Decompositions of continuity for multifunctions

Marian Przemski

Abstract
Investigations of decompositions of continuity for functions between topological spaces has a large literature, but extension of this topic to multifunctions has not yet been investigated. The aim of the present note is to introduce the study of decompositions of continuity for multifunctions. For this purpose we will generalize the methods introduced by the author in [27] and later used in [9] and in many papers including for example [10] and [23].

Keywords: \( D(c, \alpha)-, D(c, s)-, D(c, s)\)-set, \( \alpha \)-continuity, decomposition of continuity, upper semi continuity, quasi-continuity, multifunction.

2000 AMS Classification: 54C05, 54C08, 54C60, 58C07.

Received: 21.03.2016 Accepted: 20.06.2016 Doi: 10.15672/HJMS.2017.492

1. Introduction and preliminaries
Throughout the present paper, \((X, \tau)\) and \((Y, \xi)\) will denote a topological space with no separation properties assume. Given a nonempty set \(Z \subset X\), we denote by \(P(Z)\) the power set of \(Z\). For a subset \(A\) of a topological space \((X, \tau)\) we denote by \(\text{Cl}(A)\) and \(\text{Int}(A)\) the closure and the interior of \(A\), respectively. A subset \(A \subset X\) is said to be \(\alpha\)-open [21] (resp. semi-open [8], pre-open [16], b-open [2] (or \(\gamma\)-open [3], or sp-open [9]), \(\beta\)-open [17] (or ps-open [1])) if \(A \subset \text{Int}(\text{Cl}(\text{Int}(A)))\) (resp. \(A \subset \text{Cl}(\text{Int}(A))\), \(A \subset \text{Int}(\text{Cl}(A))\), \(A \subset \text{Cl}(\text{Int}(A)) \cup \text{Int}(\text{Cl}(A))\), \(A \subset \text{Cl}(\text{Int}(\text{Cl}(A)))\). The family of all \(\alpha\)-open (resp. semi-open, pre-open, \(\gamma\)-open, \(\beta\)-open) sets in \((X, \tau)\) is denoted by \(\alpha(X, \tau)\) (resp. \(\text{SO}(X, \tau), \text{PO}(X, \tau), \gamma(X, \tau), \beta(X, \tau)\)). The union of all \(\alpha\)-open (resp. semi-open, pre-open, \(\gamma\)-open, \(\beta\)-open) sets of \(X\) contained in \(A\) is denoted by \(\alpha(\text{Int}(A))\) (resp. \(\text{SO}(\text{Int}(A)), \text{PO}(\text{Int}(A)), \gamma(\text{Int}(A)), \beta(\text{Int}(A))\)). The following results will be useful later.

Lemma 1.1[2]. The following hold for a subset \(A\) of a topological space \((X, \tau)\):

(a) \(\alpha(\text{Int}(A)) = A \cap \text{Int}(\text{Cl}(\text{Int}(A)))\);
(b) \(\text{s.Int}(A) = A \cap \text{Cl}(\text{Int}(A))\);

*Lomza State University of Applied Sciences (LSUAS) Institute of Computer Science and Automation Akademicka Street 14, 18-400 Lomnica, Poland
Email: mprzemski@pwsip.edu.pl
By a multifunction \(F: X \rightarrow Y\) we mean a map defined on \(X\) with values being nonempty subsets of \(Y\). Following [4] we shall denote the upper and lower inverse images of a set \(B \subset Y\) by \(F^+(B)\) and \(F^-(B)\), respectively, that is, \(F^+(B) = \{x \in X : F(x) \subset B\}\) and \(F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}\). A multifunction \(F: (X, \tau) \rightarrow (Y, \xi)\) is called upper semi continuous (briefly u.s.c.) (resp. lower semi continuous (briefly l.s.c.)) [13, 22] at a point \(x \in X\) if, \(x \in \text{Int}(F^+(W))\) (resp. \(x \in \text{Int}(F^-(W))\)) for each open subset \(W\) of \(Y\) such that \(x \in F^+(W)\) (resp. \(x \in F^-(W)\)). It is called u.s.c. (resp. l.s.c.) if \(F\) is u.s.c. (resp. l.s.c.) at each point of \(X\).

There are many generalizations and modifications of continuity. The basic of these are as follows:

A multifunction \(F: (X, \tau) \rightarrow (Y, \xi)\) is said to be u.o.c. (or l.o.c.) [20] (resp. u.q.c. (or l.q.c.) [25], u.p.c. (or l.p.c.) [24], u.\(\gamma\).c. (or l.\(\gamma\).c.) [18], u.\(\beta\).c. (or l.\(\beta\).c.) [26]) at a point \(x \in X\) if, \(x \in \alpha.\text{Int}(F^+(W))\) (resp. \(x \in \alpha.\text{Int}(F^-(W))\)) (resp. \(x \in \alpha.\text{Int}(F^+(W))\) (resp. \(x \in \alpha.\text{Int}(F^-(W))\)) (resp. \(x \in \alpha.\text{Int}(F^+(W))\) (resp. \(x \in \alpha.\text{Int}(F^-(W))\))) for each \(W\) such that \(x \in F^+(W)\) (resp. \(x \in F^-(W)\)).

A multifunction \(F\) is called u.o.c. (or l.o.c.), (resp. u.q.c. (or l.q.c.), u.p.c. (or l.p.c.), u.\(\gamma\).c. (or l.\(\gamma\).c.), u.\(\beta\).c. (or l.\(\beta\).c.) if it has this property at each point of \(X\), that is, \(A \subset \alpha.\text{Int}(A)\) (or \(B \subset \alpha.\text{Int}(B)\)) (resp. \(A \subset s.\text{Int}(A)\) (or \(B \subset s.\text{Int}(B)\)), \(A \subset p.\text{Int}(A)\) (or \(B \subset p.\text{Int}(B)\)), \(A \subset \gamma.\text{Int}(A)\) (or \(B \subset \gamma.\text{Int}(B)\)), \(A \subset \beta.\text{Int}(A)\) (or \(B \subset \beta.\text{Int}(B)\)) for any pair \((A, B) \in \mathcal{P}(X) \times \mathcal{P}(X)\) of the form \((A, B) = (F^+(W), F^-(W))\), where \(W \in \xi\); equivalently, \(A \subset \mathcal{P}(\text{Cl}(\text{Int}(A)))\) (or \(B \subset \mathcal{P}(\text{Cl}(\text{Int}(B)))\)), (resp. \(A \subset \mathcal{P}(\text{Cl}(\text{Int}(A)))\), \(B \subset \mathcal{P}(\text{Cl}(\text{Int}(B)))\)) (resp. \(A \subset \mathcal{P}(\text{Cl}(\text{Int}(A)))\), \(B \subset \mathcal{P}(\text{Cl}(\text{Int}(B)))\)) (resp. \(A \subset \mathcal{P}(\text{Cl}(\text{Int}(A)))\), \(B \subset \mathcal{P}(\text{Cl}(\text{Int}(B)))\)) for any pair \((A, B) \in \mathcal{P}(X) \times \mathcal{P}(X)\) of the form \((A, B) = (F^+(W), F^-(W))\), where \(W \in \xi\).

Of course, if a single-valued function \(f: (X, \tau) \rightarrow (Y, \xi)\) is treated as a multifunction \(F\) given by \(F(x) = \{f(x)\}\) for all \(x \in X\), then the multifunction \(F\) is u.s.c. l.s.c. (resp. u.o.c. l.o.c., u.q.c. l.q.c., u.p.c. l.p.c., u.\(\gamma\).c. l.\(\gamma\).c., u.\(\beta\).c. l.\(\beta\).c.) if and only if the function \(f\) is continuous (resp. \(\alpha\)-continuous [19], semi-continuous [11, 14], pre-continuous [16], \(\gamma\)-continuous [9], \(\beta\)-continuous [17]) because of the simple fact that in this case we have \(F^+(B) = F^-(B) = f^+(B)\) for any \(B \subset Y\).

Since any open subset \(W \subset Y\) designate a pair \((F^+(W), F^-(W)) \in \mathcal{P}(X) \times \mathcal{P}(X)\), it is convenient to use the following general concept:

**Definition 1.2.** Let \(\mathcal{R}\) be a binary relation on \(\mathcal{P}(X)\). We say that a multifunction \(F: (X, \tau) \rightarrow (Y, \xi)\) is \(\mathcal{R}\)-continuous if \((F^+(W), F^-(W)) \in \mathcal{R}\) for any \(W \in \xi\).

**Remark 1.3.** If we denote

- \(\tau^c\) - \(\tau \times \mathcal{P}(X)\) (or \(\tau^c_\tau - \mathcal{P}(X) \times \tau\)) (resp.
- \(\tau^o\) - \(\alpha(X, \tau) \times \mathcal{P}(X)\) (or \(\tau^o_\alpha - \mathcal{P}(X) \times \alpha(X, \tau)\)),
- \(\tau^s\) - \(\text{SO}(X, \tau) \times \mathcal{P}(X)\) (or \(\tau^s_\text{SO} - \mathcal{P}(X) \times \text{SO}(X, \tau)\)),
- \(\tau^p\) - \(\text{PO}(X, \tau) \times \mathcal{P}(X)\) (or \(\tau^p_\text{PO} - \mathcal{P}(X) \times \text{PO}(X, \tau)\)),
- \(\tau^\gamma\) - \(\gamma \text{O}(X, \tau) \times \mathcal{P}(X)\) (or \(\tau^\gamma_\gamma - \mathcal{P}(X) \times \gamma \text{O}(X, \tau)\)),
- \(\tau^\beta\) - \(\beta \text{O}(X, \tau) \times \mathcal{P}(X)\) (or \(\tau^\beta_\beta - \mathcal{P}(X) \times \beta \text{O}(X, \tau)\)),

then \(\tau^c\)-continuity (or \(\tau^c_\tau\)-continuity) (resp. \(\tau^c\)-continuity (or \(\tau^c_\tau\)-continuity)), \(\tau^o\)-continuity (or \(\tau^o_\alpha\)-continuity), \(\tau^s\)-continuity (or \(\tau^s_\text{SO}\)-continuity), \(\tau^p\)-continuity (or \(\tau^p_\text{PO}\)-continuity), \(\tau^\gamma\)-continuity (or \(\tau^\gamma_\gamma\)-continuity), \(\tau^\beta\)-continuity (or \(\tau^\beta_\beta\)-continuity) is equivalent to u.s.c. (or l.s.c.) (resp. u.o.c. (or l.o.c.), u.q.c. (or l.q.c.), u.p.c. (or l.p.c.).
Definition 1.4. A multifunction $F : (X, \tau) \to (Y, \xi)$ is said to be:

(a) u.s.c. if $F^{-1}(W) \subseteq \text{Int}(F^+(W))$ for each $W \in \xi$;
(b) u.o.c. [28] if $F^{-1}(W) \subseteq \text{Int}((\text{Int}(F^+(W))))$ for each $W \in \xi$;
(c) u.q.c. [7, 5, 6, 12, 19] if $F^{-1}(W) \subseteq \text{Cl}(\text{Int}(F^+(W)))$ for each $W \in \xi$;
(d) u.p.c. [28] if $F^{-1}(W) \subseteq \text{Cl}(\text{Int}(F^+(W)))$ for each $W \in \xi$;
(e) u.c. if $F^{-1}(W) \subseteq \text{Cl}(\text{Int}(F^+(W))) \cup \text{Int}(\text{Cl}(F^+(W)))$ for each $W \in \xi$;
(f) $\gamma$-c. [28] if $F^{-1}(W) \subseteq \text{Cl}(\text{Int}(\text{Cl}(F^+(W))))$ for each $W \in \xi$.

The property $\gamma$-c. has been investigated in [7, Theorem 5.2] and in [5, 6, 12, 19] under the name of minimality of multifunctions.

2. Decompositions of continuity of type "lu"

In this paper we consider a special type of continuity for multifunctions, named in the literature "minimalit" or type "lu" and defined as follows.

Definition 2.1. Given a topological space $(X, \tau)$ we define the following operators $\mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$:

(a) $\text{Int}_u(A,B) = B \cap \text{Int}(A)$;
(b) $\alpha \cdot \text{Int}_u(A,B) = B \cap \text{Int}(\text{Cl}(\text{Int}(A)))$;
(c) $s \cdot \text{Int}_u(A,B) = B \cap \text{Cl}(\text{Int}(A))$;
(d) $p \cdot \text{Int}_u(A,B) = B \cap \text{Cl}(\text{Cl}(A))$;
(e) $\gamma \cdot \text{Int}_u(A,B) = s \cdot \text{Int}_u(A,B) \cup p \cdot \text{Int}_u(A,B)$;
(f) $\beta \cdot \text{Int}_u(A,B) = B \cap \text{Cl}(\text{Int}(\text{Cl}(A)))$.

It is easy to see that the operators in Lemma 1.1 are compositions of the above operators and the diagonal operator $\Delta_{\mathcal{P}(X)} : \mathcal{P}(X) \to \mathcal{P}(X) \times \mathcal{P}(X)$ given by $\Delta_{\mathcal{P}(X)}(A) = (A, A)$ for any $A \in \mathcal{P}(X)$. More precisely:

Remark 2.2. The following hold for a subset $A$ of a topological space $(X, \tau)$:

(a) $\text{Int}_u(A,A) = \text{Int}(A)$;
(b) $\alpha \cdot \text{Int}_u(A,A) = \alpha \cdot \text{Int}(A)$;
(c) $s \cdot \text{Int}_u(A,A) = s \cdot \text{Int}(A)$;
(d) $p \cdot \text{Int}_u(A,A) = p \cdot \text{Int}(A)$;
(e) $\gamma \cdot \text{Int}_u(A,A) = \gamma \cdot \text{Int}(A)$;
(f) $\beta \cdot \text{Int}_u(A,A) = \beta \cdot \text{Int}(A)$.

Definition 2.3. For a topological space $(X, \tau)$ we denote:

(a) $\pi^u = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : B = \text{Int}_u(A, B)\}$;
(b) $\pi^{\alpha} = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : B = \alpha \cdot \text{Int}_u(A, B)\}$;
(c) $\pi^{s} = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : B = s \cdot \text{Int}_u(A, B)\}$;
(d) $\pi^{p} = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : B = p \cdot \text{Int}_u(A, B)\}$;
(e) $\pi^{\gamma} = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : B = \gamma \cdot \text{Int}_u(A, B)\}$;
(f) $\pi^{\beta} = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : B = \beta \cdot \text{Int}_u(A, B)\}$.

Remark 2.4. The property of being u.s.c. (resp. u.o.c., u.q.c., u.p.c., u.c., lu.c., lu.c.)
is equivalent to $\tau^{lu}_u$-continuity (resp. $\tau^{lu}_\alpha$-continuity, $\tau^{lu}_s$-continuity, $\tau^{lu}_p$-continuity, $\tau^{lu}_\beta$-continuity).

The following families of subsets were used to obtain standard types of decompositions of continuity and generalized continuity.

**Definition 2.5.** [27, 9] For a topological space $(X, \tau)$ we denote:

(a) $D(c, \alpha) = \{ A \in \mathcal{P}(X) : \text{Int}(A) = \alpha \cdot \text{Int}(A) \}$;
(b) $D(c, s) = \{ A \in \mathcal{P}(X) : \text{Int}(A) = s \cdot \text{Int}(A) \}$;
(c) $D(c, p) = \{ A \in \mathcal{P}(X) : \text{Int}(A) = p \cdot \text{Int}(A) \}$;
(d) $D(c, \beta) = \{ A \in \mathcal{P}(X) : \text{Int}(A) = \beta \cdot \text{Int}(A) \}$;
(e) $D(c, s) = \{ A \in \mathcal{P}(X) : \alpha \cdot \text{Int}(A) = s \cdot \text{Int}(A) \}$;
(f) $D(c, p) = \{ A \in \mathcal{P}(X) : \alpha \cdot \text{Int}(A) = p \cdot \text{Int}(A) \}$;
(g) $D(c, \beta) = \{ A \in \mathcal{P}(X) : \alpha \cdot \text{Int}(A) = \beta \cdot \text{Int}(A) \}$;
(h) $D(c, \gamma) = \{ A \in \mathcal{P}(X) : \gamma \cdot \text{Int}(A) = \beta \cdot \text{Int}(A) \}$.

Each of the above families is of the form \{ $A \in \mathcal{P}(X) : O_1(A) = O_2(A)$ \}, that is, the set of coincidence points of a pair $(O_1, O_2)$ of different operators $O_1, O_2 : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ which are members of the collection \{ $\text{Int}, \alpha \cdot \text{Int}, p \cdot \text{Int}, s \cdot \text{Int}, \gamma \cdot \text{Int}, \beta \cdot \text{Int}$ \}. In the case of operators $O : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ introduced in Definition 2.1, we get the general result. Before starting, we need a specific operator $I : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ defined by $I(\text{Int}(A), B) \rightarrow B$.

**Theorem 2.6.** Let $(X, \tau)$ be a topological space. Any set of coincidence points of a pair $(O_1, O_2)$ of different operators belonging to the following collection

\{ $\text{Int}, \alpha \cdot \text{Int}, p \cdot \text{Int}, \gamma \cdot \text{Int}, \beta \cdot \text{Int}, I$ \} is equal to one of the following:

(a) $\tau^{lu}_u \cdot \gamma \cdot \alpha \cdot \tau^{lu}_s \cdot \tau^{lu}_p \cdot \tau^{lu}_\beta$;
(b) $\tau^{lu}(c, \alpha)$
(c) $\tau^{lu}(c, s)$
(d) $\tau^{lu}(c, p)$
(e) $\tau^{lu}(c, \beta)$
(f) $\tau^{lu}(c, s)$
(g) $\tau^{lu}(c, p)$
(h) $\tau^{lu}(c, \beta)$
(i) $\tau^{lu}(c, \gamma)$
(j) $\tau^{lu}(c, \beta)$
(k) $\tau^{lu}(c, \gamma)$

Proof. It is clear that the sets in (a) are designated by the pairs $(I, \text{Int}_u)$, $(I, \alpha \cdot \text{Int}_u)$, $(I, s \cdot \text{Int}_u)$, $(I, p \cdot \text{Int}_u)$, $(I, \gamma \cdot \text{Int}_u)$ and $(I, \beta \cdot \text{Int}_u)$, respectively. Analogously, the sets $\tau^{lu}(c, \alpha)$, $\tau^{lu}(c, s)$, $\tau^{lu}(c, p)$ and $\tau^{lu}(c, \beta)$ are designated by the pairs $(\text{Int}_u, \alpha \cdot \text{Int}_u)$, $(\text{Int}_u, s \cdot \text{Int}_u)$, $(\text{Int}_u, p \cdot \text{Int}_u)$ and $(\text{Int}_u, \beta \cdot \text{Int}_u)$ respectively. Now we show that the pairs $(\text{Int}_u, \beta \cdot \text{Int}_u)$ and $(\text{Int}_u, \gamma \cdot \text{Int}_u)$ have the same set of coincidence points. Indeed, if $\text{Int}_u(A, B) \subset \text{Int}_u(A, B)$, i.e., $\text{Int}_u(A, B) \subset \text{Int}_u(A, B) \cup \text{Int}_u(A, B) \cap \text{Int}_u(A, B)$, then $\text{Int}_u(A, B) \subset \text{Int}_u(A, B) \cup \text{Int}_u(A, B) \cap \text{Int}_u(A, B)$ and consequently, $\text{Int}(\text{Int}(A)) = \text{Int}(\text{Int}(A))$ because of $\text{Int}(\text{Int}(A)) \subset \text{Int}(\text{Int}(A)) \subset \text{Int}(\text{Int}(A))$ and $\text{Int}(\text{Int}(A)) \subset \text{Int}(\text{Int}(A))$. As a result, we obtain $\text{Int}(\text{Int}(A)) \subset \text{Int}(\text{Int}(A)) \subset \text{Int}(\text{Int}(A))$. This means that $\text{Int}_u(A, B) \subset \beta \cdot \text{Int}_u(A, B)$. So, the sets in (b), (c), (d) and (e) are the only sets of coincidence points that can be obtained by use of the operator $\text{Int}_u$. Since $\text{Int}(\text{Int}(A)) \subset \text{Int}(\text{Int}(A))$, analogously one can show that $\alpha \cdot \text{Int}_u(A, B) \subset \beta \cdot \text{Int}_u(A, B)$ and
\( \alpha \text{Int}_u(A,B) = \gamma \text{Int}_u(A,B) \) are equivalent. So, the sets in (f), (g), (h) and (b) are the only sets of coincidence points that can be obtained by use of the operator \( \alpha \text{Int}_u \). To prove that the sets in (i), (j), (d) and (g) are the only sets of coincidence points that can be obtained by use of the operator \( p \text{Int}_u \), we will show that \( p \text{Int}_u(A,B) = s \text{Int}_u(A,B) \) and \( \alpha \text{Int}_u(A,B) = \beta \text{Int}_u(A,B) \) are equivalent. If \( p \text{Int}_u(A,B) = s \text{Int}_u(A,B) \), then of course,\( \text{Cl}(p \text{Int}_u(A,B)) = \text{Cl}(s \text{Int}_u(A,B)) \). It is clear that \( \text{Cl}(p \text{Int}_u(A,B)) = \text{Cl}(\text{Int}(\text{Cl}(A))) \) as shown in the previous part of the proof. Now we show that \( \text{Cl}(s \text{Int}_u(A,B)) = \text{Cl}(\text{Int}(A)) \). Indeed, \( \text{Cl}(B \cap \text{Int}(A)) \subset \text{Cl}(s \text{Int}_u(A,B)) \subset \text{Cl}(\text{Int}(A)) \) and of course, \( \text{Cl}(B \cap \text{Int}(A)) = \text{Cl}(\text{Int}(A)) \) and therefore the equality \( p \text{Int}_u(A,B) = s \text{Int}_u(A,B) \) implies that \( \alpha \text{Int}_u(A,B) = \beta \text{Int}_u(A,B) \). Conversely, the last equality implies that \( p \text{Int}_u(A,B) \subset \beta \text{Int}_u(A,B) \subset \alpha \text{Int}_u(A,B) \subset s \text{Int}_u(A,B) \) and \( s \text{Int}_u(A,B) \subset \beta \text{Int}_u(A,B) \subset \alpha \text{Int}_u(A,B) \) and \( p \text{Int}_u(A,B) \). So, \( p \text{Int}_u(A,B) = s \text{Int}_u(A,B) \). Finally, we will show that the sets in (f) and (c) are the only sets of coincidence points that can be obtained by use of the operator \( s \text{Int}_u \). Since \( p \text{Int}_u(A,B) = s \text{Int}_u(A,B) \) and \( \alpha \text{Int}_u(A,B) = \beta \text{Int}_u(A,B) \) and \( \alpha \text{Int}_u(A,B) = p \text{Int}_u(A,B) \) are equivalent, it is enough to prove that \( s \text{Int}_u(A,B) = \gamma \text{Int}_u(A,B) \), \( s \text{Int}_u(A,B) = \beta \text{Int}_u(A,B) \) and \( \alpha \text{Int}_u(A,B) = s \text{Int}_u(A,B) \) are equivalent. If \( s \text{Int}_u(A,B) = \gamma \text{Int}_u(A,B) \), then \( \text{Cl}(s \text{Int}_u(A,B)) \subset \text{Cl}(s \text{Int}_u(A,B)) \) which means \( \text{Cl}(B \cap \text{Int}(\text{Cl}(A))) \subset \text{Cl}(B \cap \text{Int}(\text{Cl}(A))) \) and implies that \( \text{Cl}(\text{Int}(\text{Cl}(A))) \cap \text{Int}(\text{Cl}(A)) \) and \( \text{Cl}(B \cap \text{Int}(\text{Cl}(A))) \). Consequently, \( B \cap \text{Int}(\text{Cl}(\text{Cl}(A))) \subset B \cap \text{Int}(\text{Cl}(A)) \) and \( (B \cap \text{Int}(\text{Int}(A))) \cap (B \cap \text{Int}(\text{Cl}(A))) \) and \( \beta \text{Int}_u(A,B) \) and consequently, \( s \text{Int}_u(A,B) = \beta \text{Int}_u(A,B) \). The last equality implies that \( \text{Cl}(B \cap \text{Int}(\text{Cl}(A))) \subset \text{Cl}(B \cap \text{Int}(\text{Cl}(A))) \), so \( \text{Cl}(\text{Int}(\text{Cl}(A))) \subset \text{Cl}(\text{Int}(A)) \) or equivalently, \( \text{Int}(\text{Cl}(A)) \subset \text{Int}(\text{Cl}(A)) \). Thus, \( B \cap \text{Int}(\text{Cl}(A)) \cap B \cap \text{Int}(\text{Cl}(\text{Cl}(A))) \), i.e. \( \alpha \text{Int}_u(A,B) \). Finally, using the last equality we have \( \gamma \text{Int}_u(A,B) = (B \cap \text{Int}(\text{Cl}(A))) \cup (B \cap \text{Int}(\text{Cl}(\text{Cl}(A)))) \) and the proof is complete.

The following diagram shows the relationship between the families of sets considered in the above theorem.

Diagram 2.7.
The next result follows directly from the definitions of the families of sets considered in Theorem 2.6 and, in the case (d), from the fact that the equalities $s.\text{Int}_u(A,B) - \beta.\text{Int}_u(A,B)$ and $\alpha.\text{Int}_u(A,B) - p.\text{Int}_u(A,B)$ are equivalent as is shown in the proof of this theorem.

**Theorem 2.8.** For any topological space $(X, \tau)$, the following hold:

(a) $\tau^l_{c_u} - D^{lu}(c,\alpha) \cap \tau^l_{s_u} = D^{lu}(c,\alpha) \cap \tau^l_{p_u} - D^{lu}(c,\beta) \cap \tau^l_{\beta_u};$

(b) $\tau^l_{\alpha_u} - D^{lu}(\alpha, s) \cap \tau^l_{p_u} = D^{lu}(\alpha, p) \cap \tau^l_{p_u} - D^{lu}(\alpha, \beta) \cap \tau^l_{\beta_u};$

(c) $\tau^l_{p_u} - D^{lu}(p, \gamma) \cap \tau^l_{\beta_u} = D^{lu}(p, \beta) \cap \tau^l_{\beta_u};$

(d) $\tau^l_{s_u} = D^{lu}(\alpha, p) \cap \tau^l_{\beta_u};$

(e) $\tau^l_{\gamma_u} - D^{lu}(\gamma, \beta) \cap \tau^l_{\beta_u}.$

As a consequence of the above result we offer the following five decomposition theorems for multifunctions.

**Theorem 2.9.** For any multifunction $F:(X, \tau) \rightarrow (Y, \xi)$, the following statements are equivalent:

(a) $F$ is lu.s.c.;

(b) $F$ is lu.\(\alpha\).c. and $D^{lu}(\alpha, c, \alpha)$-continuous;

(c) $F$ is lu.q.c. and $D^{lu}(c, s)$-continuous;

(d) $F$ is lu.p.c. and $D^{lu}(c, p)$-continuous;

(e) $F$ is lu.\(\beta\).c. and $D^{lu}(c, \beta)$-continuous.

**Theorem 2.10.** For any multifunction $F:(X, \tau) \rightarrow (Y, \xi)$, the following statements are equivalent:

(a) $F$ is lu.\(\alpha\).c.;

(b) $F$ is lu.q.c. and $D^{lu}(\alpha, s)$-continuous;

(c) $F$ is lu.p.c. and $D^{lu}(\alpha, p)$-continuous;

(d) $F$ is lu.\(\beta\).c. and $D^{lu}(\alpha, \beta)$-continuous.

**Theorem 2.11.** For any multifunction $F:(X, \tau) \rightarrow (Y, \xi)$, the following statements are equivalent:

(a) $F$ is lu.p.c.;

(b) $F$ is lu.\(\gamma\).c. and $D^{lu}(\gamma, p)$-continuous;

(c) $F$ is lu.\(\beta\).c. and $D^{lu}(\gamma, \beta)$-continuous.

**Theorem 2.12.** For any multifunction $F:(X, \tau) \rightarrow (Y, \xi)$, the following statements are equivalent:

(a) $F$ is lu.q.c.;

(b) $F$ is lu.\(\beta\).c. and $D^{lu}(\gamma, p)$-continuous.

**Theorem 2.13.** For any multifunction $F:(X, \tau) \rightarrow (Y, \xi)$, the following statements are equivalent:

(a) $F$ is lu.\(\gamma\).c.;

(b) $F$ is lu.\(\beta\).c. and $D^{lu}(\gamma, \beta)$-continuous.

**Remark 2.14.** The classes of multifunctions corresponding to appropriate families of sets shown in the Diagram 2.7 are strictly different.

Proof. Let us consider some examples of multifunctions $F:(X, \tau) \rightarrow (R, \xi)$, where $R$ is the set of all real numbers, $\tau$ denotes the natural topology on $R$ and $\xi$ is generated by the
basis $\mathfrak{B}_\xi = \{(-\infty, r) : r \in R\}$. For a multifunction $F$ and $W \in \mathfrak{B}_\xi$ we use the following notation: $A_W = F^*(W)$ and $B_W = F^-(W)$.

(i) Let $F$ be defined as follows: $F(0) = R$, $F(x) = \{-\ln(x)\}$ if $x > 0$ and $F(x) = \{-\ln(-x)\}$ if $x < 0$. If $W \in \mathfrak{B}_\xi$, then we have $A_W = (-\infty, -\exp(-r)) \cup (\exp(-r), +\infty)$ and $B_W = A_W \cup \{0\}$. Then $\text{Int}(\text{Cl}(A_W)) = A_W$. So, $B_W \cap \text{Int}(\text{Cl}(A_W)) = A_W$ but $B_W \not\subseteq \text{Cl}(\text{Int}(A_W))$ and therefore $F$ is $\mathcal{D}^{iu}(c, \beta)$-continuous but not $\text{lu.c.}$. 

(ii) Let $F$ be defined by $F(0) = R$, $F(x) = \{-\ln(x)\}$ if $x > 0$ and $F(x) = \{-\ln(-x)\}$ if $x < 0$ and let $W \in \mathfrak{B}_\xi$. Then we have $A_W = (-\exp(r), 0) \cup (0, \exp(r))$, $B_W = A_W \cup \{0\}$, $\text{Cl}(\text{Int}(A_W)) = A_W$ but $B_W \not\subseteq \text{Int}(A_W)$. So, $F$ is $\text{lu.c.}$ but not $\mathcal{D}^{iu}(c, \alpha)$-continuous.

(iii) Let us define $F$ by $F(0) = R$, $F(x) = \{-\ln(x)\}$ if $x > 0$, $F(x) = (-\infty, -\ln(-x))$ if $x \in Q \cap (-\infty, 0)$ and $F(x) = (-\ln(-x), \infty)$ if $x \in (-\infty, 0) \setminus Q$, where $Q$ is the set of all rational numbers, and let $W \in \mathfrak{B}_\xi$. Then we have $A_W = (\{0\} \cup (\exp(-r), +\infty), B_W = (-\infty, -\exp(-r)) \cup (\exp(-r), +\infty))$ and $\text{Int}(\text{Cl}(A_W)) = (\exp(-r), \infty)$ and $\text{Cl}(\text{Int}(A_W)) = (-\exp(-r), \infty)$. So $B_W \cap \text{Cl}(\text{Int}(A_W)) \subseteq \text{Int}(A_W)$ but $B_W \cap \text{Int}(\text{Cl}(A_W)) \not\subseteq \text{Int}(A_W)$ and $B_W \cap \text{Cl}(\text{Int}(A_W)) \not\subseteq \text{Int}(\text{Cl}(A_W))$. Therefore, $F$ is $\mathcal{D}^{iu}(c, s)$-continuous but not $\mathcal{D}^{iu}(p, \beta)$-continuous and not $\mathcal{D}^{iu}(p, \beta)$-continuous.

(iv) Let us define $F$ as follows: $F(0) = R$, $F(x) = \{-\ln(x)\}$ if $x > 0$, $F(x) = (-\ln(-x))$ if $x \in Q \cap (-\infty, 0)$ and $F(x) = (-\ln(-x), \infty)$ if $x \in (-\infty, 0) \setminus Q$, where $Q$ is the set of all rational numbers, and let $W \in \mathfrak{B}_\xi$. Then we have $A_W = (-\exp(r), 0) \cup (0, \exp(r))$, $B_W = (-\exp(r), \exp(r))$, $\text{Int}(A_W) = (0, \exp(r))$ and $\text{Cl}(A_W) = (-\exp(r), \exp(r))$. So $B_W \cap \text{Cl}(\text{Int}(A_W)) \not\subseteq \text{Int}(A_W)$ and $B_W \cap \text{Cl}(\text{Int}(A_W)) \not\subseteq \text{Int}(\text{Cl}(A_W))$. Therefore, $F$ is $\text{lu.c.}$ but not $\mathcal{D}^{iu}(c, \alpha)$-continuous and $F$ is not $\mathcal{D}^{iu}(c, \beta)$-continuous.

(v) Let us define $F$ as follows: $F(0) = R$, $F(x) = \{-\ln(k)\}$ if $x \in \{k - 1, k\}$, where $k = -1, -2, \ldots$, $F(x) = \{\ln(-\frac{1}{\pi^2})\}$ if $x \in \left[-\frac{1}{\pi^2}, -\frac{1}{\pi^2}\right]$, $\text{Int}(\text{Cl}(A_W)) = (-\exp(r), -\xi), (\exp(\xi), \exp(r) + \xi)$ and $\text{Cl}(A_W) = \text{Cl}(\text{Int}(A_W))$ for some $\xi \in [0, 1)$. So $B_W \subseteq \text{Cl}(\text{Int}(A_W))$ but $B_W \cap \text{Cl}(\text{Int}(A_W)) \not\subseteq \text{Int}(\text{Cl}(A_W))$, and thus $F$ is $\text{lu.c.}$ but not $\mathcal{D}^{iu}(p, \beta)$-continuous.

(vi) Define $F$ by $F(x) = (-\infty, -x)$ for all $x \in R$ and let $W \in \mathfrak{B}_\xi$. Then $A_W = [-r, +\infty)$, $B_W = R$, $\text{Int}(A_W) = (-\infty, -r)$ and $\text{Cl}(A_W) = A_W$. Consequently, $B_W \cap \text{Cl}(\text{Int}(A_W)) \subseteq \text{Int}(A_W)$ but $B_W \cap \text{Cl}(\text{Int}(A_W)) \not\subseteq \text{Int}(\text{Cl}(A_W))$. So, $F$ is $\mathcal{D}^{iu}(c, \alpha)$-continuous but not $\mathcal{D}^{iu}(\alpha, \beta)$-continuous.

(vii) Let $F$ be defined by $F(0) = R$, $F(x) = \{\ln(x)\}$ if $x \in Q \cap (0, \infty)$, $F(x) = \{-\ln(x)\}$ if $x \in (0, \infty) \setminus Q$ and $F(x) = \{-\ln(-x)\}$ if $x < 0$. If $W \in \mathfrak{B}_\xi$, then we have $A_W = (-\infty, -\exp(-x)) \cup (0, \exp(x))$ and $\text{Int}(A_W) = (-\infty, -\exp(-x)) \cup [0, \exp(x))$ and $\text{Cl}(A_W) = (-\infty, -\exp(-x)) \cup [0, \exp(x))$ and $B_W = (-\infty, -\exp(-x)) \cup [0, \exp(x))$. Thus $B_W \subseteq \text{Cl}(\text{Int}(A_W))$ but $B_W \not\subseteq \text{Cl}(\text{Int}(A_W)) \cup \text{Int}(\text{Cl}(A_W))$, so $F$ is $\text{lu.c.}$ but not $\text{lu.c.}$

References

A.A. El-Atik, A study of some types of mappings on topological spaces, Master's Thesis. Faculty of Science, Tanta University, (1997).


V. Popa, Some properties of \(H\)-almost continuous multifunctions, Problem. Mat. 10(1990), 9-36.


