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Research Article

An Incircle, Circumcircle, Excircle and Apollonius Circle of a Triangle in the Maximum Plane

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Abstract
Classical Euclidean geometry places significant emphasis on circles related to triangles, such as the incircle, circumcircle, excircle, and Apollonius circles. Each of these circles shows important features of the triangle. As new types of geometry were developed, these classic shapes were looked at again in different ways, leading to new mathematical ideas. One of these new geometries is called maximum plane geometry, which uses a different way to measure distances. In this geometry, circles take the form of axes-aligned squares. This creates both similarities and differences compared to circles in regular Euclidean geometry. This paper investigates the existence and uniqueness of these types of circles in maximum plane geometry and analyzes their properties. By clearly defining them and looking at their effects, the paper tries to build on old results, show how they are different, and find uses in areas like computational geometry and discrete mathematics.

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1. Introduction

The exploration of circles related to triangles has long occupied a central position in classical Euclidean geometry. Constructions such as the *incircle*, *circumcircle*, *excircle*, and *Apollonius circle* provide not only elegant representations of triangular structures but also essential tools for uncovering deeper geometric relationships. Over the centuries, these notions have inspired extensive generalizations and motivated the search for analogues within broader geometric settings.

More recently, the rise of non-Euclidean and metric-based geometries has offered new perspectives on these traditional concepts. In particular, *maximum plane geometry*, determined by the maximum metric

$$d_\infty(A,B) = \max\{|x_A - x_B|, |y_A - y_B|\},$$

for points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ in $\mathbb{R}^2$, presents a fertile ground for investigation. Unlike the Euclidean case, the unit circle in this metric is represented by an axis-aligned square, leading to geometric behaviors that are often unexpected. Constructions such as tangency, centers of triangles, and circle-related loci thus acquire new and distinctive forms.

In this framework, the classical circles of a triangle require significant reinterpretation. The incircle, for example, may fail to be unique, and its tangency conditions are closely tied to the orientation and relative position of the triangle. The circumcircle, typically defined by a square enclosing the vertices, alters familiar relations such as those involving Euler's line or the nine-point circle. Likewise, Apollonius circles, which in Euclidean geometry describe loci of constant distance ratios to two vertices, take on polyhedral analogues in the maximum metric, revealing further interactions between distance functions and geometric configurations.

Beyond their theoretical importance, these reinterpretations also have practical implications. Metrics of this kind naturally arise in contexts such as optimization, digital and discrete geometry, and computer vision, where axis-aligned and grid-based structures are prevalent. A refined understanding of classical geometric objects within the maximum metric can therefore enrich algorithmic approaches and computational models.

The aim of this study is to investigate the existence, uniqueness, and properties of the incircle, circumcircle, and Apollonius circles in maximum plane geometry. Through a systematic comparison with their Euclidean analogues, we seek to highlight the structural differences and identify new geometric patterns that emerge in this setting. In doing so, the work extends the boundaries of triangle geometry while also strengthening the connections between metric geometry, polyhedral structures, and applications in the mathematical sciences.

2. Preliminaries

Before presenting the main results, we recall some basic notions concerning metric spaces and maximum plane geometry. A *metric space* (X,d) consists of a set X together with a function $d: X\times X\to\mathbb{R}_{\geq 0}$ that satisfies the properties of non-negativity, symmetry, and the triangle inequality. In the Euclidean plane, the standard distance function is defined by

$$d_2(A,B)=\sqrt{(x_A-x_B)^2+(y_A-y_B)^2},$$

for two points $A=(x_A,y_A)$ and $B=(x_B,y_B)$ in $\mathbb{R}^2$.

In contrast, the *maximum metric*, also referred to as the Chebyshev metric or the L_∞ metric, is given by

$$d_\infty(A,B)=\max\{|x_A-x_B|,|y_A-y_B|\}.$$

The unit circle with respect to d_∞ is the set

$$C(0,1)=\{(x,y)\in\mathbb{R}^2:\max\{|x|,|y|\}=1\},$$

which geometrically corresponds to a square centered at the origin with sides parallel to the coordinate axes. This departure from the Euclidean case results in new behaviors of classical constructions. Concepts such as tangency, inscribed and circumscribed figures, and distance ratios require reformulation in order to properly reflect the geometry determined by d_∞.

The study of metric spaces beyond the Euclidean framework has a long and rich tradition. The foundational work of Fréchet and Hausdorff established the basis of metric and topological spaces. Subsequently, alternative distance functions such as the taxicab metric (L_1) and the maximum metric (L_∞) have attracted considerable attention, both for their distinctive geometric properties and for their wide range of applications in mathematics and related fields (see [1, 2, 3, 4]).

In the context of polyhedral geometry, it has been shown that the unit spheres corresponding to the Euclidean, taxicab, and maximum metrics can be represented by the circle, the octahedron, and the cube, respectively. This connection has stimulated extensive research into the relationship between metrics and polyhedral structures. In particular, studies have examined the behavior of triangle centers, bisectors, and other classical constructs under these alternative metrics, leading to generalizations of well-known Euclidean theorems (see [5, 6, 7, 8]).

Recent work has focused specifically on the reinterpretation of incircles, circumcircles, and related loci within discrete and non-Euclidean geometries. For instance, research in digital geometry has considered axis-aligned approximations of circular arcs, while investigations in computational geometry have explored optimization-based characterizations of metric circles. However, the systematic study of incircle, circumcircle, and Apollonius circles in the maximum plane remains relatively underdeveloped, leaving significant room for new contributions.

This paper aims to build upon these foundations by offering a comprehensive examination of triangle-associated circles in maximum plane geometry. By doing so, we contribute both to the theoretical advancement of metric geometry and to the broader discourse linking classical geometric ideas with modern applications. The results and examples given in the Chapters 3 and 4 are from the master's thesis of Aylin Palazoğlu, given in reference [8], and have been published in an article in order to reach a wider audience.

3. Incircle and circumcircle of a triangle in the maximum plane

In this section, we examine the concepts of the incircle and circumcircle of an arbitrary triangle considered in the maximum plane. Since the points and lines of the maximum plane coincide with those of the Euclidean plane, the triangles in the maximum plane are identical to the ones in the Euclidean plane. However, as the notion of distance is defined differently, the concept of a circle also changes. As stated in the section on fundamental concepts, a circle in the maximum plane corresponds to a Euclidean square whose sides are parallel to the coordinate axes. It is well known that any triangle in the Euclidean plane admits both an incircle and a circumcircle. However, this becomes a nontrivial problem in the maximum plane and therefore deserves careful investigation. Within this framework, we shall explore the conditions under which a triangle in the maximum plane possesses an incircle and a circumcircle. To this end, we first provide the formal definitions of the incircle and circumcircle in this context.

Definition 1. *Let a triangle be given in the maximum plane. The maximum circle tangent to all three sides of the triangle and lying entirely within the triangle is called the* incircle *of the triangle.*

Definition 2. *Let a triangle be given in the maximum plane. The maximum circle passing through all three vertices of the triangle and containing the triangle within it is called the* circumcircle *of the triangle.*

These definitions can also be expressed from another perspective:

i. The incircle is the largest circle that can be inscribed in the triangle.

ii. The circumcircle is the smallest circle that can circumscribe the triangle.

Accordingly, a crucial concept that emerges in both the incircle and circumcircle in the maximum plane is the notion of *tangency*, which requires careful consideration.

In Euclidean geometry, the tangency of a line and a circle is defined by the existence of a single common point between the line and the circle. However, this notion is modified in maximum geometry. Since a circle in the maximum plane corresponds to a Euclidean square whose sides are parallel to the coordinate axes, tangency between a line and a maximum circle occurs not only when they share exactly one common point but also when the line coincides with one side of the square.

Figure 1 below illustrates and characterizes this situation.

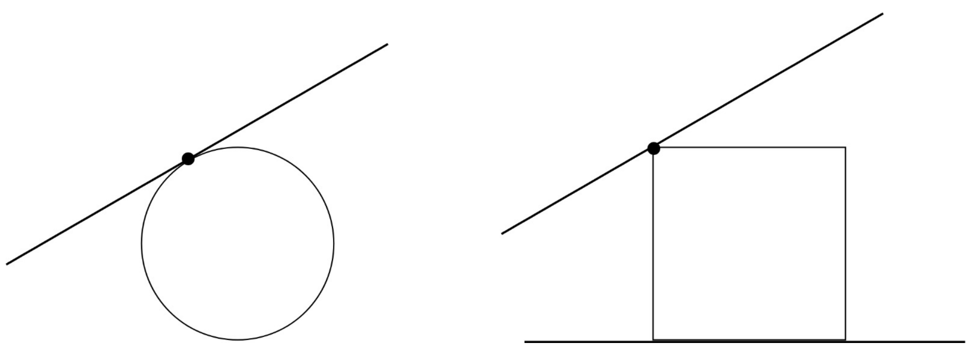

Fig. 1. Tangent to circle in Euclidean and maximum planes

In this section, the vertices of the considered triangles will be labeled in a clockwise manner. Since all translations are isometries, the vertex C will be placed at the origin without loss of generality. When C is taken as the origin, the other two vertices A and B are symmetric in roles; that is, although a particular orientation is adopted, interchanging A and B merely results in an exchange of their roles without affecting generality.

Accordingly, we set

$$A = (x_a, y_a), \quad B = (x_b, y_b), \quad C = (x_c, y_c) = (0, 0),$$

with the additional assumption $y_a > y_b$ without loss of generality.

Furthermore, in maximum plane geometry, reflections with respect to lines of slope $m \in \{-1, 0, 1\}$ and $m \to \infty$, as well as rotations through angles $\theta = k\frac{\pi}{2}$ for $k \in \{0, 1, 2, 3\}$, are isometries. Therefore, it suffices to restrict our attention to the configurations illustrated below, see in Figure 2. Triangles in such a configuration will be referred to as *triangles in the fundamental position*.

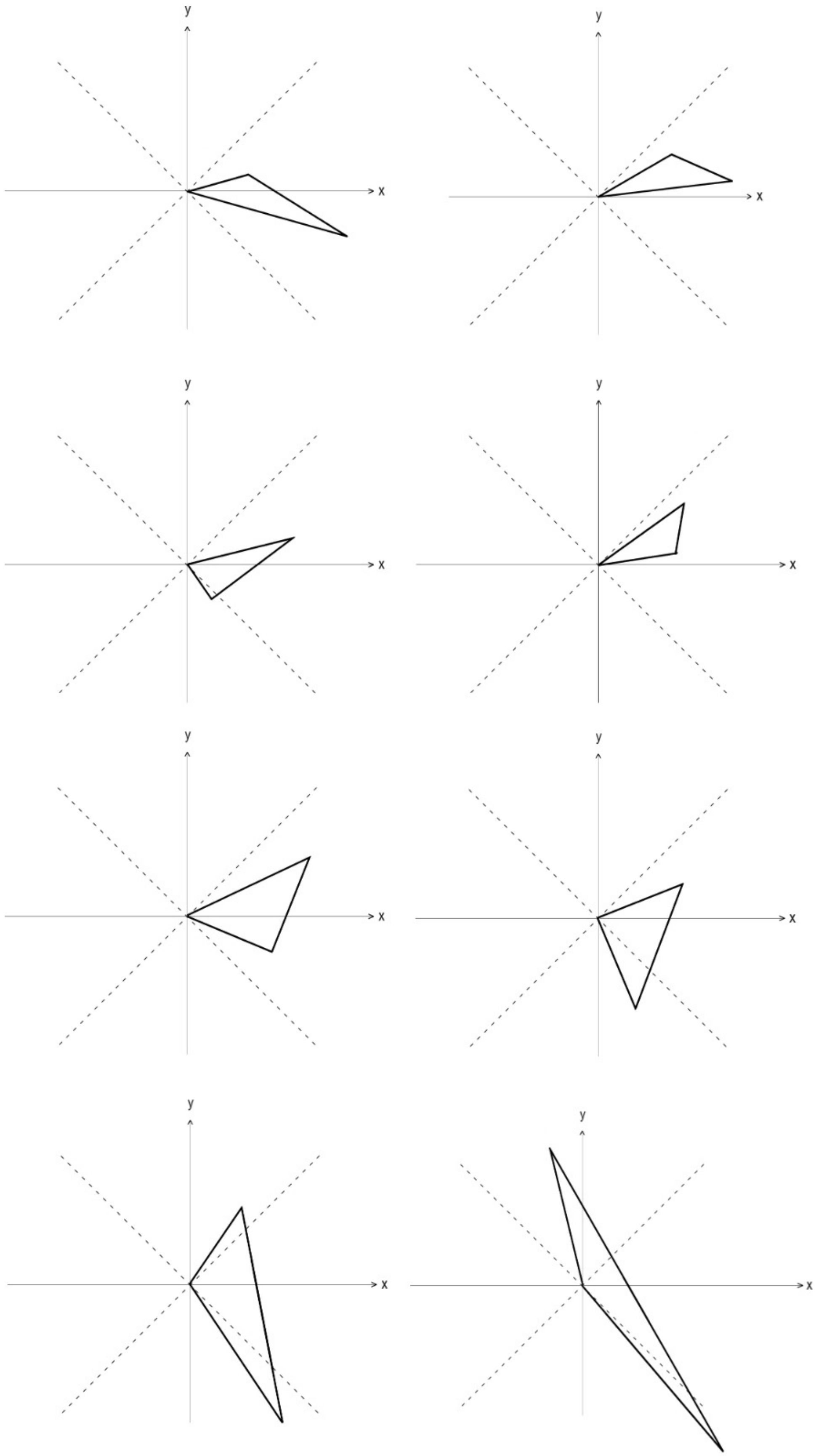

Fig. 2. Triangles in fundamental position in maximum plane

Theorem 3. *In the maximum plane, every triangle admits a unique incircle. Moreover, the center of the incircle coincides with the intersection point of the maximum internal angle bisectors of the triangle.*

Proof. Let $\triangle ABC$ be a triangle in the maximum plane in the fundamental position. Without loss of generality, we take

$$A=(x_a,y_a),\quad B=(x_b,y_b),\quad C=(0,0).$$

The equations of the lines containing the sides of the triangle are

$$\ell_a: x_by-y_bx=0,\quad \ell_b: x_ay-y_ax=0,\quad \ell_c:(x_b-x_a)y-(y_b-y_a)x+x_a(y_b-y_a)-y_a(x_b-x_a)=0.$$

Using the definition of distance in the maximum plane and ensuring the bisectors remain inside the triangle, the equations of the internal angle bisectors are obtained. Setting

$$m_a=\frac{y_b}{x_b},\quad m_b=\frac{y_a}{x_a},\quad m_c=\frac{y_b-y_a}{x_b-x_a},$$

the bisector at vertex C is given by

$$\frac{|x_by-y_bx|}{|x_b|+|y_b|}=\frac{|x_ay-y_ax|}{|x_a|+|y_a|},$$

which simplifies to

$$y=\frac{(1+|m_b|)m_a+(1+|m_a|)m_b}{(1+|m_b|)+(1+|m_a|)}x.$$

Similarly, the bisector at vertex A is expressed as

$$\frac{|x_ay-y_ax|}{|x_a|+|y_a|}=\frac{|(x_b-x_a)y-(y_b-y_a)x+x_a(y_b-y_a)-y_a(x_b-x_a)|}{|x_b-x_a|+|y_b-y_a|},$$

which, after simplification, becomes

$$y=\frac{(1+|m_b|)m_c+(1+|m_c|)m_b}{|m_b|-|m_c|}x+\frac{x_a(1+|m_b|)(m_c-m_b)}{|m_b|-|m_c|}.$$

For the bisector at vertex B, one similarly obtains

$$y=\frac{(1+|m_c|)m_a+(1+|m_a|)m_c}{1+|m_a|+|m_c|}x+\frac{x_a\big((1+|m_a|)m_c+(1+|m_a|)m_b\big)}{1+|m_a|+|m_c|}.$$

Solving this linear system, and recalling $m_a=\frac{y_b}{x_b}$, $m_b=\frac{y_a}{x_a}$, and $m_c=\frac{y_b-y_a}{x_b-x_a}$, the coordinates of the intersection point are found as

$$x=\frac{x_a(|x_b|+|y_b|)+x_b(|x_a|+|y_a|)}{|x_a|+|y_a|+|x_b|+|y_b|+|x_b-x_a|+|y_b-y_a|},$$

$$y=\frac{y_a(|x_b|+|y_b|)+y_b(|x_a|+|y_a|)}{|x_a|+|y_a|+|x_b|+|y_b|+|x_b-x_a|+|y_b-y_a|}.$$

This point lies inside $\triangle ABC$ and is equidistant from all three sides, hence it is the center of the incircle. Consequently, the unique incircle of $\triangle ABC$ exists, and its radius is

$$r=\frac{y_ax_b-x_ay_b}{|x_a|+|y_a|+|x_b|+|y_b|+|x_b-x_a|+|y_b-y_a|}.$$

Figure 3 illustrates the incircles of triangles in the fundamental position.

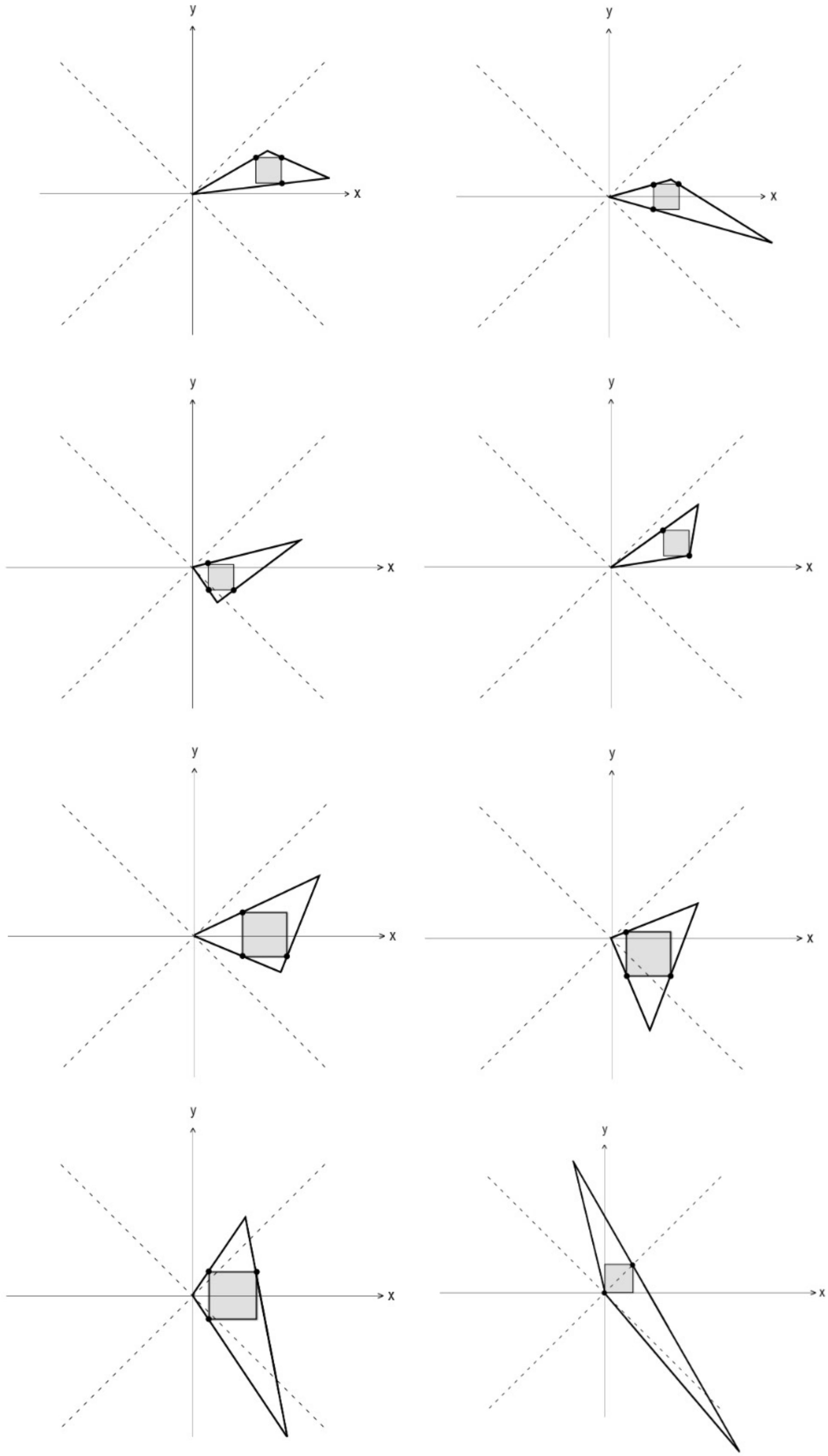

Fig. 3. The incircles of triangles in the fundamental position

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Let the ordering of the projections of given points onto the x-axis, or equivalently the ordering of their abscissae, be called the *horizontal ordering*. Similarly, let the ordering of the projections of points onto the y-axis, or equivalently the ordering of their ordinates, be called the *vertical ordering*. This ordering will play a crucial role in determining the existence of the circumcircle of a triangle in the maximum plane, as expressed in the following theorem.

Theorem 4. *In the maximum plane, if one of the three vertices of a triangle occupies the middle position in both horizontal and vertical ordering, then the triangle has no circumcircle. In all other cases, the triangle admits a circumcircle.*

Proof. Let $\triangle ABC$ be a triangle in the maximum plane in the fundamental position. Since a maximum circle is a Euclidean square with sides parallel to the coordinate axes, the circumcircle of $\triangle ABC$ must be such a square passing through its three vertices.

If none of the edges of the triangle are horizontal or vertical, then each vertex must lie on a distinct side of the square. Hence, two of the square's sides must be parallel to the x-axis and the other two parallel to the y-axis. Conversely, if at least one side of the triangle is horizontal or vertical, then two vertices of the triangle lie on two adjacent sides of the square. In this case, the circumcircle can be constructed by drawing horizontal and/or vertical lines passing through the vertices of the triangle, provided these lines do not intersect the interior of the triangle.

Now suppose one vertex of the triangle lies in the middle position with respect to both the horizontal and vertical ordering. In that case, no horizontal or vertical line passing through that vertex can avoid intersecting the interior of the triangle (i.e., such a line cannot be tangent to the triangle). Therefore, the circumcircle of the triangle does not exist.

For triangles in the fundamental position, this situation occurs precisely when

$$0 < y_b < y_a, \quad 0 < x_b < x_a, \quad \text{or} \quad y_b < 0 < y_a,\ x_a < 0 < x_b.$$

In these cases, the circumcircle cannot be drawn.

On the other hand, if vertex C admits a vertical (or horizontal) line that does not intersect the interior of the triangle, then the lines through vertices A and B must be chosen of the opposite orientation (horizontal or vertical). To determine the correct configuration, one measures the lengths of the sides AC and BC; the more distant vertex from C dictates the orientation of the line opposite to C. If multiple configurations are possible, the type is fixed by requiring that the longest side of the triangle must lie on parallel sides of the square.

Consequently, whenever none of the three vertices occupies the middle position in both horizontal and vertical ordering, a circumcircle exists for the triangle in the maximum plane.

Figure 4 illustrates the circumcircles of triangles in the fundamental position. ■

Once a triangle in the maximum plane is known to admit a circumcircle, determining its center and radius is straightforward. First, the longest side of the triangle is identified.

i. If the longest side is horizontal, then the abscissa of the center is given by the arithmetic mean of the abscissae of the two vertices on that side. If the remaining vertex lies above (resp. below) this side, then the ordinate of the center is obtained by subtracting (resp. adding) half of the length of the longest side from the ordinate of the third vertex.

ii. If the longest side is vertical, then the ordinate of the center is given by the arithmetic mean of the ordinates of the two vertices on that side. If the remaining vertex lies to the right (resp. to the left) of this side, then the abscissa of the center is obtained by subtracting (resp. adding) half of the length of the longest side from the abscissa of the third vertex.

In either case, the radius of the circumcircle is equal to half of the length of the longest side.

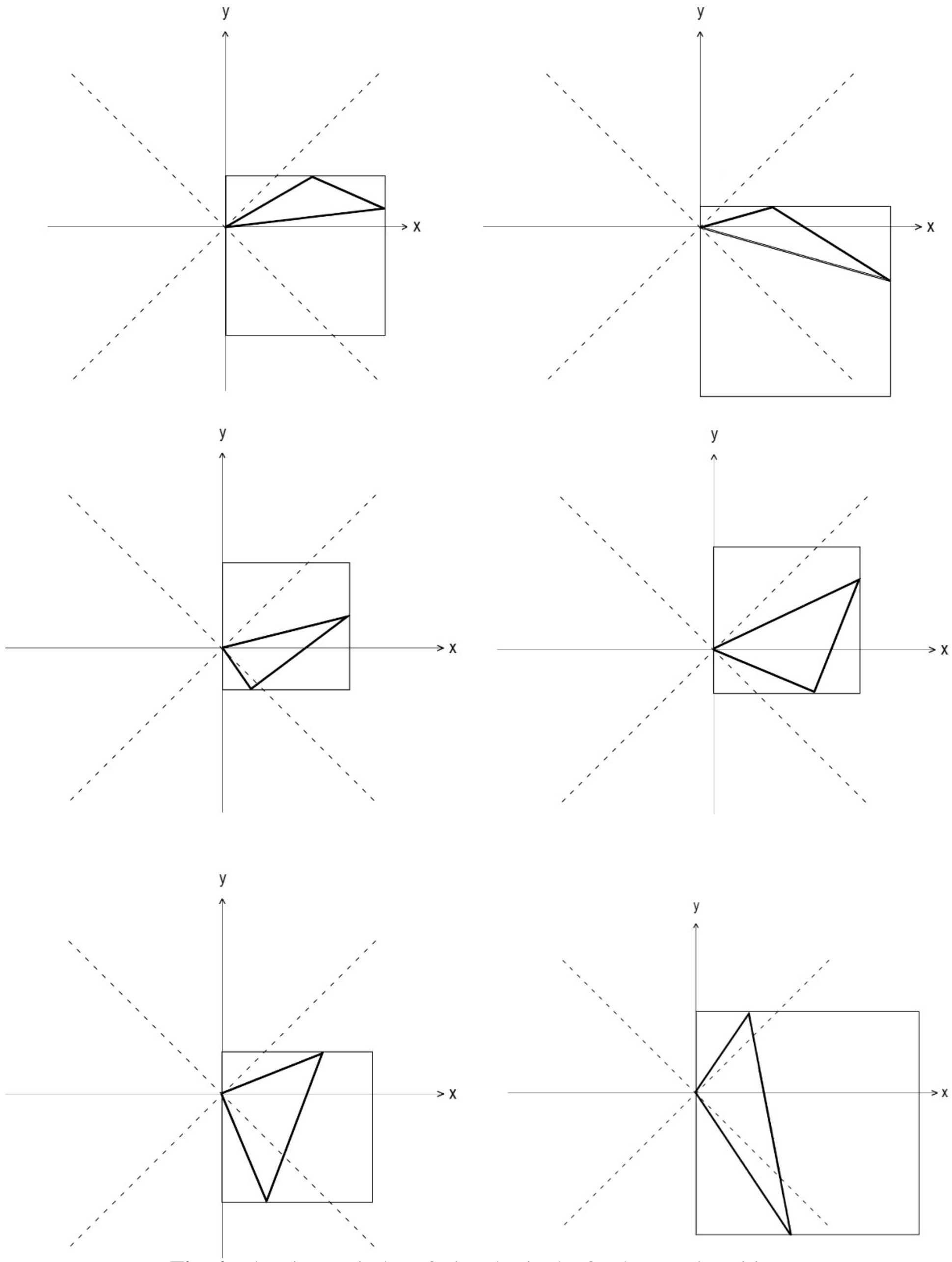

Fig. 4. The circumcircles of triangles in the fundamental position

Example 5. *Consider the triangle $\triangle ABC$ with vertices*

$$A=(3,2),\quad B=(7,1),\quad C=(0,0).$$

The side lengths in the maximum metric are

$$|a|=d_M(B,C)=\max\{|7-0|,|1-0|\}=7,$$

$$|b|=d_M(A,C)=\max\{|3-0|,|2-0|\}=3,$$

$$|c|=d_M(A,B)=\max\{|7-3|,|1-2|\}=4.$$

Thus, the longest side is BC, with length 7*. Its slope is*

$$m_{BC}=\frac{1-0}{7-0}=\frac{1}{7},$$

showing that BC is essentially horizontal. Therefore, the abscissa of the center is the midpoint of B and C:

$$x_M = \frac{0+7}{2} = \frac{7}{2}.$$

Since vertex A lies above the line BC, the ordinate of the center is given by

$$y_M = y_A - \frac{|BC|}{2} = 2 - \frac{7}{2} = -\frac{3}{2}.$$

Hence, as also shown in the Figure 5, the center of the circumcircle is

$$M = \left(\tfrac{7}{2}, -\tfrac{3}{2}\right),$$

and the radius is

$$r = \frac{|BC|}{2} = \frac{7}{2}.$$

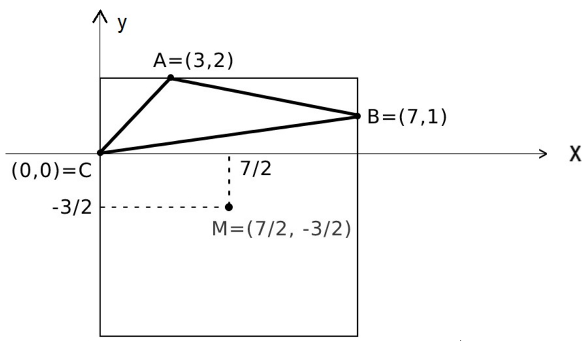

Fig. 5. The circumcircle of the triangle $\triangle ABC$

In the maximum plane, if $\triangle ABC$ has at least one horizontal or vertical side, or if two of its sides are horizontal and vertical, then the triangle may admit infinitely many circumcircles. The following theorem makes this situation precise.

Theorem 6. *Let $\triangle ABC$ be a triangle in the maximum plane. Then:*

i. *If one side of $\triangle ABC$ lies on a horizontal (or vertical) line and this side is not the longest side of the triangle, then $\triangle ABC$ has infinitely many circumcircles.*

ii. *If two sides of $\triangle ABC$ lie on horizontal and vertical lines, then $\triangle ABC$ has infinitely many circumcircles.*

Proof. Suppose $\triangle ABC$ has one side lying on a horizontal or vertical line. Without loss of generality, let BC be horizontal.

If BC is the longest side, then by the fundamental position assumption we must have

$$0 \le x_a \le x_b, \qquad 0 \le y_a \le x_b.$$

In this case, the horizontal side of the circumcircle must pass through vertex A. Consequently, the vertical sides of the circumcircle are determined uniquely by vertices B and C, and thus the circumcircle is uniquely defined.

If, however, $y_a > x_b$ with $0 \le x_a \le x_b$, then clearly the horizontal side of the circumcircle must pass through vertex A. In this configuration, AC and AB are the longest sides, and the radius of the circumcircle is $y_a/2$. The abscissa of the lower-left corner of the square may take any value in the interval $[x_b - y_a, 0]$. Hence, there exist infinitely many circumcircles.

On the other hand, if $x_a > x_b$ or $x_a < 0$, then the vertical side of the circumcircle must pass through vertex A. In this case, the minimal possible radius is

$$r_{\min} = \max\left\{\frac{|x_a|}{2}, \frac{|y_a|}{2}\right\},$$

and a circumcircle exists for every radius $r \ge r_{\min}$. Therefore, there are infinitely many circumcircles.

Finally, if $\triangle ABC$ has two sides lying on horizontal and vertical lines, then necessarily two sides of the circumcircle coincide with these sides. The minimal possible radius in this case is

$$r_{\min} = \max\left\{\frac{|y_a|}{2}, \frac{|x_b|}{2}\right\},$$

and again, for every radius $r \geq r_{\min}$, a circumcircle exists. Thus, there are infinitely many circumcircles in this case as well. Figure 6 shows some examples of this situation.

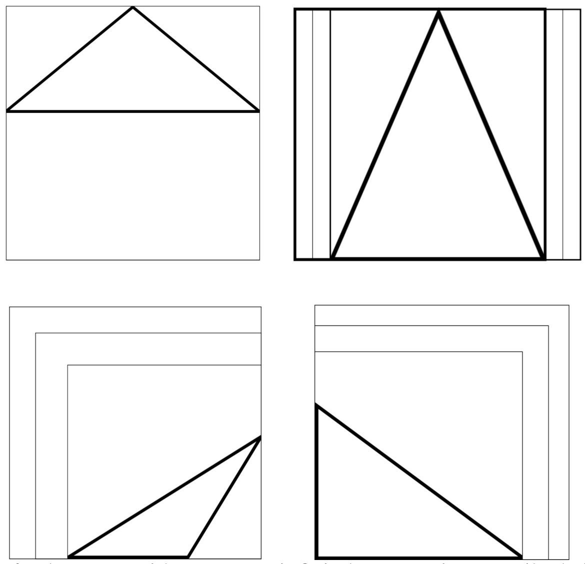

Fig. 6. The cases with respect to infinitely many circumscribed circle

■

4. The Apollonius circle of a triangle in the maximum plane

In this section, we investigate the notion of a circle in the maximum plane that contains the three excircles of a given triangle and is tangent to each of them. This concept corresponds to a well-known configuration in the Euclidean plane, where such a circle is called the *Apollonius circle*. Figure 7 illustrates this situation in the Euclidean setting.

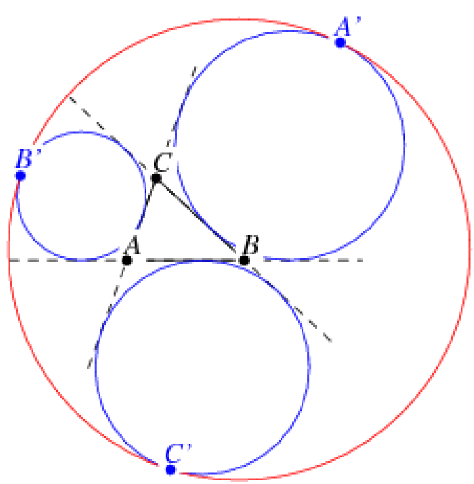

Fig. 7. Apollonius circle in the Euclidean plane

It is known that every triangle in the Euclidean plane admits an Apollonius circle in this sense. However, whether such a construction holds in the maximum plane constitutes a problem worth examining. Therefore, we begin by introducing the definitions of an excircle and the Apollonius circle in the maximum plane.

Definition 7. *Let $\triangle ABC$ be a triangle in the maximum plane. A* maximum excircle *of $\triangle ABC$ is defined as a maximum circle that is tangent to one side of the triangle and to the extensions of the other two sides.*

Definition 8. *Let $\triangle ABC$ be a triangle in the maximum plane. The* Apollonius circle *of $\triangle ABC$ is defined as a maximum circle that contains the three maximum excircles of the triangle and is tangent to each of them.*

As in Section 3, the notion of tangency here is not limited to the case when a line and a circle (or two circles) have exactly one common point. Instead, tangency will also include the situation in which a line and a circle, or two circles, intersect along a line segment. Furthermore, the assumption that triangles are in the fundamental position will again be employed.

Theorem 9. *Let $\triangle ABC$ be a triangle in the maximum plane in the fundamental position with vertices*

$$A = (x_a, y_a), \quad B = (x_b, y_b), \quad C = (0,0).$$

Then:

i. *If* $|x_a| + |y_a| + |x_b| + |y_b| - |x_b - x_a| - |y_b - y_a| \neq 0$, *then $\triangle ABC$ admits an excircle tangent to the side AB.*

ii. *If* $|x_a| + |y_a| + |x_b - x_a| + |y_b - y_a| - |x_b| - |y_b| \neq 0$, *then $\triangle ABC$ admits an excircle tangent to the side BC.*

iii. *If* $|x_a| + |y_a| - |x_b| - |y_b| - |x_b - x_a| - |y_b - y_a| \neq 0$, *then $\triangle ABC$ admits an excircle tangent to the side AC.*

Proof. Consider $\triangle ABC$ in the maximum plane with vertices $A = (x_a, y_a)$, $B = (x_b, y_b)$, and $C = (0,0)$ in the fundamental position. For an excircle tangent to the side AB and the extensions of AC and BC to exist, there must be a point equidistant from these three lines. Thus, one must examine the external angle bisectors at the vertices A and C, together with the internal angle bisector at the vertex B. If these three bisectors intersect at a point, then this point is the center of the desired excircle.

The equations of the lines containing the sides are given by:

$$\overleftrightarrow{BC} : x_b y - y_b x = 0,$$

$$\overleftrightarrow{AC} : x_a y - y_a x = 0,$$

$$\overleftrightarrow{AB} : (x_b - x_a)y - (y_b - y_a)x + x_a(y_b - y_a) - y_a(x_b - x_a) = 0.$$

Using the distance formula in the maximum plane and taking orientations into account, the corresponding bisectors can be written as follows. For instance, the internal angle bisector at C is

$$\frac{|x_b y - y_b x|}{|x_b| + |y_b|} = \frac{|x_a y - y_a x|}{|x_a| + |y_a|},$$

which can be simplified to

$$\big[x_b(|x_a| + |y_a|) + x_a(|x_b| + |y_b|)\big]y - \big[y_b(|x_a| + |y_a|) + y_a(|x_b| + |y_b|)\big]x = 0.$$

Similarly, the external angle bisector at A becomes

$$\begin{aligned}[x_a(|x_b - x_a| + |y_b - y_a|) \quad &+ \quad (x_b - x_a)(|x_a| + |y_a|)]y - [y_a(|x_b - x_a| + |y_b - y_a|) + (y_b - y_a)(|x_a| + |y_a|)]x \\ &+ \quad (|x_a| + |y_a|)[x_a(y_b - y_a) - y_a(x_b - x_a)] = 0\end{aligned}$$

while the external angle bisector at B is given by

$$\begin{aligned}[x_b(|x_b - x_a| + |y_b - y_a|) \quad &- \quad (x_b - x_a)(|x_b| + |y_b|)]y - [y_b(|x_b - x_a| + |y_b - y_a|) - (y_b - y_a)(|x_b| + |y_b|)]x \\ &+ \quad (|x_b| + |y_b|)[x_a(y_b - y_a) - y_a(x_b - x_a)] = 0.\end{aligned}$$

Solving this system of three linear equations yields the coordinates of the intersection point:

$$x = \frac{x_b(|x_a| + |y_a|) + x_a(|x_b| + |y_b|)}{|x_a| + |y_a| + |x_b| + |y_b| - |x_b - x_a| - |y_b - y_a|},$$

$$y = \frac{y_b(|x_a| + |y_a|) + y_a(|x_b| + |y_b|)}{|x_a| + |y_a| + |x_b| + |y_b| - |x_b - x_a| - |y_b - y_a|}.$$

This point is equidistant from side the AB and the extensions of AC and BC, and thus is the center of the excircle tangent to AB. Its radius is

$$r = \frac{y_a x_b - x_a y_b}{|x_a| + |y_a| + |x_b| + |y_b| - |x_b - x_a| - |y_b - y_a|}.$$

If the denominator vanishes, the center and radius cannot be defined, and hence no such excircle exists. The existence and formulas for the excircles tangent to the sides BC and AC follow analogously, he and this yields the remaining parts of the theorem. ■

Example 10. *Let $\triangle ABC$ be a triangle in the maximum plane with vertices $A = (3,9)$, $B = (5,1)$, and $C = (0,0)$. Accordingly, as seen in Figure 8, the excircle tangent to the side AB has center*

$$M_1 = \left(\tfrac{39}{4}, \tfrac{33}{4}\right), \quad r_1 = \tfrac{21}{4},$$

the excircle tangent to the side BC has center

$$M_2 = \left(\tfrac{21}{8}, -\tfrac{21}{8}\right), \quad r_2 = \tfrac{21}{8},$$

and the excircle tangent to the side AC has center

$$M_3 = \left(-\tfrac{21}{2}, \tfrac{21}{2}\right), \quad r_3 = \tfrac{21}{2}.$$

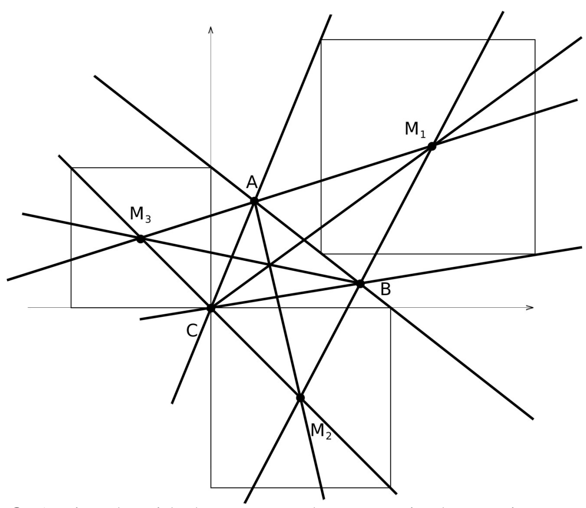

Fig. 8. A triangle with three external tangents in the maximum plane

Example 11. *Let $\triangle ABC$ be a triangle in the maximum plane with vertices $A = (8,24)$, $B = (5,1)$, and $C = (0,0)$. Then, the excircle tangent to the side AB has center*

$$M_1 = \left(\tfrac{52}{3}, \tfrac{44}{3}\right), \quad r_1 = \tfrac{28}{3}.$$

Similarly, the excircle tangent to the side BC has center

$$M_2 = \left(\tfrac{28}{13}, -\tfrac{28}{13}\right), \quad r_2 = \tfrac{28}{13}.$$

However, for the excircle tangent to the side AC, we obtain

$$|x_a| + |y_a| - |x_b| - |y_b| - |x_b - x_a| - |y_b - y_a| = 8 + 24 - 5 - 1 - 3 - 23 = 0,$$

which implies that such an excircle does not exist. Figure 9 illustrates this situation.

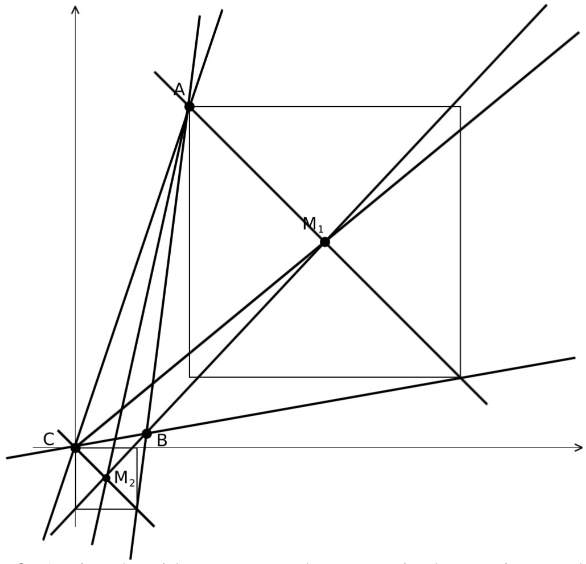

Fig. 9. A triangle with two external tangents in the maximum plane

Theorem 12. *Let $\triangle ABC$ be a triangle in the maximum plane in standard position with vertices $A=(x_a,y_a)$, $B=(x_b,y_b)$, and $C=(0,0)$. If at least one of the following conditions holds:*

$$|x_a|+|y_a|+|x_b|+|y_b|-|x_b-x_a|-|y_b-y_a|=0,$$

$$|x_a|+|y_a|+|x_b-x_a|+|y_b-y_a|-|x_b|-|y_b|=0,$$

$$|x_a|+|y_a|-|x_b|-|y_b|-|x_b-x_a|-|y_b-y_a|=0,$$

then the Apollonius circle of $\triangle ABC$ does not exist.

Proof. Let $\triangle ABC$ be given in the maximum plane in standard position with $A=(x_a,y_a)$, $B=(x_b,y_b)$, and $C=(0,0)$. By Theorem 9, under the above conditions, $\triangle ABC$ has at least one side without an excircle. Consequently, it is immediate that an Apollonius circle cannot exist. ∎

Theorem 13. *Let $\triangle ABC$ be a triangle in the maximum plane in the fundamental position with vertices $A=(x_a,y_a)$, $B=(x_b,y_b)$, and $C=(0,0)$. Suppose further that all three excircles of the triangle exist. Then the Apollonius circle of $\triangle ABC$ does not exist if at least one of the following groups of inequalities holds:*

i.

$$\frac{y_b(|x_a|+|y_a|)+y_a(|x_b|+|y_b|)+y_ax_b-x_ay_b}{|x_a|+|y_a|+|x_b|+|y_b|-|x_b-x_a|-|y_b-y_a|}<\frac{y_b(|x_a|+|y_a|)-y_a(|x_b|+|y_b|)+x_ay_b-y_ax_b}{|x_a|+|y_a|-|x_b|-|y_b|-|x_b-x_a|-|y_b-y_a|}$$

and

$$\frac{x_b(|x_a|+|y_a|)+x_a(|x_b|+|y_b|)+y_ax_b-x_ay_b}{|x_a|+|y_a|+|x_b|+|y_b|-|x_b-x_a|-|y_b-y_a|}<\frac{x_b(|x_a|+|y_a|)-x_a(|x_b|+|y_b|)+x_by_a-x_ay_b}{|x_a|+|y_a|+|x_b-x_a|+|y_b-y_a|-|x_b|-|y_b|}.$$

ii.

$$\frac{y_b(|x_a|+|y_a|)-y_a(|x_b|+|y_b|)+x_ay_b-x_by_a}{|x_a|+|y_a|+|x_b-x_a|+|y_b-y_a|-|x_b|-|y_b|}>\frac{y_b(|x_a|+|y_a|)+y_a(|x_b|+|y_b|)+x_ay_b-y_ax_b}{|x_a|+|y_a|+|x_b|+|y_b|-|x_b-x_a|-|y_b-y_a|}$$

and

$$\frac{x_b(|x_a|+|y_a|)-x_a(|x_b|+|y_b|)+x_ay_b-x_by_a}{|x_a|+|y_a|+|x_b-x_a|+|y_b-y_a|-|x_b|-|y_b|}>\frac{x_b(|x_a|+|y_a|)-x_a(|x_b|+|y_b|)+x_by_a-x_ay_b}{|x_a|+|y_a|-|x_b|-|y_b|-|x_b-x_a|-|y_b-y_a|}.$$

iii.

$$\frac{y_b(|x_a|+|y_a|)-y_a(|x_b|+|y_b|)+x_ay_b-y_ax_b}{|x_a|+|y_a|-|x_b|-|y_b|-|x_b-x_a|-|y_b-y_a|}<\frac{y_b(|x_a|+|y_a|)+y_a(|x_b|+|y_b|)+y_ax_b-y_bx_a}{|x_a|+|y_a|+|x_b|+|y_b|-|x_b-x_a|-|y_b-y_a|}$$

and

$$\frac{x_b(|x_a|+|y_a|)-x_a(|x_b|+|y_b|)+x_by_a-y_bx_a}{|x_a|+|y_a|-|x_b|-|y_b|-|x_b-x_a|-|y_b-y_a|}>\frac{x_b(|x_a|+|y_a|)-x_a(|x_b|+|y_b|)+x_ay_b-y_ax_b}{|x_a|+|y_a|+|x_b-x_a|+|y_b-y_a|-|x_b|-|y_b|}.$$

If none of the above conditions (1), (2), or (3) hold, then $\triangle ABC$ admits an Apollonius circle in the maximum plane.

Proof. Let $\triangle ABC$ be a triangle in the maximum plane in the fundamental position with vertices $A=(x_a,y_a)$, $B=(x_b,y_b)$, and $C=(0,0)$, and assume that all three excircles exist. Clearly, the following inequalities must be satisfied:

$$|x_a|+|y_a|+|x_b|+|y_b|-|x_b-x_a|-|y_b-y_a|\neq 0,$$

$$|x_a|+|y_a|+|x_b-x_a|+|y_b-y_a|-|x_b|-|y_b|\neq 0,$$

$$|x_a|+|y_a|-|x_b|-|y_b|-|x_b-x_a|-|y_b-y_a|\neq 0.$$

By Theorem 9, the excircle tangent to the side AB has center

$$x=\frac{x_b(|x_a|+|y_a|)+x_a(|x_b|+|y_b|)}{|x_a|+|y_a|+|x_b|+|y_b|-|x_b-x_a|-|y_b-y_a|},\quad y=\frac{y_b(|x_a|+|y_a|)+y_a(|x_b|+|y_b|)}{|x_a|+|y_a|+|x_b|+|y_b|-|x_b-x_a|-|y_b-y_a|},$$

and radius

$$r=\frac{y_ax_b-x_ay_b}{|x_a|+|y_a|+|x_b|+|y_b|-|x_b-x_a|-|y_b-y_a|}.$$

Similarly, the excircle tangent to the side BC has center

$$x=\frac{x_b(|x_a|+|y_a|)-x_a(|x_b|+|y_b|)}{|x_a|+|y_a|+|x_b-x_a|+|y_b-y_a|-|x_b|-|y_b|},\quad y=\frac{y_b(|x_a|+|y_a|)-y_a(|x_b|+|y_b|)}{|x_a|+|y_a|+|x_b-x_a|+|y_b-y_a|-|x_b|-|y_b|},$$

and radius

$$r=\frac{y_ax_b-x_ay_b}{|x_a|+|y_a|+|x_b-x_a|+|y_b-y_a|-|x_b|-|y_b|}.$$

Finally, the excircle tangent to the side AC has center

$$x=\frac{x_b(|x_a|+|y_a|)-x_a(|x_b|+|y_b|)}{|x_a|+|y_a|-|x_b|-|y_b|-|x_b-x_a|-|y_b-y_a|},\quad y=\frac{y_b(|x_a|+|y_a|)-y_a(|x_b|+|y_b|)}{|x_a|+|y_a|-|x_b|-|y_b|-|x_b-x_a|-|y_b-y_a|},$$

and radius

$$r=\frac{x_ay_b-y_ax_b}{|x_a|+|y_a|-|x_b|-|y_b|-|x_b-x_a|-|y_b-y_a|}.$$

Denote these three excircles by $\mathscr{C}_{AB}$, $\mathscr{C}_{BC}$, and $\mathscr{C}_{AC}$. Since both the excircles and the Apollonius circle are maximum circles, tangency between them corresponds to alignment of sides.

If, for instance, the upper edge of $\mathscr{C}_{AB}$ lies below that of $\mathscr{C}_{AC}$ and simultaneously the right edge of $\mathscr{C}_{AB}$ lies to the left of the right edge of $\mathscr{C}_{BC}$, then no maximum circle can exist that simultaneously contains and is tangent to $\mathscr{C}_{AB}$. This situation is captured by the inequalities in Case (1). One can see for examining to related case Figure 10.

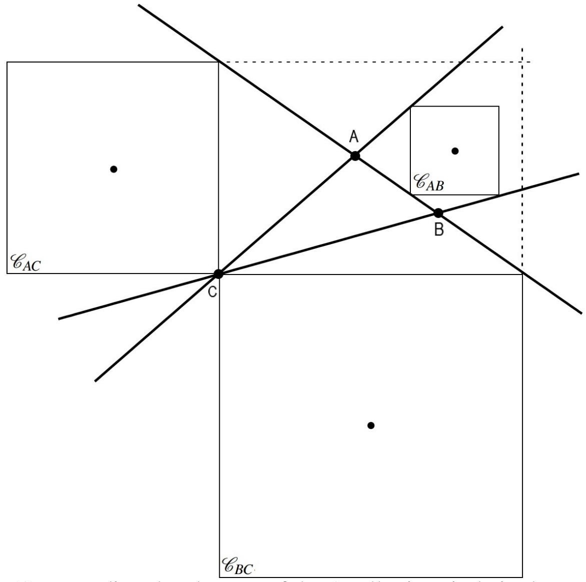

Fig. 10. Case (1) Regarding the absence of the Apollonius circle in the maximum plane

Analogous reasoning shows that if the lower edge of $\mathscr{C}_{BC}$ lies above that of $\mathscr{C}_{AB}$ and the left edge of $\mathscr{C}_{BC}$ lies to the right of that of $\mathscr{C}_{AC}$, then no Apollonius circle can be constructed. This corresponds to Case (2). One can see for examining to related case Figure 11.

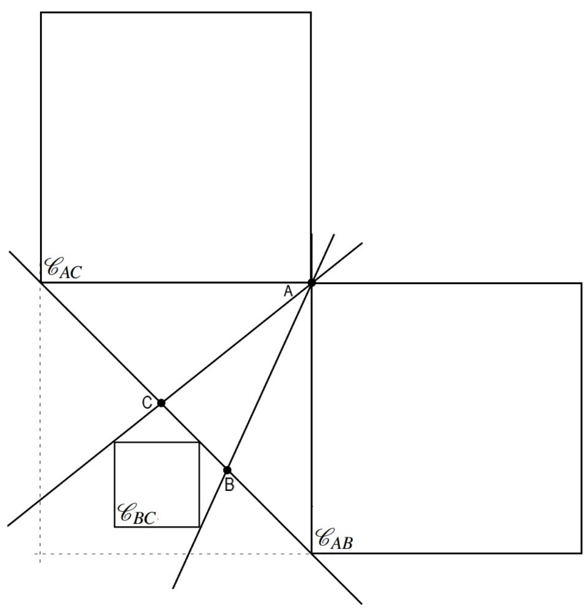

Fig. 11. Case (2) Regarding the absence of the Apollonius circle in the maximum plane

Similarly, if the upper edge of $\mathscr{C}_{AC}$ lies below that of $\mathscr{C}_{AB}$ and its left edge lies to the right of that of $\mathscr{C}_{BC}$, then again no Apollonius circle exists, as described in Case (3). One can see for examining to related case Figure 12.

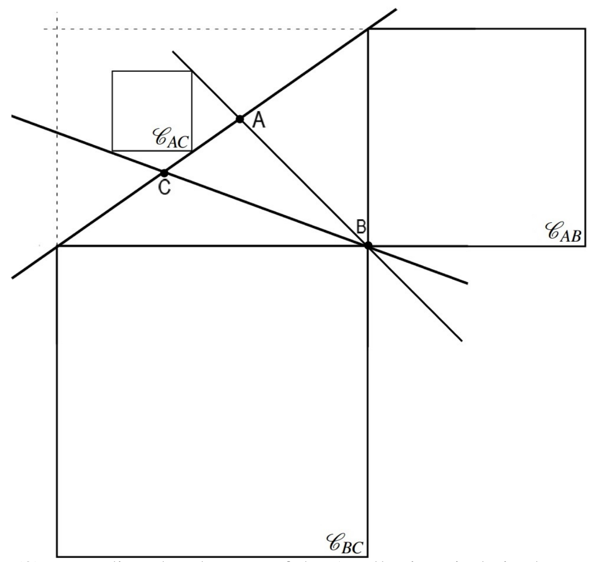

Fig. 12. Case (3) Regarding the absence of the Apollonius circle in the maximum plane

Therefore, whenever at least one of the inequality groups (1), (2), or (3) is satisfied, the Apollonius circle does not exist. Otherwise, the triangle $\triangle ABC$ admits an Apollonius circle in the maximum plane. ∎

Example 14. *Consider again the triangle $\triangle ABC$ with vertices $A = (3,9)$, $B = (5,1)$, and $C = (0,0)$, as discussed in Example 10. From Example 10, it is known that this triangle admits three external tangent circles. Observing the coordinate values of the vertices, one finds that none of the inequality groups given in Theorem 13 are satisfied. In this case, the Apollonius circle has its center at $\left(-3, \frac{51}{4}\right)$ and radius $r = 18$. One can see for examining to related case in Figure 13.*

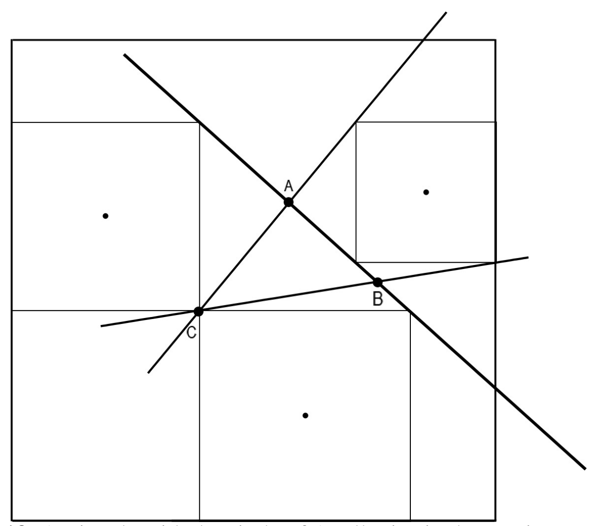

Fig. 13. A triangle with the circle of Apollonius in the maximum plane

5. Conclusion

In this work, we have undertaken a comprehensive study of the incircle, circumcircle, excircles, and the Apollonius circle of a triangle within the framework of maximum plane geometry. Unlike the Euclidean plane, where the existence and uniqueness of these circles are classical and well-established, the maximum metric introduces new structural conditions and, occasionally, limitations to these constructions.

We established that every triangle in the maximum plane admits a unique incircle, whose center coincides with the intersection point of the internal angle bisectors. This result not only extends a central Euclidean property to the maximum setting but also provides a solid foundation for understanding tangency in non-Euclidean metrics. In contrast, the circumcircle demonstrates a more delicate behavior: depending on the horizontal and vertical ordering of the vertices, a triangle may fail to admit any circumcircle, or in certain degenerate configurations, admit infinitely many. These results highlight the intricate relationship between metric definitions of distance and classical notions of circumscription.

The study of excircles uncovered similar subtle properties, as their presence depends on particular algebraic requirements tied to the positions of the triangle's corners. Based on this, it was found that the Apollonius circle only exists under extra conditions, which include the presence of all three excircles and the absence of specific inequality requirements. These results show that the Apollonius configuration, which is generally valid in the Euclidean plane, becomes dependent on certain conditions in the maximum plane, highlighting the strict nature of the L_∞ metric.

In addition to their theoretical importance, these findings emphasize the significance of maximum plane geometry in practical and computational settings. Because the maximum metric is commonly found in grid systems, optimization processes, and digital geometry, reinterpreting traditional geometric elements like circles associated with triangles offers useful perspectives for algorithmic geometry and discrete modeling. Moreover, the close relationship between metric geometry and polyhedral structures points to exciting opportunities for future research, such as examining higher-dimensional versions and linking them to Minkowski geometries.

In conclusion, this research shows that by looking at classical geometric ideas through the perspective of different metrics, new behaviors and structural details emerge. The findings not only enhance the understanding of maximum geometry but also create opportunities for broader use in both theoretical and practical areas of mathematics.

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