty)$$

be the truncated truncated cube distance function. Consider three distinct points $P_1 = (x_1,y_1,z_1)$, $P_2 = (x_2,y_2,z_2)$, and $P_3 = (x_3,y_3,z_3)$ in $\mathbb{R}^3$.

For any pair of points P_i and P_j, denote by U_{ij}, V_{ij}, and W_{ij} the maximum, the middle, and the minimum, respectively, of the set

$$\left\{|x_i-x_j|, |y_i-y_j|, |z_i-z_j|\right\}.$$

We verify the metric axioms:

(M1) Non-negativity and identity of indiscernibles: Since absolute values are always nonnegative, $d_{TTC}(P_1,P_2) \geq 0$. If $d_{TTC}(P_1,P_2) = 0$, then

$$\max\{U,\ k_1(U+V)+k_2W,\ k_3(U+V+W)\} = 0,$$

where U, V, W correspond to P_1 and P_2. This implies $U = 0$, $V = 0$, and $W = 0$, hence $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$, and therefore $P_1 = P_2$.

(M2) Symmetry: Since $|x_1 - x_2| = |x_2 - x_1|$ (and similarly for y and z coordinates), it follows immediately that

$$d_{TTC}(P_1,P_2) = d_{TTC}(P_2,P_1).$$

(M3) Triangle inequality: From the definition,

$$\begin{aligned} d_{TTC}(P_1,P_3) &= \max\{U_{13},\ k_1(U_{13}+V_{13})+k_2W_{13},\ k_3(U_{13}+V_{13}+W_{13})\} \\ &\leq \max\left\{U_{12}+U_{23},\ k_1(U_{12}+U_{23}+V_{12}+V_{23})+k_2(W_{12}+W_{23}), k_3(U_{12}+U_{23}+V_{12}+V_{23}+W_{12}+W_{23})\right\} \\ &=: I. \end{aligned}$$

By Lemma 2, $I \leq d_{TTC}(P_1,P_2) + d_{TTC}(P_2,P_3)$. Therefore, the triangle inequality holds.

Thus, d_{TTC} satisfies all metric axioms on $\mathbb{R}^3$. Geometrically, the unit sphere corresponds to a truncated truncated cube in $\mathbb{R}^3$, as illustrated in Figure 3. ■

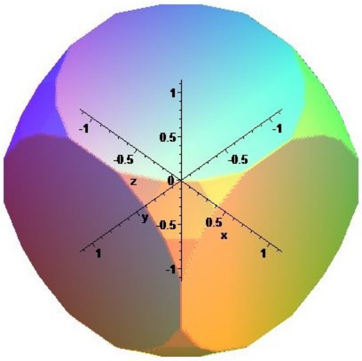

Fig. 3. The unit sphere in terms of d_{TTC}: truncated truncated cube

Corollary 4. *The equation defining the truncated truncated cube centered at* (x_0,y_0,z_0) *with radius* r *is given by*

$$\max\{U_0,\ k_1(U_0+V_0)+k_2W_0,\ k_3(U_0+V_0+W_0)\} = r,$$

where U_0, V_0, *and* W_0 *denote the maximum, median, and minimum, respectively, of the set* $\{|x-x_0|,|y-y_0|,|z-z_0|\}$. *This equation characterizes a polyhedron with* 38 *faces and* 72 *vertices.*

The vertices' coordinates are obtained by translating the center (x_0,y_0,z_0) *by all permutations of the components and all possible sign changes of the vectors*

$$\left(\frac{2\sqrt{2}-1}{7}r,\quad r,\quad r\right) \quad \textit{and} \quad \left(\frac{2\sqrt{2}-1}{3}r,\quad \frac{\sqrt{2}+1}{3}r,\quad r\right).$$

Lemma 5. *Let*

$$k_1 = \frac{20+2\sqrt{2}}{49}, \quad k_2 = \frac{-1+2\sqrt{2}}{7}, \quad k_3 = \sqrt{2}-1.$$

Consider the line l *passing through points* $P_1 = (x_1,y_1,z_1)$ *and* $P_2 = (x_2,y_2,z_2)$ *in the three-dimensional Euclidean space, and denote by* d_E *the Euclidean metric. If* l *has direction vector* (p,q,r), *then the truncated truncated cube distance satisfies*

$$d_{TTC}(P_1,P_2) = \mu(P_1P_2)\, d_E(P_1,P_2),$$

where

$$\mu(P_1P_2) = \frac{\max\{U_d, k_1(U_d+V_d)+k_2W_d, k_3(U_d+V_d+W_d)\}}{\sqrt{p^2+q^2+r^2}},$$

and U_d, V_d, W_d *denote the maximum, median, and minimum of the set* $\{|p|,|q|,|r|\}$*, respectively.*

Proof. The parametric equations of l yield

$$x_1 - x_2 = \lambda p, \quad y_1 - y_2 = \lambda q, \quad z_1 - z_2 = \lambda r, \quad \lambda \in \mathbb{R}.$$

Consequently,

$$d_{TTC}(P_1,P_2) = |\lambda| \cdot \max\{U_d, k_1(U_d+V_d)+k_2W_d, k_3(U_d+V_d+W_d)\},$$

where U_d, V_d, and W_d are used by the same meaning like as the Lemma 5. Moreover, the Euclidean distance is

$$d_E(P_1,P_2) = |\lambda|\sqrt{p^2+q^2+r^2},$$

which directly implies the stated result. ■

The above lemma demonstrates that the d_{TTC}-distance along any line is a positive scalar multiple of the Euclidean distance on the same line. From this, the following corollaries immediately follow:

Corollary 6. *If* P_1, P_2*, and* X *are three collinear points in* $\mathbb{R}^3$*, then*

$$d_E(P_1,X) = d_E(P_2,X) \quad \text{if and only if} \quad d_{TTC}(P_1,X) = d_{TTC}(P_2,X).$$

Corollary 7. *For any three distinct collinear points* P_1, P_2*, and* X *in* $\mathbb{R}^3$*, the following ratio holds:*

$$\frac{d_{TTC}(X,P_1)}{d_{TTC}(X,P_2)} = \frac{d_E(X,P_1)}{d_E(X,P_2)}.$$

In other words, the ratios of the Euclidean and d_{TTC} *distances along any line coincide.*

3. Truncated truncated octahedron metric and some properties

The truncated truncated octahedron is a polyhedron derived through the successive application of the truncation operation to the regular octahedron. In the first stage, truncating the octahedron at its vertices produces the truncated octahedron, a well-known Archimedean solid characterized by regular hexagonal and square faces. Applying a second truncation to each vertex of this intermediate solid yields the truncated truncated octahedron, a more complex polyhedron exhibiting a richer face structure.

This solid is composed of eight tri-symmetric dodecagonal faces, six regular octagonal faces, and twenty-four isosceles triangular faces. In total, it possesses seventy-two vertices and one hundred eight edges. An interesting combinatorial property is that the truncated truncated octahedron shares the same face count, vertex count, and edge count as the truncated truncated cube, despite the differences in face geometry and symmetry group.

From a symmetry perspective, the truncated truncated octahedron retains aspects of the octahedral symmetry group O_h, although the double truncation process alters the arrangement and proportions of its faces, leading to a more intricate yet still highly symmetrical structure.

Figure 4 depicts, in order, the regular octahedron, its truncated form, the truncated truncated octahedron, and a transparent rendering of the latter, illustrating the progression of transformations and the resulting geometric complexity.

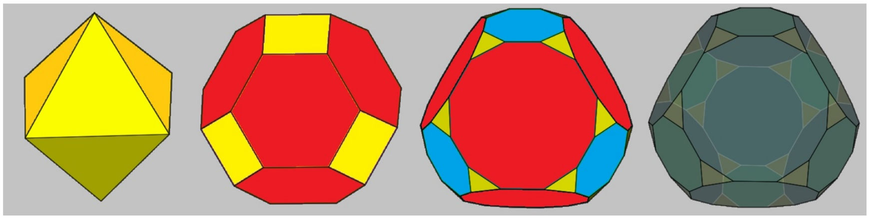

Fig. 4. Octahedron, truncated octahedron and truncated truncated octahedron

The metric that unit sphere is the truncated truncated octahedron is described as follows:

Definition 8. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. U, V, W are used by the same meaning like as the Definition 1. The distance function $d_{TTO} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ truncated truncated octahedron distance between P_1 and P_2 is defined by*

$$d_{TTO}(P_1, P_2) = \max\left\{U, k_1\left(U + V + W\right), k_2 U + k_3 V\right\}$$

where $k_1 = \dfrac{52 + 7\sqrt{10} - 9\sqrt{2} - 4\sqrt{5}}{82}$, $k_2 = \dfrac{62 - 2\sqrt{10} - 17\sqrt{2} + 12\sqrt{5}}{71}$ *and* $k_3 = \dfrac{63 + 7\sqrt{10} - 23\sqrt{2} - 13\sqrt{5}}{71}$.

According to the truncated truncated octahedron metric, there exist three distinct types of paths connecting the points P_1 and P_2. These paths are classified as follows:

1. A single line segment parallel to one of the coordinate axes,
2. A concatenation of three line segments, each parallel to a coordinate axis,
3. A union of two line segments, where one segment is parallel to a coordinate axis, and the other forms an angle of

$$\arctan\left(\frac{35087490 - 35144881\sqrt{10} + 40409025\sqrt{2} + 433431354\sqrt{5}}{913258232}\right)$$

with another coordinate axis.

Consequently, the *truncated truncated octahedron distance* between two points P_1 and P_2 can be expressed in terms of the Euclidean lengths of specific segments, depending on the geometric configuration under consideration. In case (i), this distance coincides exactly with the Euclidean length of the straight-line segment joining P_1 and P_2. In case (ii), the distance is given by k_1 times the sum of the Euclidean lengths of three consecutive segments, each aligned with a distinct principal direction determined by the underlying polyhedral metric. Finally, in case (iii), the distance equals k_2 times the sum of the Euclidean lengths of two such segments, corresponding to a reduced set of directional transitions.

Figure 5 presents representative examples of *truncated truncated octahedron paths* connecting P_1 and P_2, constructed under the coordinate ordering condition

$$|y_1 - y_2| \;\geq\; |x_1 - x_2| \;\geq\; |z_1 - z_2|.$$

This inequality reflects a specific hierarchy among the coordinate differences, thereby determining which of the above cases applies. The figure also illustrates how the metric adapts to this ordering, resulting in path structures that differ significantly from those induced by the Euclidean metric, while still preserving the characteristic symmetries of the truncated truncated octahedron.

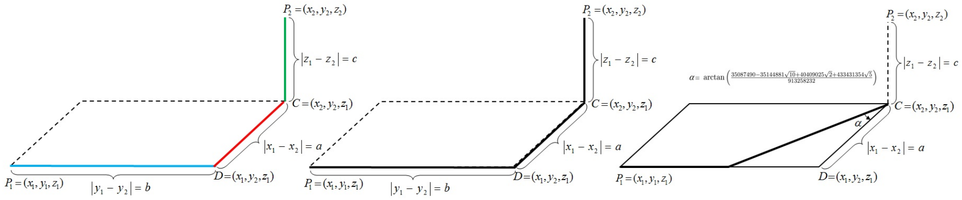

Fig. 5. TTO way from P_1 to P_2

It is a classical result that the unit sphere in the three-dimensional Euclidean space $(\mathbb{R}^3, d_T)$ endowed with the taxicab metric

$$d_T(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|$$

is an octahedron. This correspondence arises from the fact that the taxicab norm measures distance as the sum of absolute coordinate differences, thereby inducing a polyhedral geometry whose unit ball is the convex hull of the coordinate axes' extremal points.

In [16], the authors extend this framework by introducing alternative metrics on $\mathbb{R}^3$ whose unit spheres are realized as *truncated octahedra*. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two arbitrary points in $\mathbb{R}^3$. U, V, W are used by the same meaning like as the Definition1. Here, U, V, and W respectively represent the largest, intermediate, and smallest coordinate differences between P_1 and P_2.

The two polyhedral metrics considered in [16] are then defined as

$$d_M(P_1,P_2) = U+V+W,$$

$$d_{TC}(P_1,P_2) = \max\left\{U,\ \frac{2}{3}\left(U+V+W\right)\right\}.$$

The metric d_M generalizes the taxicab distance by weighting each coordinate difference equally, yet in an ordered fashion based on magnitude, while d_{TC} incorporates a balancing mechanism between the maximal coordinate difference and a scaled sum of all differences. Geometrically, these constructions deform the octahedral unit sphere of the taxicab metric into a truncated octahedron, while preserving convexity and central symmetry.

Theorem 9. *The function d_{TTO} defines a metric on $\mathbb{R}^3$. Moreover, the unit sphere with respect to d_{TTO} is a truncated truncated octahedron in $\mathbb{R}^3$.*

Proof. The verification that d_{TTO} satisfies the metric axioms can be carried out analogously to the proof of Theorem 3. The geometric representation of the unit sphere with respect to d_{TTO} is illustrated in Figure 6: ■

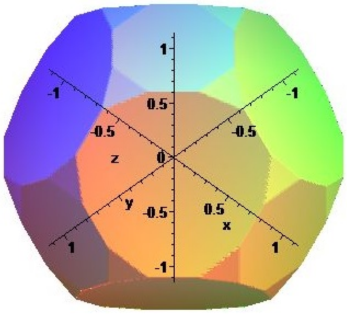

Fig. 6. The unit sphere with respect to d_{TTO}, namely the truncated truncated octahedron

Corollary 10. *The equation of a truncated truncated octahedron centered at (x_0,y_0,z_0) with radius r is given by*

$$\max\left\{U_0, k_1\left(U_0+V_0+W_0\right), k_2U_0+k_3V_0\right\} = r,$$

where U_0,V_0,W_0 are used by the same meaning like as the Corollary 4. This defines a polyhedron with 38 faces and 72 vertices. The vertices are obtained by translating all permutations and all possible sign variations of the components of $\left(\frac{1+2\sqrt{2}-\sqrt{5}}{4}r, \frac{3-\sqrt{2}+\sqrt{5}-\sqrt{10}}{4}r, r\right)$ and $\left(\frac{5-\sqrt{2}+\sqrt{5}-\sqrt{10}}{4}r, \frac{3+2\sqrt{2}-\sqrt{5}}{4}r, 0\right)$ to the center (x_0,y_0,z_0).

Lemma 11. *Let*

$$k_1 = \frac{52+7\sqrt{10}-9\sqrt{2}-4\sqrt{5}}{82}, \quad k_2 = \frac{62-2\sqrt{10}-17\sqrt{2}+12\sqrt{5}}{71}, \quad k_3 = \frac{63+7\sqrt{10}-23\sqrt{2}-13\sqrt{5}}{71}.$$

Suppose l is the line passing through points $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$ in $\mathbb{R}^3$, and let d_E indicate the Euclidean metric. If l has direction vector (p,q,r), then

$$d_{TTO}(P_1,P_2) = \mu(P_1P_2)\, d_E(P_1,P_2),$$

where

$$\mu(P_1P_2) = \frac{\max\left\{U_d,\ k_1(U_d+V_d+W_d),\ k_2U_d+k_3V_d\right\}}{\sqrt{p^2+q^2+r^2}},$$

and U_d,V_d,W_d are used by the same meaning like as the Lemma 5.

Proof. The parametric equation of the line l yields

$$x_1-x_2=\lambda p, \quad y_1-y_2=\lambda q, \quad z_1-z_2=\lambda r, \quad \lambda\in\mathbb{R}.$$

Thus,

$$d_{TTO}(P_1,P_2) = |\lambda| \cdot \max\left\{U_d,\, k_1(U_d+V_d+W_d),\, k_2U_d+k_3V_d\right\},$$

where U_d, V_d, W_d are as defined above, and the Euclidean distance satisfies

$$d_E(P_1,P_2) = |\lambda|\sqrt{p^2+q^2+r^2}.$$

This completes the proof. ■

The prior lemma suggests that the d_{TTO}-distance measured along any line is a positive scalar multiple of the Euclidean distance along that identical line. As a result, the ensuing corollaries naturally follow:

Corollary 12. *Let P_1, P_2, and X be three collinear points in $\mathbb{R}^3$. Then*

$$d_E(P_1,X) = d_E(P_2,X) \quad \text{if and only if} \quad d_{TTO}(P_1,X) = d_{TTO}(P_2,X).$$

Corollary 13. *If P_1, P_2, and X are three distinct collinear points in $\mathbb{R}^3$, then*

$$\frac{d_{TTO}(X,P_1)}{d_{TTO}(X,P_2)} = \frac{d_E(X,P_1)}{d_E(X,P_2)}.$$

That is, the ratios of the Euclidean distances and the d_{TTO}-distances along a line coincide.

4. Isometry group of truncated truncated octahedron and truncated truncated cube spaces

Three fundamental approaches form the basis of geometric investigations: the *synthetic*, *metric*, and *group-theoretic* methods. The synthetic approach focuses on axiomatic reasoning and direct geometric constructions, the metric approach emphasizes the quantitative study of distances and angles, while the group-theoretic method centers on the analysis of isometry groups associated with a given geometry. Within the group-theoretic framework, convex sets play a pivotal role in characterizing isometry groups, as they frequently serve as fundamental domains or unit balls for the underlying metric space. Consequently, the primary objects of investigation in this context are geometric properties invariant under the group of motions. A considerable body of literature has explored the structure and classification of isometry groups in various spaces (see [7, 11, 12]).

As noted in the Introduction, *Minkowski geometry* preserves the same linear structure as Euclidean space, yet it is distinguished by the presence of an *anisotropic* metric. In this framework, the role of the Euclidean sphere as the unit ball is replaced by a specific *symmetric*, *closed*, and *convex* set. The geometric and symmetry properties of this set determine many of the fundamental characteristics of the associated geometry.

In [25], the author establishes the following theorem, which provides a foundational result connecting the geometry of the unit ball with the structure of the isometry group, thereby deepening the understanding of Minkowski spaces through a group-theoretic lens.

Theorem 14. *If the unit ball C in the normed space $(V, \|\cdot\|)$ does not intersect any two-dimensional subspace in an ellipse, then the isometry group $I(3)$ of $(V, \|\cdot\|)$ is isomorphic to the semidirect product of the translation group $T(3)$ of $\mathbb{R}^3$ and a finite subgroup of the group of linear transformations with determinant ± 1.*

This theorem naturally raises a fundamental question: which specific subgroup corresponds to the linear component of this semidirect product?

In this paper, we demonstrate that the isometry groups of the three-dimensional spaces endowed with the TTC-metric and TTO-metric are precisely the semidirect product of the octahedral group O_h and the translation group $T(3)$. Here, O_h denotes the Euclidean symmetry group of the octahedron, while $T(3)$ consists of all translations in $\mathbb{R}^3$. For the remainder of this article, we denote by $\triangle$ either TTC or TTO, i.e., $\triangle \in \{TTC, TTO\}$.

Definition 15. *Let $P, Q \in \mathbb{R}^3_{\triangle}$. The* minimum distance set *between P and Q is defined as*

$$[PQ] := \{X \mid d_{\triangle}(P,X) + d_{\triangle}(Q,X) = d_{\triangle}(P,Q)\}.$$

In general, $[PQ]$ corresponds to a hexagonal dipyramid in $\mathbb{R}^3_{TTC}$, which need not be uniform, as illustrated in Figure 7-(a). In contrast, $[PQ]$ represents an octagonal dipyramid with diagonal PQ in $\mathbb{R}^3_{TTO}$, as shown in Figure 7-(b).

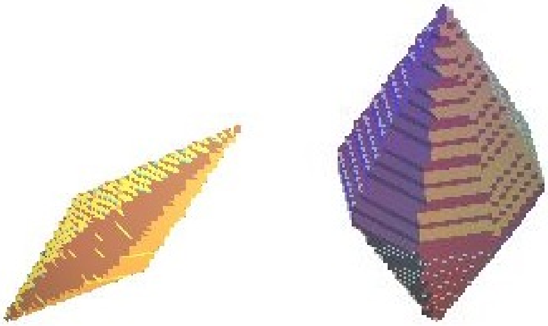
(a) (b)

Fig. 7. (a) The minimum distance sets of spaces of truncated truncated cube, (b) truncated truncated octahedron

Proposition 16. *Let $\phi : \mathbb{R}^3_{\triangle} \to \mathbb{R}^3_{\triangle}$ be an isometry, and let $[PQ]$ denote the minimum distance set between points P and Q. Then,*

$$\phi([PQ]) = [\phi(P)\phi(Q)].$$

Proof. For any $Y \in \phi([PQ])$, there exists $X \in [PQ]$ such that $Y = \phi(X)$. By definition,

$$d_{\triangle}(P,X) + d_{\triangle}(Q,X) = d_{\triangle}(P,Q).$$

Applying the isometry ϕ, we obtain

$$d_{\triangle}(\phi(P),\phi(X)) + d_{\triangle}(\phi(Q),\phi(X)) = d_{\triangle}(\phi(P),\phi(Q)),$$

which implies

$$Y = \phi(X) \in [\phi(P)\phi(Q)].$$

■

Corollary 17. *Let $\phi : \mathbb{R}^3_{\triangle} \to \mathbb{R}^3_{\triangle}$ be an isometry and $[PQ]$ the corresponding minimum distance set. Then ϕ maps vertices of $[PQ]$ to vertices and preserves the lengths of its edges.*

Proposition 18. *Let $\phi : \mathbb{R}^3_{\triangle} \to \mathbb{R}^3_{\triangle}$ be an isometry such that $\phi(O) = O$. Then $\phi \in O_h$.*

Proof. Since $\triangle \in \{TTC, TTO\}$, we consider the two cases separately.

First, suppose $\triangle = TTC$. Define the constants

$$C_1 = \frac{2\sqrt{2}-1}{3}, \quad C_2 = \frac{\sqrt{2}+1}{3}, \quad C_3 = 1,$$

and points

$$P_1 = (C_1,C_2,C_3), \quad P_2 = (C_1,C_3,C_2), \quad P_3 = (C_2,C_1,C_3), \quad P_4 = (C_2,C_3,C_1), \quad P_5 = (C_3,C_1,C_2), \quad P_6 = (C_3,C_2,C_1),$$

and

$$R = \left(\frac{\sqrt{2}+1}{3}, \frac{\sqrt{2}+1}{3}, \frac{\sqrt{2}+1}{3}\right)$$

in $\mathbb{R}^3_{TTC}$. Consider the minimum distance set $[OR]$, which forms a hexagonal dipyramid (see Figure 8-(a)).

The points $P_1,\ldots,P_6$ lie both on $[OR]$ and on the unit sphere centered at the origin. These six points correspond to the vertices of a hexagonal face of the truncated truncated cube. By Corollary 17, the isometry ϕ maps these points to vertices of a truncated truncated cube. Since ϕ preserves edge lengths, and the truncated truncated cube has 8 hexagonal faces, there are 8 possible images for these points on the hexagonal faces, each with 6 permutations of vertices. Therefore, there are $8 \times 6 = 48$

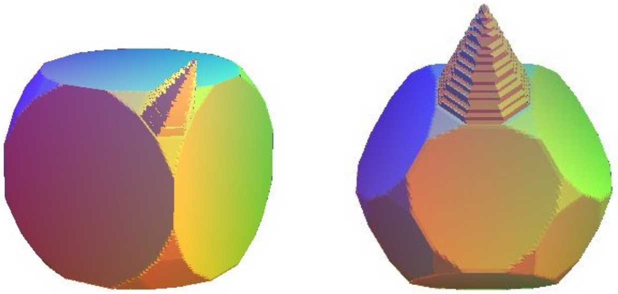
(a) (b)

Fig. 8. (a) The intersection of the sphere and minimum distance set of spaces of truncated truncated cube, (b) truncated truncated octahedron

possible mappings. Examining these cases individually reveals that these correspond precisely to the elements of the subgroup O_h.

Next, suppose $\triangle = TTO$. Define

$$C_1 = \frac{1+2\sqrt{2}-\sqrt{5}}{4}, \quad C_2 = \frac{3-\sqrt{2}+\sqrt{5}-\sqrt{10}}{4}, \quad C_3 = 1,$$

and points

$$\begin{aligned} &P_1 = (C_1, C_2, C_3), \quad P_2 = (C_1, -C_2, C_3), \quad P_3 = (-C_1, C_2, C_3), \quad P_4 = (-C_1, -C_2, C_3), \\ &P_5 = (C_2, C_1, C_3), \quad P_6 = (C_2, -C_1, C_3), \quad P_7 = (-C_2, C_1, C_3), \quad P_8 = (-C_2, -C_1, C_3), \end{aligned}$$

and

$$R = (0, 0, 2)$$

in $\mathbb{R}^3_{TTO}$. The minimum distance set $[OR]$ is an octagonal dipyramid with diagonal OR (see Figure 8-(b)).

The points P_i lie on both $[OR]$ and the unit sphere centered at the origin. These eight points are the vertices of an octagonal face of the truncated truncated octahedron. By Corollary 17, ϕ maps these points to vertices of a truncated truncated octahedron. Since ϕ preserves edge lengths, and the truncated truncated octahedron has 6 octagonal faces, there are $6 \times 8 = 48$ possible images. Examining these cases similarly shows these form the subgroup O_h.

Thus, in either case, ϕ belongs to the octahedral group O_h. ■

Theorem 19. *Let* $\phi : \mathbb{R}^3_\triangle \to \mathbb{R}^3_\triangle$ *be an isometry. Then there exist a unique translation* $T_A \in T(3)$ *and an element* $\psi \in O_h$ *such that*

$$\phi = T_A \circ \psi.$$

Proof. Let $A = \phi(O) = (a_1, a_2, a_3)$. Define $\psi = T_{-A} \circ \phi$. Then ψ is an isometry fixing the origin: $\psi(O) = O$. By Proposition 18, it follows that $\psi \in O_h$. Hence, $\phi = T_A \circ \psi$. The uniqueness of this decomposition is straightforward. ■

5. Conclusion

In this study, we introduce three-dimensional metric spaces whose unit spheres take the form of the *truncated truncated cube* and the *truncated truncated octahedron*. We present the construction of the metrics that generate these spaces and examine several of their fundamental geometric properties. Furthermore, we determine the isometry groups associated with these metrics.

It is shown that, in both cases, the group of isometries is the *semi-direct product* of the *octahedral symmetry group* O_h and the translation group $T(3)$ of the three-dimensional Euclidean space, i.e.,

$$\mathrm{Iso}(\mathbb{R}^3, d) \cong O_h \ltimes T(3),$$

where O_h denotes the full (Euclidean) symmetry group of the cube and the octahedron, and $T(3)$ represents the group of all translations in $\mathbb{R}^3$. The symmetry group O_h plays a crucial role in the characterization of these spaces, as it preserves the structure of both the truncated truncated cube and the truncated truncated octahedron.

The approach adopted in this work highlights the interplay between polyhedral geometry and metric space theory, demonstrating how altering the unit sphere to a convex polyhedron with a high degree of symmetry yields non-Euclidean yet highly symmetric metric spaces. In future work, it would be natural to extend this framework by considering other Archimedean and Catalan solids exhibiting octahedral symmetry, thereby constructing new metric spaces and exploring the corresponding isometry groups, geometric invariants, and potential applications in both pure and applied mathematics.

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