Sufficient Efficiency Conditions Associated with a Multidimensional Multiobjective Fractional Variational Problem

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Abstract — This paper presents a study on sufficient efficiency conditions for a class of multidimensional vector ratio optimization problems, identified by (MFP), of minimizing a vector of path-independent curvilinear integral functionals quotients subject to PDE and/or PDI constraints involving higher-order partial derivatives. Under generalized \((\rho, b)\)-quasiinvexity assumptions, sufficient conditions of efficiency are formulated for a feasible solution in (MFP).

Keywords: Efficient solution, Vector fractional optimization problem, Higher-order jet bundle, Generalized \((\rho, b)\)-quasiinvexity.

Mathematics Subject Classification: 65K10, 90C29, 26B25, 53C65.

1 Introduction and Problem Description

It is very well known the importance of convexity in optimization theory. But, the concept of convexity does no longer suffice for many mathematical models coming from engineering, economics, decision sciences, mechanics. Consequently, Hanson [6] introduced a significant generalization of convexity, called invexity. A generalization of invexity is the notion of preinvexity, introduced by Weir and Mond [29]. In this regard, more contributions and various approaches refer, for instance, to Jeyakumar [8], Arana-Jíménez et al. [4], Noor and Noor [12], Tang and Yang [18], Antczak [3], Mititelu and Treanță [10], Treanță [26, 27]. Moreover, a generalization of convexity on manifolds has been proposed by Udriște [28] and Rapcsák [16]. Also, Pini [13] introduced the notion of invex function on Riemannian manifolds. Other approaches have been well documented in Barani and Pouryayevali [2], Agarwal et al. [1] and Treanță and Arana-Jíménez [25].

In this paper, the goal is to establish some new results, associated with the nonlinear optimization theory on higher-order jet bundles, which extend and further develop some results obtained in previous works, such as Jagannathan [7], Tanino and Sawaragi [19], Mond and Husain [11], Weir and Mond [30], Preda [14, 15], Liang et al. [9], Chinchuluun and Pardalos [5], Treanță and Udriște [20] and Treanță [21, 22, 23, 24]. This work comes as a natural continuation of a recent paper, Treanță [23], where a study on nec-
sary efficiency conditions associated to (MFP) is introduced. Due to their physical meaning (mechanical work), the curvilinear integral cost functionals become very important in applications. In physical terms, we are given a number of sources (producing mechanical work) which have to be minimized on a set of limited resources (namely, the set of feasible solutions). More concretely, in this paper, we are looking for efficient efficiency conditions in the following multidimensional multiobjective fractional variational problem

\[
(MFP) \quad \min_{x(\cdot)} \left( \frac{F^1(x(\cdot))}{W^1(x(\cdot))}, \frac{F^2(x(\cdot))}{W^2(x(\cdot))}, \ldots, \frac{F^r(x(\cdot))}{W^r(x(\cdot))} \right)
\]

subject to \( x(\cdot) \in F(\Omega_{t_0,t_1}) \),

where the mathematical tools used are given briefly below (for more details, see Treanță [23]):

- the path-independent curvilinear integral functionals

\[
F^l(x(\cdot)) := \int_{\Gamma_{t_0,t_1}} f^l_\beta \left( \chi_{x_{a_1\ldots a_{s-1}}}(t) \right) dt^\beta, \quad l = 1, \ldots, n, \quad \beta = \frac{1}{1}, \ldots, \frac{1}{m},
\]

\[
W^l(x(\cdot)) := \int_{\Gamma_{t_0,t_1}} w^l_\beta \left( \chi_{x_{a_1\ldots a_{s-1}}}(t) \right) dt^\beta > 0, \quad l = 1, \ldots, n, \quad \beta = \frac{1}{1}, \ldots, \frac{1}{m},
\]

generated by the (higher-order) closed Lagrange 1-form densities of \( C^\infty \)-class

\[
 f_\beta = (f^l_\beta) : J^{s-1}(T, M) \to R^r, \quad l = 1, \ldots, n, \quad \beta = \frac{1}{1}, \ldots, \frac{1}{m},
\]

\[
w_\beta = (w^l_\beta) : J^{s-1}(T, M) \to R^r, \quad l = 1, \ldots, n, \quad \beta = \frac{1}{1}, \ldots, \frac{1}{m};
\]

- the notations

\[
\chi_{x_{a_1\ldots a_{s-1}}}(t) := (t, x(t), x_{a_1}(t), \ldots, x_{a_1\ldots a_{s-1}}(t)), \quad t \in \Omega_{t_0,t_1},
\]

with \( x_{a_1}(t) := \frac{\partial x}{\partial t^{a_1}}(t), \ldots, x_{a_1\ldots a_{s-1}}(t) := \frac{\partial^{s-1}x}{\partial t^{a_1}\partial t^{a_2}\ldots\partial t^{a_{s-1}}}(t) \), and

\[
\alpha_j \in \{1, 2, \ldots, m\}, \quad j = 1, s - 1, \quad x = (x^1, \ldots, x^n) = (x^i), \quad i = 1, n;
\]

- \( t = (t^\beta), \quad \beta = \frac{1}{1}, \ldots, \frac{1}{m} \), and \( x = (x^i), \quad i = 1, n \), are the local coordinates on the Riemannian manifolds \((T, h)\) and \((M, g)\), respectively; in addition, \(M\) is a complete manifold;

- \( \Gamma_{t_0,t_1} \) represents a piecewise \( C^{s-1} \)-class curve joining the diagonally opposite points \( t_0 = (t^0_1, \ldots, t^0_m) \) and \( t_1 = (t^1_1, \ldots, t^1_m) \) of the hyper-parallelepiped \( \Omega_{t_0,t_1} \subset R^m \);

- throughout this work, there are used the customary relations between two vectors of the same dimension;

- the set \( F(\Omega_{t_0,t_1}) \) of all feasible solutions in (MFP) is

\[
x \in C^\infty(\Omega_{t_0,t_1}, M), \quad g \left( \chi_{x_{a_1\ldots a_{s-1}}}(t) \right) \leq 0, \quad h \left( \chi_{x_{a_1\ldots a_{s-1}}}(t) \right) = 0, \quad t \in \Omega_{t_0,t_1}
\]

\[
x(t_\xi) = x_\xi, \quad x_{a_1\ldots a_j}(t) = \tilde{x}_{a_1\ldots a_j}, \quad \alpha_j \in \{1, \ldots, m\}, \quad j = 1, s - 2, \quad \xi \in \{0, 1\},
\]
Sufficient Efficiency Conditions

where
\[ g \left( x_{\alpha_1 \ldots \alpha_{s-1}}(t) \right) \leq 0, \quad h \left( x_{\alpha_1 \ldots \alpha_{s-1}}(t) \right) = 0, \quad t \in \Omega_{t_0,t_1}, \]

are partial differential inequations (PDIs), respectively partial differential equations (PDEs) of evolution, generated by the \( C^\infty \)-class Lagrange matrix densities

\[ g = (g^h_\alpha) : J^{s-1}(T,M) \to \mathbb{R}^q, \quad a = \overline{1},q, \quad b = \overline{1},p, \quad p < n, \]
\[ h = (h^b_\alpha) : J^{s-1}(T,M) \to \mathbb{R}^d, \quad a = \overline{1},d, \quad b = \overline{1},d, \quad d < n, \]

and

\[ C^\infty (\Omega_{t_0,t_1}, M) := \{ x: \Omega_{t_0,t_1} \to M; \ x \text{ of } C^\infty - \text{class} \} \]

is equipped with the distance

\[ d \left( x, x^0 \right) = d \left( x(\cdot), x^0(\cdot) \right) = \sup_{t \in \Omega} d_g \left( x(t), x^0(t) \right), \]

where \( d_g \left( x(t), x^0(t) \right) \) is geodesic distance in \((M,g)\).

Also, in this paper, we shall use the multi-index notation (see Saunders [17]). Saunders defines a multi-index as an \( m \)-tuple \( I \) of natural numbers. Its components are denoted \( I(\alpha) \), where \( \alpha \) is an ordinary index, \( 1 \leq \alpha \leq m \). For instance, the multi-index \( 1_\alpha \) is defined as follows: \( 1_\alpha(\alpha) = 1, \ 1_\alpha(\beta) = 0 \) for \( \alpha \neq \beta \). Define on components the addition and the substraction of the multiindexes (although the result of a substraction might not be a multi-index): \( (I \pm J)(\alpha) = I(\alpha) \pm J(\alpha) \). We call the length of a multi-index the following number \( |I| = \sum_{\alpha=1}^m I(\alpha) \), and its factorial is \( I! = \prod_{\alpha=1}^m (I(\alpha))! \). The number of distinct indices represented by \( \{\alpha_1, \alpha_2, ..., \alpha_k\} \), \( \alpha_j \in \{1,2, ..., m\}, j = 1, k \), is

\[ n(\alpha_1, \alpha_2, ..., \alpha_k) := \frac{! \alpha_1 + \alpha_2 + ... + \alpha_k !}{(\alpha_1 + \alpha_2 + ... + \alpha_k)!}. \]

2 Preliminaries

To make complete our presentation, we recall and introduce some definitions and preliminary results.

**Definition 1** A feasible solution \( x^0(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1}) \) of the problem (MFP) is called efficient solution if there exists no other feasible solution \( x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1}) \) such that \( K(x(\cdot)) \leq K(x^0(\cdot)) \), where

\[ K(x(\cdot)) := \left( \int_{\Gamma_{t_0,t_1}} f^1_{\beta} \left( x_{\alpha_1 \ldots \alpha_{s-1}}(t) \right) dt^3, \ldots, \int_{\Gamma_{t_0,t_1}} f^r_{\beta} \left( x_{\alpha_1 \ldots \alpha_{s-1}}(t) \right) dt^3 \right) \]

\[ = \left( \int_{\Gamma_{t_0,t_1}} w^1_{\beta} \left( x_{\alpha_1 \ldots \alpha_{s-1}}(t) \right) dt^3, \ldots, \int_{\Gamma_{t_0,t_1}} w^r_{\beta} \left( x_{\alpha_1 \ldots \alpha_{s-1}}(t) \right) dt^3 \right). \]
In Treanţă [23], the following result is proved: if \( x^0(\cdot) \in F(\Omega_{t_0,t_1}) \) is [normal] efficient solution of the problem (MFP), then there exist the multipliers \( \lambda \in R^r \), \( \mu \) and \( \nu \) such that the following conditions are fulfilled:

\[
\sum_{c=1}^{r} \lambda_c \left[ \frac{\partial f^c_\beta}{\partial x} \left( x^0_{\alpha_1...\alpha_{s-1}}(t) \right) - R^c_0 \frac{\partial w^c_\beta}{\partial x} \left( x^0_{\alpha_1...\alpha_{s-1}}(t) \right) \right] + \mu_\beta(t) \frac{\partial g}{\partial x} \left( x^0_{\alpha_1...\alpha_{s-1}}(t) \right) + \nu_\beta(t) \frac{\partial h}{\partial x} \left( x^0_{\alpha_1...\alpha_{s-1}}(t) \right) - D_{\alpha_1} \sum_{c=1}^{r} \lambda_c \left[ \frac{\partial f^c_\beta}{\partial x_{\alpha_1}} \left( x^0_{\alpha_1...\alpha_{s-1}}(t) \right) - R^c_0 \frac{\partial w^c_\beta}{\partial x_{\alpha_1}} \left( x^0_{\alpha_1...\alpha_{s-1}}(t) \right) \right] \\
= \sum_{c=1}^{r} \lambda_c \left[ \frac{\partial f^c_\beta}{\partial x_{\alpha_1}} \left( x^0_{\alpha_1...\alpha_{s-1}}(t) \right) - R^c_0 \frac{\partial w^c_\beta}{\partial x_{\alpha_1}} \left( x^0_{\alpha_1...\alpha_{s-1}}(t) \right) \right] \\
+ (-1)^{s-1} \frac{1}{n(\alpha_1,...,\alpha_{s-1})} D_{\alpha_1...\alpha_{s-1}} \sum_{c=1}^{r} \lambda_c \left[ \frac{\partial f^c_\beta}{\partial x_{\alpha_1...\alpha_{s-1}}} \left( x^0_{\alpha_1...\alpha_{s-1}}(t) \right) \right] \\
+ (-1)^{s} \frac{1}{n(\alpha_1,...,\alpha_{s-1})} D_{\alpha_1...\alpha_{s-1}} \sum_{c=1}^{r} \lambda_c R^c_0 \frac{\partial w^c_\beta}{\partial x_{\alpha_1...\alpha_{s-1}}} \left( x^0_{\alpha_1...\alpha_{s-1}}(t) \right) \\
+ (-1)^{s-1} \frac{1}{n(\alpha_1,...,\alpha_{s-1})} D_{\alpha_1...\alpha_{s-1}} \left\{ \mu_\beta(t) \frac{\partial g}{\partial x_{\alpha_1...\alpha_{s-1}}} \left( x^0_{\alpha_1...\alpha_{s-1}}(t) \right) \right\} = 0
\]

\((\text{higher order Euler-Lagrange PDEs}), \quad \beta = \bar{1}, \bar{m}\) \\
\(\mu_\beta(t) g \left( x^0_{\alpha_1...\alpha_{s-1}}(t) \right) = 0, \quad \mu_\beta(t) \geq 0, \quad t \in \Omega_{t_0,t_1}, \quad \beta = \bar{1}, \bar{m}\) \\
\(\lambda \geq 0, \quad e^t \lambda = 1, \quad e^t := (1,1,...,1) \in R^r\).

Further, we shall introduce a generalized \((\rho, b)\)-quasiinvexity associated with the aforementioned optimization problem involving path-independent curvilinear integral functionals. The concept of \((\rho, b)\)-quasiinvexity, associated with simple integral functionals, was also used in recent works for the study of some multiobjective variational problems (see Treanţă [20, 22]).

Let \( \rho \) be a real number and \( b : [C^\infty(\Omega_{t_0,t_1}, M)]^{2s} \to [0, \infty) \) a functional. In the following, we consider the notations:

\[
b(\mathbf{x}(\cdot), x_{\alpha_1}(\cdot), \ldots, x_{\alpha_1...\alpha_{s-1}}(\cdot), x^0(\cdot), x^0_{\alpha_1}(\cdot), \ldots, x^0_{\alpha_1...\alpha_{s-1}}(\cdot)) := b_{xx^0}
\]

\[
\eta(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1...\alpha_{s-1}}(t), x^0(t), x^0_{\alpha_1}(t), \ldots, x^0_{\alpha_1...\alpha_{s-1}}(t)) := \eta_{xx^0}, \quad t \in \Omega_{t_0,t_1}.
\]

Also, let \( a = (a_\beta) : J^{s-1}(T, M) \to R^m \) be a closed Lagrange 1-form that determines the following path-independent curvilinear integral functional

\[
A(x(t)) = \int_{\Gamma_{t_0,t_1}} a_\beta \left( x_{\alpha_1...\alpha_{s-1}}(t) \right) dt^\beta.
\]
Theorem 3.1 (Sufficient efficiency conditions for (MFP)). Let
\[ \alpha \in \{1, \ldots, m\}, \quad \zeta = \overline{1, s - 2}, \quad t \in \Omega_{t_0, t_1}, \]
and \( \theta : [C^\infty (\Omega_{t_0, t_1}, M)]^{2s} \to \mathbb{R}^n \) such that, for any \( x \neq x^0 \), we have the following implication:

\[ [A(x) \leq A(x^0)] \]

\[ \implies \sum_{i=0}^{2s} \left[ \int_{t_0}^{t_1} \left( \frac{1}{n} D_{\alpha_1, \ldots, \alpha_{s-1}} \eta_{xx0} \right) \left( \frac{\partial a_{\alpha}}{\partial x} \left( \chi_{x_{\alpha_1} \ldots x_{\alpha_s}} (t) \right) \right) dt^\beta \right] \]

\[ \begin{align*}
\left( b_{x^0} + b_{x^0} \right) & \leq -\rho b_{x^0} \| \theta_{xx0} \|^2.
\end{align*} \]

Example 1 Consider

\[ x : [0, 1] \to \mathcal{M} \subseteq \mathbb{R}^2, \quad x(t) = (x_1(t), x_2(t)), \]

a \( C^2 \)-class function defined on the real interval \([0, 1]\). Let \( h : [0, 1] \times \mathcal{M} \to \mathbb{R} \) be a continuously differentiable function. The following functional of curvilinear integral type

\[ H (x(\cdot)) = \int_0^1 h(t, \bar{x}(t)) dt \]

is, as it can be verified, \((\rho, 1)\)-quasiinvex, for \( \rho \leq 0 \) and any \( \theta \), at \( x^0(\cdot) \) with respect to

\[ \eta(t, x(t), \dot{x}(t), \ddot{x}(t), x_0(t), \dot{x}_0(t), \ddot{x}_0(t)) \]

\[ = \left( \eta_1(t, x(t), \dot{x}(t), \ddot{x}(t), x^0(t), \dot{x}_0(t), \ddot{x}_0(t)), \eta_2(t, x(t), \dot{x}(t), \ddot{x}(t), x^0(t), \dot{x}_0(t), \ddot{x}_0(t)) \right) \]

\[ = H (x(\cdot)) - H (x^0(\cdot)) \left( D^2 \frac{\partial h}{\partial x^1} (t, \bar{x}^0(t)), D^2 \frac{\partial h}{\partial x^2} (t, \bar{x}^0(t)) \right). \]

The previous example can be easily extended to \( n \)-dimensional vector valued functions and, by using normal coordinates, to the multidimensional case.

3 Main result

The next theorem is the main result of this paper.

**Theorem 3.1** (Sufficient efficiency conditions for (MFP)). Let \( x^0(\cdot) \in F (\Omega_{t_0, t_1}) \), \( \lambda \in \mathbb{R}^\nu \), \( \mu \) and \( \nu \) satisfying (1). As well, assume that the following hypotheses are fulfilled:

1. the functionals

\[ \int_{t_0}^{t_1} \left[ f_{\beta} (\chi_{x_{\alpha_1} \ldots x_{\alpha_s}} (t)) - R_{0, \alpha} (\chi_{x_{\alpha_1} \ldots x_{\alpha_s}} (t)) \right] dt^{\beta}, \]

\[ l = \overline{1, r}, \quad \beta = \overline{1, m}, \] are (\( \rho \), \( b \))-quasiinvex at \( x^0(\cdot) \) with respect to \( \eta \) and \( \theta \);
b) \( \int_{\Gamma_{t_0, t_1}} \mu_\beta(t) g \left( x_{a_1 \ldots a_{s-1}}(t) \right) dt^\beta \) is \((\rho_2, b)\)-quasiinvex at \( x^0(\cdot) \) with respect to \( \eta \) and \( \theta \);

c) \( \int_{\Gamma_{t_0, t_1}} \nu_\beta(t) h \left( x_{a_1 \ldots a_{s-1}}(t) \right) dt^\beta \) is \((\rho_3, b)\)-quasiinvex at \( x^0(\cdot) \) with respect to \( \eta \) and \( \theta \);

d) at least one of the integrals of a) - c) is strictly \((\rho, b)\)-quasiinvex at the point \( x^0(\cdot) \) with respect to \( \eta \) and \( \theta \); (see \( \rho = \rho^1, \rho^2 \) or \( \rho^3 \))

e) \( \sum_{l=1}^r \lambda_l \rho^1_l + \rho_2 + \rho_3 \geq 0 \) \((\rho^1, \rho_2, \rho_3 \in \mathbb{R})\).

Then the point \( x^0(\cdot) \) is an efficient solution of the problem (MFP).

Proof. Consider, by reductio ad absurdum, that \( x^0(\cdot) \) is not an efficient solution of (MFP). Then, for \( l = 1, r \), there exists \( x(\cdot) \in \mathcal{F}(\Omega_{t_0, t_1}) \) such that

\[
\int_{\Gamma_{t_0, t_1}} \left[ f^l_\beta \left( x_{a_1 \ldots a_{s-1}}(t) \right) - R^l_0 w^l_\beta \left( x_{a_1 \ldots a_{s-1}}(t) \right) \right] dt^\beta \leq \int_{\Gamma_{t_0, t_1}} \left[ f^k_\beta \left( x_{a_1 \ldots a_{s-1}}(t) \right) - R^k_0 w^k_\beta \left( x_{a_1 \ldots a_{s-1}}(t) \right) \right] dt^\beta
\]

and there exists at least \( k \in \{1, 2, \ldots, r\} \) with

\[
\int_{\Gamma_{t_0, t_1}} \left[ f^k_\beta \left( x_{a_1 \ldots a_{s-1}}(t) \right) - R^k_0 w^k_\beta \left( x_{a_1 \ldots a_{s-1}}(t) \right) \right] dt^\beta < \int_{\Gamma_{t_0, t_1}} \left[ f^k_\beta \left( x_{a_1 \ldots a_{s-1}}(t) \right) - R^k_0 w^k_\beta \left( x_{a_1 \ldots a_{s-1}}(t) \right) \right] dt^\beta.
\]

Using the hypothesis a) and setting \( X^l_\beta := f^l_\beta - R^l_0 w^l_\beta \), we have

\[
b_{xx^0} \int_{\Gamma_{t_0, t_1}} \left[ \eta_{xx^0} \frac{\partial X^l_\beta}{\partial x} \left( x_{a_1 \ldots a_{s-1}}(t) \right) + \left( D_{a_1} \eta_{xx^0} \right) \frac{\partial X^l_\beta}{\partial x_{a_1}} \left( x_{a_1 \ldots a_{s-1}}(t) \right) \right] dt^\beta + \ldots + b_{xx^0} \int_{\Gamma_{t_0, t_1}} \left[ \left( \frac{1}{n(\alpha_1, \ldots, \alpha_{s-1})} D^s_{\alpha_1 \ldots a_{s-1}} \eta_{xx^0} \right) \frac{\partial X^l_\beta}{\partial x_{a_1 \ldots a_{s-1}}} \left( x_{a_1 \ldots a_{s-1}}(t) \right) \right] dt^\beta \leq -\rho^1_l b_{xx^0} \left\| \theta_{xx^0} \right\|^2.
\]

Multiplying by \( \lambda_l \geq 0 \) and making the sum from \( l = 1 \) to \( l = r \), we obtain

\[
b_{xx^0} \int_{\Gamma_{t_0, t_1}} \left[ \eta_{xx^0} \lambda \frac{\partial X^l_\beta}{\partial x} \left( x_{a_1 \ldots a_{s-1}}(t) \right) \right] dt^\beta + \ldots + b_{xx^0} \int_{\Gamma_{t_0, t_1}} \left[ \left( D_{a_1} \eta_{xx^0} \right) \lambda \frac{\partial X^l_\beta}{\partial x_{a_1}} \left( x_{a_1 \ldots a_{s-1}}(t) \right) \right] dt^\beta \]

(2)
The following inequality
\[
\int_{\Gamma_{t_0}, t_1} \mu_\beta(t) g \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) dt^\beta \leq \int_{\Gamma_{t_0}, t_1} \mu_\beta(t) g \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) dt^\beta,
\]
gives
\[
\frac{1}{n(\alpha_1, \ldots, \alpha_{s-1})} D^{s-1}_{\alpha_1, \ldots, \alpha_{s-1}} \eta_{xx0} \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) \frac{\partial X_\beta}{\partial x_{\alpha_1, \ldots, \alpha_{s-1}}} \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) \right] dt^\beta
\leq - \left( \sum_{i=1}^r \lambda_i \rho_i^2 \right) b_{xx0} \|| \theta_{xx0} \|^2.
\]
Also, the equality (see c))
\[
\int_{\Gamma_{t_0}, t_1} \nu_\beta(t) h \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) dt^\beta = \int_{\Gamma_{t_0}, t_1} \nu_\beta(t) h \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) dt^\beta
\]
gives
\[
\int_{\Gamma_{t_0}, t_1} \eta_{xx0} \nu_\beta(t) \frac{\partial h}{\partial x} \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) dt^\beta
\leq - \rho_3 b_{xx0} \|| \theta_{xx0} \|^2.
\]
Making the sum (2) + (3) + (4), side by side, and taking into account d), we get
\[
\int_{\Gamma_{t_0}, t_1} \eta_{xx0} \left[ \frac{\partial X_\beta}{\partial x} \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) + \mu_\beta(t) \frac{\partial g}{\partial x} \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) \right] dt^\beta
\]
\[
+ b_{xx0} \int_{\Gamma_{t_0}, t_1} \nu_\beta(t) \frac{\partial h}{\partial x} \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) dt^\beta
\]
\[
+ b_{xx0} \int_{\Gamma_{t_0}, t_1} \left( D_{\alpha_1, \alpha_{s-1}} \eta_{xx0} \right) \nu_\beta(t) \frac{\partial h}{\partial x} \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) dt^\beta
\]
\[
+ b_{xx0} \int_{\Gamma_{t_0}, t_1} \left( D_{\alpha_1, \alpha_{s-1}} \eta_{xx0} \right) \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) \frac{\partial h}{\partial x} \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) dt^\beta
\]
\[
+ b_{xx0} \int_{\Gamma_{t_0}, t_1} \left( D_{\alpha_1, \alpha_{s-1}} \eta_{xx0} \right) \nu_\beta(t) \frac{\partial h}{\partial x} \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) dt^\beta
\]
\[
+ b_{xx0} \int_{\Gamma_{t_0}, t_1} \left( D_{\alpha_1, \alpha_{s-1}} \eta_{xx0} \right) \nu_\beta(t) \frac{\partial h}{\partial x} \left( \chi_{x_{\alpha_1, \ldots, \alpha_{s-1}}} (t) \right) dt^\beta
\]
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\[ + b_{xx^0} \int_{\Gamma_{10, t_1}} (D_{\alpha_1} \eta_{xx^0}) \left[ \nu_\beta(t) \frac{\partial h}{\partial x_{\alpha_1}} \left( \chi_{x_{\alpha_1} \ldots x_{s_1+1}}(t) \right) \right] dt^3 \]

\[ + \ldots + b_{xx^0} \int_{\Gamma_{10, t_1}} \left( \frac{1}{n(\alpha_1, \ldots, \alpha_{s_1-1})} D^{s_1-1}_{\alpha_1 \ldots \alpha_{s_1-1}} \eta_{xx^0} \right) \left[ \mu_\beta(t) \frac{\partial g}{\partial x_{\alpha_1 \ldots \alpha_{s_1-1}}} \left( \chi_{x_{\alpha_1} \ldots x_{s_1-1}}(t) \right) \right] dt^3 \]

\[ + b_{xx^0} \int_{\Gamma_{10, t_1}} \left( \frac{1}{n(\alpha_1, \ldots, \alpha_{s_1-1})} D^{s_1-1}_{\alpha_1 \ldots \alpha_{s_1-1}} \eta_{xx^0} \right) \left[ \nu_\beta(t) \frac{\partial h}{\partial x_{\alpha_1 \ldots \alpha_{s_1-1}}} \left( \chi_{x_{\alpha_1} \ldots x_{s_1-1}}(t) \right) \right] dt^3 \]

\[ < - \left( \sum_{l=1}^r \lambda_l \rho_l + \rho_2 + \rho_3 \right) b_{xx^0} \| \theta_{xx^0} \|^2 . \]

This implies that \( b_{xx^0} > 0 \) and the foregoing inequality can be rewritten as

\[ \int_{\Gamma_{10, t_1}} \eta_{xx^0} \left[ \lambda \frac{\partial X_\beta}{\partial x} \left( \chi_{x_{\alpha_1} \ldots x_{s_1-1}}(t) \right) \right] dt^3 \]

\[ + \int_{\Gamma_{10, t_1}} \eta_{xx^0} \left[ \nu_\beta(t) \frac{\partial h}{\partial x} \left( \chi_{x_{\alpha_1} \ldots x_{s_1-1}}(t) \right) \right] dt^3 \]

\[ + \int_{\Gamma_{10, t_1}} (D_{\alpha_1} \eta_{xx^0}) \left[ \lambda \frac{\partial X_\beta}{\partial x_{\alpha_1}} \left( \chi_{x_{\alpha_1} \ldots x_{s_1-1}}(t) \right) \right] dt^3 \]

\[ + \ldots + \int_{\Gamma_{10, t_1}} \left( \frac{1}{n(\alpha_1, \ldots, \alpha_{s_1-1})} D^{s_1-1}_{\alpha_1 \ldots \alpha_{s_1-1}} \eta_{xx^0} \right) \left[ \lambda \frac{\partial X_\beta}{\partial x_{\alpha_1 \ldots \alpha_{s_1-1}}} \left( \chi_{x_{\alpha_1} \ldots x_{s_1-1}}(t) \right) \right] dt^3 \]

\[ + \left( \sum_{l=1}^r \lambda_l \rho_l + \rho_2 + \rho_3 \right) \| \theta_{xx^0} \|^2 , \]

or, after integrating by parts, we get

\[ \int_{\Gamma_{10, t_1}} \eta_{xx^0} \left[ \lambda \frac{\partial X_\beta}{\partial x} \left( \chi_{x_{\alpha_1} \ldots x_{s_1-1}}(t) \right) \right] dt^3 \]

\[ + \int_{\Gamma_{10, t_1}} \eta_{xx^0} \left[ \nu_\beta(t) \frac{\partial h}{\partial x} \left( \chi_{x_{\alpha_1} \ldots x_{s_1-1}}(t) \right) \right] dt^3 \]
The above given computation is obtained by using the boundary conditions

\[-\int_{\Gamma_{0,t_1}} \eta_{\theta \xi} \partial D_{\alpha_1} \left[ \lambda \frac{\partial X_\beta}{\partial x_{\alpha_1}} (\chi^0_{\alpha_1...\alpha_{s-1}} (t)) + \mu_\beta (t) \frac{\partial g}{\partial x_{\alpha_1}} (\chi^0_{\alpha_1...\alpha_{s-1}} (t)) \right] dt^\beta \]

\[+ \cdots + (-1)^{s-1} \int_{\Gamma_{0,t_1}} \eta_{\theta \xi} \frac{1}{n (\alpha_1, ..., \alpha_{s-1})} D^{s-1}_{\alpha_1...\alpha_{s-1}} \left[ \lambda \frac{\partial X_\beta}{\partial x_{\alpha_1...\alpha_{s-1}}} (\chi^0_{\alpha_1...\alpha_{s-1}} (t)) \right] dt^\beta \]

\[+ (-1)^{s-1} \int_{\Gamma_{0,t_1}} \eta_{\theta \xi} \frac{1}{n (\alpha_1, ..., \alpha_{s-1})} D^{s-1}_{\alpha_1...\alpha_{s-1}} \left[ \mu_\beta (t) \frac{\partial g}{\partial x_{\alpha_1...\alpha_{s-1}}} (\chi^0_{\alpha_1...\alpha_{s-1}} (t)) \right] dt^\beta \]

\[< - \left( \sum_{l=1}^r \lambda_l \rho_l^1 + \rho_2 + \rho_3 \right) \| \theta_{x^0} \|^2 . \]

The above given computation is obtained by using the boundary conditions

\[x(t_\xi) = x_\xi, \ x_{\alpha_1...\alpha_j} (t_\xi) = \tilde{x}_{\alpha_1...\alpha_j} \xi, \ \alpha_\xi \in \{1, ..., m\}, \ \zeta, j = 1, s-2, \ \xi \in \{0, 1\}, \]

(see \[x(t_\xi) = x_\xi = x^0(t_\xi), \ x_{\alpha_1...\alpha_j} (t_\xi) = \tilde{x}_{\alpha_1...\alpha_j} \xi = x^0_{\alpha_1...\alpha_j} (t_\xi)\]), and the following conditions (see Definition 2),

\[\eta_{x^0_x^0} = 0, \ D_{\alpha_1} \eta_{x^0_x^0} = 0, \cdots, D^{s-2}_{\alpha_1...\alpha_{s-2}} \eta_{x^0_x^0} = 0 \]

\[\alpha_\xi \in \{1, ..., m\}, \ \zeta = 1, s-2, \ \xi \in \Omega_{t_0,t_1}. \]

The conditions (1) lead us to

\[0 < - \left( \sum_{l=1}^r \lambda_l \rho_l^1 + \rho_2 + \rho_3 \right) \| \theta_{x^0} \|^2 . \]

Applying the hypothesis e) and \( \| \theta_{x^0} \|^2 \geq 0 \), we get a contradiction. Thus, the point \( x^0 \) is an efficient solution for (MFP) and the proof is complete.

**Corollary 1** (Sufficient efficiency conditions for (MFP)). Let \( x^0(\cdot) \) be a feasible solution of the problem (MFP) and assume that (1) is fulfilled. Also, consider the following properties hold:

a) the functionals \( \int_{\Gamma_{0,t_1}} \left[ f^l_\beta (\chi_{\alpha_1...\alpha_{s-1}} (t)) - R^l_0 u^l_\beta (\chi_{\alpha_1...\alpha_{s-1}} (t)) \right] dt^\beta, \ l = 1, r, \ \beta = 1, m \) are \((\rho_l^1, b)\)-quasiinvex at the point \( x^0(\cdot) \) with respect to \( \eta \) and \( \theta \);

b') the functional \( \int_{\Gamma_{0,t_1}} \left[ \mu_\beta (t) g (\chi_{\alpha_1...\alpha_{s-1}} (t)) + \nu_\beta (t) h (\chi_{\alpha_1...\alpha_{s-1}} (t)) \right] dt^\beta \) is \((\rho_2, b)\)-quasiinvex at the point \( x^0(\cdot) \) with respect to \( \eta \) and \( \theta \);
d’) at least one of the integrals \( \int_{t_0,t_1} \left[ f^l_\beta \left( x_{a_1 \ldots a_s-1} (t) \right) - R^l_0 w^l_\beta \left( x_{a_1 \ldots a_s-1} (t) \right) \right] dt^\beta \),

\[ l = 1, \ldots, \beta = 1, \ldots, m, \int \Gamma_{t_0,t_1} \left[ \mu_\beta (t) g \left( x_{a_1 \ldots a_s-1} (t) \right) + \nu_\beta (t) h \left( x_{a_1 \ldots a_s-1} (t) \right) \right] dt^\beta, \]

is strictly \((\rho_1, b)\) or \((\rho_2, b)\)-quasiinvex at the point \( x^0 (\cdot) \) with respect to \( \eta \) and \( \theta \);

e’) \[ \sum_{l=1}^r \lambda_l \rho_1^l + \rho_2 \geq 0 \quad (\rho_1^l, \rho_2 \in R). \]

Then the point \( x^0 (\cdot) \) is an efficient solution of the problem \( \text{MFP} \).

**Proof.** The proof follows in the same manner as in Theorem 3.1. The functionals

\[ \int \Gamma_{t_0,t_1} \left[ \mu_\beta (t) g \left( x_{a_1 \ldots a_s-1} (t) \right) \right] dt^\beta, \int \Gamma_{t_0,t_1} \left[ \nu_\beta (t) h \left( x_{a_1 \ldots a_s-1} (t) \right) \right] dt^\beta \]

are replaced by

\[ \int \Gamma_{t_0,t_1} \left[ \mu_\beta (t) g \left( x_{a_1 \ldots a_s-1} (t) \right) + \nu_\beta (t) h \left( x_{a_1 \ldots a_s-1} (t) \right) \right] dt^\beta. \]

**Corollary 2 (Sufficient efficiency conditions for \( \text{MFP} \)).** Consider that (1) and the following hypotheses are fulfilled:

a’) the functionals

\[ \int \Gamma_{t_0,t_1} \left[ W^l (x^0 (t)) f^l_\beta \left( x_{a_1 \ldots a_s-1} (t) \right) - F^l (x^0 (t)) w^l_\beta \left( x_{a_1 \ldots a_s-1} (t) \right) \right] dt^\beta, \]

\[ l = 1, r, \beta = 1, m, \int \Gamma_{t_0,t_1} \left[ \mu_\beta (t) g \left( x_{a_1 \ldots a_s-1} (t) \right) + \nu_\beta (t) h \left( x_{a_1 \ldots a_s-1} (t) \right) \right] dt^\beta \]

is \((\rho_2, b)\)-quasiinvex at the point \( x^0 (\cdot) \) with respect to \( \eta \) and \( \theta \);

b’) the functional

\[ \int \Gamma_{t_0,t_1} \left[ \mu_\beta (t) g \left( x_{a_1 \ldots a_s-1} (t) \right) + \nu_\beta (t) h \left( x_{a_1 \ldots a_s-1} (t) \right) \right] dt^\beta \]

\[(\rho_1^l, b)\) or \((\rho_2, b)\)-quasiinvex at the point \( x^0 (\cdot) \) with respect to \( \eta \) and \( \theta \);

e’’) \[ \sum_{l=1}^r \lambda_l \rho_1^l + \rho_2 \geq 0 \quad (\rho_1^l, \rho_2 \in R). \]

Then the point \( x^0 (\cdot) \) is an efficient solution of the problem \( \text{MFP} \).

**Proof.** Taking into account the definition of \( R^l_0 \) and redefining \( \mu_\beta \) and \( \nu_\beta \), the functional

\[ \int \Gamma_{t_0,t_1} \left[ f^l_\beta \left( x_{a_1 \ldots a_s-1} (t) \right) - R^l_0 w^l_\beta \left( x_{a_1 \ldots a_s-1} (t) \right) \right] dt^\beta \]
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is changed into

\[
\int_{\Gamma_0,t_1} \left[ W^l \left( x^0(t) \right) j^l_{\beta} \left( \chi_{x_{a_1-\alpha s-1}}(t) \right) - F^l \left( x^0(t) \right) w^l_{\beta} \left( \chi_{x_{a_1-\alpha s-1}}(t) \right) \right] dt^\beta
\]

and the integrals

\[
\int_{\Gamma_0,t_1} \left[ \mu_{\beta}(t) g \left( \chi_{x_{a_1-\alpha s-1}}(t) \right) \right] dt^\beta, \quad \int_{\Gamma_0,t_1} \left[ \nu_{\beta}(t) h \left( \chi_{x_{a_1-\alpha s-1}}(t) \right) \right] dt^\beta
\]

are replaced by

\[
\int_{\Gamma_0,t_1} \left[ \mu_{\beta}(t) g \left( \chi_{x_{a_1-\alpha s-1}}(t) \right) + \nu_{\beta}(t) h \left( \chi_{x_{a_1-\alpha s-1}}(t) \right) \right] dt^\beta.
\]

**Corollary 3** (Sufficient efficiency conditions for (MFP)). If the conditions (1) are fulfilled and the following properties hold:

a') the functionals

\[
\int_{\Gamma_0,t_1} \left[ W^l \left( x^0(t) \right) j^l_{\beta} \left( \chi_{x_{a_1-\alpha s-1}}(t) \right) - F^l \left( x^0(t) \right) w^l_{\beta} \left( \chi_{x_{a_1-\alpha s-1}}(t) \right) \right] dt^\beta,
\]

\(l = 1, r, \beta = 1, m, \) are \((\rho_1, b)\)-quasiinvex at the point \(x^0(\cdot)\) with respect to \(\eta\) and \(\theta\);

b) \(\int_{\Gamma_0,t_1} \mu_{\beta}(t) g \left( \chi_{x_{a_1-\alpha s-1}}(t) \right) dt^\beta\) is \((\rho_2, b)\)-quasiinvex at \(x^0(\cdot)\) with respect to \(\eta\) and \(\theta\);

c) \(\int_{\Gamma_0,t_1} \nu_{\beta}(t) h \left( \chi_{x_{a_1-\alpha s-1}}(t) \right) dt^\beta\) is \((\rho_3, b)\)-quasiinvex at \(x^0(\cdot)\) with respect to \(\eta\) and \(\theta\);

d*) at least one of the integrals of a'), b), c) is strictly \((\rho, b)\)-quasiinvex at the point \(x^0(\cdot)\) with respect to \(\eta\) and \(\theta\); \(\text{see } \rho = \rho_1, \rho_2 \text{ or } \rho_3\)

e*) \(\sum_{l=1}^{r} \lambda_l \rho^l_1 + \rho_2 + \rho_3 \geq 0\) \((\rho_1^l, \rho_2, \rho_3 \in R)\),

then the point \(x^0(\cdot)\) is an efficient solution of the problem (MFP).

**Proof.** Taking into account the definition of \(R_0^l\) and redefining \(\mu_{\beta}\) and \(\nu_{\beta}\), the functional

\[
\int_{\Gamma_0,t_1} \left[ f^l_{\beta} \left( \chi_{x_{a_1-\alpha s-1}}(t) \right) - R_0^l w^l_{\beta} \left( \chi_{x_{a_1-\alpha s-1}}(t) \right) \right] dt^\beta
\]

is replaced by

\[
\int_{\Gamma_0,t_1} \left[ W^l \left( x^0(t) \right) j^l_{\beta} \left( \chi_{x_{a_1-\alpha s-1}}(t) \right) - F^l \left( x^0(t) \right) w^l_{\beta} \left( \chi_{x_{a_1-\alpha s-1}}(t) \right) \right] dt^\beta
\]

and the proof follows in the same manner as in Theorem 3.1.
4 Conclusion

In this paper, motivated by the ongoing research in this area, by using the extended notion of $(\rho, b)$-quasiinvexity, we have formulated and proved sufficient efficiency conditions for a class of multidimensional vector ratio optimization problems (MFP) of minimizing a vector of path-independent curvilinear integral functionals (mechanical work) quotients subject to PDE and/or PDI constraints involving higher-order partial derivatives. Due to physical meaning of the objective functionals, the importance of this research paper has been supported both from theoretical and practical considerations.

References

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