Erciyes Üniversitesi İktisadi ve İdari Bilimler Fakültesi Dergisi, Sayı 9, Kasım 1991

PROBABILISTIC PERT MODELS AND IMPLICIT BETA DISTRIBUTION ASSUMPTION BEHIND IT

Osman UNUTULMAZ*

INTRODUCTION

In text books and as well as in many other publications, along with the minimum and maximum activity completion times, the modal (mostlikely) activity completion time is also estimated independently. Finally, these three estimated activity completion times are being used for calculation of the expected activity completion time of the beta distribution. Yet, estimating the mostlikely or modal activity completion time along with the minimum and maximum activity completion times contradicts with the theoretical nature of beta distribution. Because, the modal or mostlikely value of beta distribution is a function of the minimum and maximum values. That is, when the minimum and the maximum values are estimated the modal value can be determined. Therefore it shall not be estimated independently.

Depending on the magnitude of the difference between the estimated and calculated modal values, which is the function of the degrees of skewness, the expected value of each activity will deviate from the true one. This in turn will result in incorrect expected project completion time and probability and/or selection of improper critical path.

In this paper, having summarized the theoretical facts and the fallacies encountered, we have tackeled a real life problem. Consequently, we have determined that the expected project completion time differs from the one obtained using freely estimated mostlikely values. Probability that the project takes a certain time for completion also differs from the real one.

*). Doç.Dr., Erciyes Üniv. İktisadi ve İdari Bilimleri Fakültesi Öğr.Üy.

BETA DENSITY FUNCTION AND THEORETICAL FACTS

The shape of the beta density is function of its shape parameters γ and η. In its common form it represents the beta density family. In connection with probabilistic PERT models it can be stated that when

1- $\gamma>1$, $\eta>1$ and $\gamma=\eta$ then the function is symmetrical,

2- $\gamma>1$, $\eta>1$ and $\gamma>\eta$ then the function is negatively skewed, and

3- $\gamma>1$, $\eta>1$ and $\gamma<\eta$ then the function is positively skewed.

The beta density function is originally defined [1] as

$$f(x,\gamma,\eta)=\frac{\Gamma(\gamma+\eta)}{\Gamma(\gamma)+\Gamma(\eta)}\times x^{\gamma-1}\times(1-x)^{\eta-1} \quad \text{for } \gamma>0,\ \eta>0 \quad \text{and } 0\leq x\leq 1$$

Mean and the variance of the above defined function [2] are

$$\mu=\frac{\gamma}{\gamma+\eta} \qquad \text{and} \qquad \sigma^2=\frac{\gamma\times\eta}{(\gamma+\eta+1)\times(\gamma+\eta)^2} \qquad \text{respectively.}$$

In a [k,b] interval the beta function is expressed as

$$f(y,\gamma,\eta)=\frac{\Gamma(\gamma+\eta)}{(b-k)\times\Gamma(\gamma)\times\Gamma(\eta)}\times\left(\frac{y-k}{b-k}\right)^{\gamma-1}\times\left(1-\frac{y-k}{b-k}\right)^{\eta-1}$$

for $\gamma>0$, $\eta>0$ and $k\leq y\leq b$.

Mean and the variance of this function may then be expressed as

$$\mu=\frac{k\times\eta+b\times\gamma}{\gamma+\eta} \qquad \text{and} \qquad \sigma^2=\frac{\gamma\times\eta}{(\gamma+\eta+1)\times(\gamma+\eta)^2}\times(b-k)^2$$

Mostlikely (modal) value of the same is

$$m=\frac{k\times(\eta-1)+b\times(\gamma-1)}{\gamma+\eta-2}$$

In terms of the modal value m and the other parameters the mean of the beta distribution can be written as

$$\mu = \frac{k + m \times (\gamma + \eta - 2) + b}{\gamma + \eta}.$$

INDEPENDENTLY ESTIMATED MOSTLIKELY VALUE AND CALCULATION OF THE MEAN AND THE VARIANCE IN PROBABILISTIC PERT MODELS

Before calculating the mean and variance of activity completion times, the minimum activity time k, the maximum activity time b and the mostlikely activity time m are estimated and are substituted in the following relationships in existing literature[2,3,4,5,6]. Where these are

$$\mu = \frac{k + 4 \times m + b}{6}.$$

and

$$\sigma^2 = \frac{(b-k)^2}{6}.$$

Yet, these relationships for mean and the variance hold only if

1- $\gamma = 3 + \sqrt{2}$
$\eta = 3 - \sqrt{2}$

or

2- $\gamma = 3 - \sqrt{2}$
$\eta = 3 + \sqrt{2}$

or

3- $\gamma = 4$
$\eta = 4$

Under such conditions, we end up with the following special beta functions. Where, the shape of each function is known.

For the first, second and third cases the probability density functions become

$$f(y,3+\sqrt{2},3-\sqrt{2})=\frac{13.008}{b-k}\times\left(\frac{y-k}{b-k}\right)^{2+\sqrt{2}}\times\left(1-\frac{y-k}{b-k}\right)^{2-\sqrt{2}}$$

$$f(y,3-\sqrt{2},3+\sqrt{2})=\frac{13.008}{b-k}\times\left(\frac{y-k}{b-k}\right)^{2-\sqrt{2}}\times\left(1-\frac{y-k}{b-k}\right)^{2+\sqrt{2}}$$

$$f(y,4,4)=\frac{140}{b-k}\times\left(\frac{y-k}{b-k}\right)^{3}\times\left(1-\frac{y-k}{b-k}\right)^{3}$$

for $k \leq y \leq b$ and $k>0$,

and are negatively skewed, positively skewed and symmetrical unimodal functions respectively. In fact these functions and their modal values are only the function of k and b.Thus, the modal value relationships for the negatively skewed, positively skewed and symmetrical functions become

$$m=\frac{(2-\sqrt{2})\times k+(2+\sqrt{2})\times b}{4}$$

$$m=\frac{(2+\sqrt{2})\times k+(2-\sqrt{2})\times b}{4} \quad \text{and}$$

$$m=\frac{k+b}{4}$$

respectively. Consequently, it can be concluded that the modal values can not be freely estimated.Because, the modal value is the function of the minimum and maximum value estimates. In other words, contrary to the practice, for a given set of minimum and maximum value estimates the modal value shall be calculated using the above relationships. Estimating the modal value independently contradicts with the theoretical facts.

· Depending on the magnitude of the difference between the estimated, which has no theoretical ground, and the calculated modal values the expected task completion times will differ from the theoretically correct one [2,3]. Under such conditions, the deviation from the real mean task completion time will be as much as

$$d = \frac{4 \times [\text{Calculated Modal Value - Estimated Modal Value}]}{6}$$

When the difference between the calculated and the estimated modal value is negligible or the density is symmetrical then this effect may be ommited. That is when the difference is considerably small the mean job completion time, the mean project completion time and as well as the critical path may not be affected. Otherwise, the results will be affected considerably; especially when the magnitudes of the minimum and maximum task completion times and the difference between them are big enough.

In fact the variance is not affected by this fallacy ; because, it is not expressed as a function of the modal value.

AN APPLICATION

It is proved that the formulas used to calculate the mean activity time and variance are valid only for some special values of the shape parameters (γ and η) of beta distribution. It is also indicated that the modal values of those special beta distributions are defined by lower and upper limits of each activity range. Therefore, the minimum and the maximum of the activity time range estimate and the information on the type of the skewness are enough to calculate the mean activity times. Otherwise, the expected project duration will not be theoretically true.

The difference between the calculated and expected modal values is also function of the length of the range. Improper application, especially for bigger ranges, may lead one to unrealistic results in projects completion time, project completion time probabilities and in an improper and unrealistic critical path selection.

We have selected a real life construction problem where minimum, maximum and mostlikely values are provided,Table 1. First, we have solved the problem using independently estimated modal values along with the estimated minimum and maximum values of each range. This is the way of solving these types of problems in practice. For this purpose, a computer program [7] is employed; program requires that the estimated minimum, maximum and mostlikely values shall be loaded. Later, the same program is modified to calculate the mean activity times using the estimated minimum and maximum activity times; the mostlikely value is calculated on the basis of the estimated minimum and maximum values, and on the type of skewness.

Consequently, we have determined that, for this specific problem, the critical path remained unchanged; it is due to the fact that the difference between the means obtained using estimated modal value and the calculated modal value was not big enough to chance the critical path. The critical path obtained passes through the nodes $0 \rightarrow 1 \rightarrow 3 \rightarrow 5 \rightarrow 8 \rightarrow 10 \rightarrow 11 \rightarrow 13 \rightarrow 14 \rightarrow 16 \rightarrow 17 \rightarrow 20 \rightarrow 27 \rightarrow 43 \rightarrow 49 \rightarrow 51$. Yet, for the first case mean project completion time is 298.9 days with variance 15.97. However, for the second case the mean project completion time is 304.5 days. The variance is the same. Theoretically correct result is the 304.5 days. The probability that the project will be completed within 298.8 days will be 0.0771. When the theoretical distortion is not realized, one might think that the probability of completing the project within 298.8 days is 0.50.

It should be realized that for calculating the mean activity and mean project completion times correctly, one shall use the calculated modal value not the estimated one.

CONCLUSION

Behind the relationships developed for calculating the mean and the variance of beta distribution there is an implicit assumption; one of the three special beta distributions are adopted without realizing. Therefore, to get theoretically correct activity and the mean project completion times the modal value shall be calculated using the minimum and maximum values. There is no need to estimate the modal value independently and it has no theoretical ground at all.

For this special case the critical path remained the same but the probability of project completion time for a certain period of time has changed. However, for some other cases the critical path may also change depending on the difference between the means obtained using the estimated and calculated modal values.

TABLE 1

Job Description For Mills Company Constructions

Event No. P.	S.	DESCRIPTION	Elapsed Times (Days) (a)	(m)	(b)
0	1	Survey and excavation	10	11	19
0	2	Re-bar	5	7	9
0	3	Form made	9	11	14
1	3	Leen conc. and Marking	4	6	7
2	3	Dummy activity	-	-	-
3	6	Bldg "C" 1st and 2nd form conc.	13	14	17
3	5	Bldg "A"1st and 2nd form conc.	15	19	21
3	4	Bldg "B" 1st and 2nd form conc.	15	17	20
4	7	Bldg "B" back fill	8	10	17
5	8	2nd form conc.	9	11	12
6	7	Bldg "C" back fill	8	10	11
7	12	Bldg "C" hard core and G/F conc.	10	12	15
7	9	Bldg "B" hard core and G/F conc.	11	13	14
8	10	Bldg "A" back fill kerp	15	19	22
9	15	Bldg "B" 1st floor P.C Erection	7	9	11
10	11	Bldg "A" hard core and G/F conc	12	14	15
11	13	Bldg "A" stail wall	11	13	14
12	14	Bldg "C" 1st floor P.C Erection	6	7	9
13	14	Bldg "A" 1st floor P.C Erection	13	15	16
14	16	Bldg "A""C" 2nd floor p.c Erection	5	7	8
15	16	Bldg "B" 2nd floor p.c Erection	5	6	8
16	17	Water tank and side wall	8	11	13
16	21	Arcade p.c and penthouse p.c Erec.	25	30	34
17	19	Bldg "A" Block work	22	26	30
17	20	Bldg "B" Block work	11	12	14
17	18	Bldg "C" Block work	16	20	23
18	20	Dummy activity	-	-	-
18	24	Bldg "C" plaster G/F	35	40	45
19	20	Dummy activity	-	-	-
19	22	Bldg "A" plaster G/F	23	29	34
20	25	Electric piping	17	19	20
20	27	Plumbing	10	12	14

23	29	Bldg "B" plaster G/F	6	8	9
21	48	Hand rail marking	18	20	22
21	47	Roof/sloop conc.	14	17	19
22	28	Bldg "A" plaster 1st	16	18	20
23	29	Bldg "B" plaster 1st	9	11	13
24	26	Bldg "C" plaster 1st	19	22	25
25	44	Electric working	26	31	35
26	30	Bldg "C" terrazzo tile	8	11	12
26	38	Bldg "C" suspended ceiling	8	11	13
26	37	Bldg "C" all door windows	13	14	16
27	43	Sanitary piping	12	15	18
28	31	Bldg "A" terrazzo tile	11	12	14
28	32	Bldg "A" suspended ceiling	10	11	12
28	33	Bldg "A" all door windows	8	10	12
29	34	Bldg "B" terrazzo tile	9	12	13
29	35	Bldg "B" suspended ceiling	13	15	17
29	36	Bldg "B" all door windows	9	10	13
30	39	Bldg "C" ceramic tile	13	14	16
30	40	Bldg "C" marble work	11	12	14
31	46	Bldg "A" ceramic tile	10	13	14
31	45	Bldg "A" marble work	16	19	21
32	46	Bldg "A" putting	15	16	18
33	46	Bldg "A" wooden door fixing	12	14	16
34	41	Bldg "B" ceramic tile	10	12	13
34	42	Bldg "B" marble work	8	10	11
35	42	Bldg "B" putty	10	12	13
36	42	Bldg "B" wooden door fixing	30	31	39
37	40	Bldg "C" wooden door fixing	18	20	22
38	40	Bldg "C" putty	14	15	17
39	40	Dummy activity	-	-	-
40	49	Bldg "C" internal paint	10	12	14
41	42	Dummy activity	-	-	-
42	49	Bldg "B" internal paint	47	52	58
43	49	Sanitary fixing	20	21	23
44	49	Lighting fixing	36	40	43
45	46	Dummy activity	-	-	-
46	49	Bldg "A" internal paint	13	14	16
47	49	Roof water proof and test	22	25	28
48	49	Bldg "A""B""C" hand rail fixing	11	12	14
49	50	External paint	14	16	18
49	51	Bldg "A""B""C" external paint test	19	20	21
50	51	Dummy activity	-	-	-
51	52	Site	8	10	11

REFERENCES

1. Hahn, Gerald J., Shapiro, Samuel S., Statistical Models in Engineering, John Wiley and Sons 1967, pp. 91-99.

2. Unutulmaz, Osman, "Olasılıklı Pert Analizinde Kullanılan Beta Dağılımı ve Uygulama Hataları" **Erciyes Üniversitesi İktisadi ve İdari Bilimler Fakültesi Dergisi**, Mayıs 1983, Sayı: 4 Sh. 181-189

3. Unutulmaz, Osman, "Olasılıklı Pert Analizinde Kullanılan Beta Dağılımının Beklenen Değerinin Hesabında Ortaya Çıkabilecek Hataların Parametrik Olarak İncelenmesi" **Erciyes Üniversitesi İktisadi ve İdari Bilimler Fakültesi Dergisi**, Kasım 1984, Sayı: 6, Sh. 107-111.

4. Sivazlian, B. D., Stanfel L. E., Optimization Techniques In Operations Research, Prentice-Hall, Inc. New Jersey 1975.

5. Bhatangar, S. K., Network Analysis Techniques, A Halsted Press Book, John Wiley and Sons Inc. New Delhi, 1986.

6. Hillier, Frederic S., Lieberman, Gerald J., Introduction to Operations Research, Holdan-day Inc., 1973.

7. Whitehouse, Gray E., Washburn, Donald A., "A Program For PERT on a Minicomputer", **Industrial Engineering**, Vol. 13, No.1, January 1981, pp. 20-21.