Journal of Multidisciplinary Modeling and Optimization 1(1) (2018), 1-13.

Sufficient Efficiency Conditions Associated with a Multidimensional Multiobjective Fractional Variational Problem

Savin TREANŢĂ[1]

[1]Department of Applied Mathematics,
University "Politehnica" of Bucharest, Bucharest, Romania
savin_treanta@yahoo.com

Abstract — *This paper presents a study on sufficient efficiency conditions for a class of multidimensional vector ratio optimization problems, identified by (MFP), of minimizing a vector of path-independent curvilinear integral functionals quotients subject to PDE and/or PDI constraints involving higher-order partial derivatives. Under generalized* (ρ, b)*-quasiinvexity assumptions, sufficient conditions of efficiency are formulated for a feasible solution in (MFP).*

Keywords: Ffficient solution, Vector fractional optimization problem, Higher-order jet bundle, Generalized (ρ, b)-quasiinvexity.
Mathematics Subject Classification: 65K10, 90C29, 26B25, 53C65.

1 Introduction and Problem Description

It is very well known the importance of convexity in optimization theory. But, the concept of convexity does no longer suffice for many mathematical models coming from engineering, economics, decision sciences, mechanics. Consequently, Hanson [6] introduced a significant generalization of convexity, called *invexity*. A generalization of invexity is the notion of *preinvexity*, introduced by Weir and Mond [29]. In this regard, more contributions and various approaches refer, for instance, to Jeyakumar [8], Arana-Jiménez *et al.* [4], Noor and Noor [12], Tang and Yang [18], Antczak [3], Mititelu and Treanţă [10], Treanţă [26, 27]. Moreover, a generalization of convexity on manifolds has been proposed by Udrişte [28] and Rapcsák [16]. Also, Pini [13] introduced the notion of invex function on Riemannian manifolds. Other approaches have been well documented in Barani and Pouryayevali [2], Agarwal *et al.* [1] and Treanţă and Arana-Jiménez [25].

In this paper, the goal is to establish some new results, associated with the nonlinear optimization theory on higher-order jet bundles, which extend and further develop some results obtained in previous works, such as Jagannathan [7], Tanino and Sawaragi [19], Mond and Husain [11], Weir and Mond [30], Preda [14, 15], Liang *et al.* [9], Chinchuluun and Pardalos [5], Treanţă and Udrişte [20] and Treanţă [21, 22, 23, 24]. This work comes as a natural continuation of a recent paper, Treanţă [23], where a study on necessary efficiency conditions associated to (MFP) is introduced. Due to their physical meaning (mechanical work), the curvilinear integral cost functionals become very important in applications. In physical terms, we are given a number of p sources (producing mechanical work) which have to be minimized on a set of limited resources (namely, the set of feasible solutions). More concretely, in this paper, we are looking for sufficient efficiency conditions in the following multidimensional multiobjective fractional variational problem

$$(MFP) \qquad \min_{x(\cdot)} \left(\frac{F^1(x(\cdot))}{W^1(x(\cdot))}, \frac{F^2(x(\cdot))}{W^2(x(\cdot))}, ..., \frac{F^r(x(\cdot))}{W^r(x(\cdot))} \right)$$

$$\text{subject to } x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1}),$$

where the mathematical tools used are given briefly below (for more details, see Treanţă [23]):

• the path-independent curvilinear integral functionals

$$F^l(x(\cdot)) := \int_{\Gamma_{t_0,t_1}} f^l_\beta \left(\chi_{x_{\alpha_1 \cdots \alpha_{s-1}}}(t) \right) dt^\beta, \quad l = \overline{1,r},\ \beta = \overline{1,m},$$

$$W^l(x(\cdot)) := \int_{\Gamma_{t_0,t_1}} w^l_\beta \left(\chi_{x_{\alpha_1 \cdots \alpha_{s-1}}}(t) \right) dt^\beta > 0, \quad l = \overline{1,r},\ \beta = \overline{1,m},$$

generated by the (higher-order) closed Lagrange 1-form densities of C^∞-class

$$f_\beta = \left(f^l_\beta\right) : J^{s-1}(T,M) \to R^r, \quad l = \overline{1,r},\ \beta = \overline{1,m},$$

$$w_\beta = \left(w^l_\beta\right) : J^{s-1}(T,M) \to R^r, \quad l = \overline{1,r},\ \beta = \overline{1,m};$$

• the notations

$$\chi_{x_{\alpha_1 \cdots \alpha_{s-1}}}(t) := \left(t, x(t), x_{\alpha_1}(t), ..., x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t)\right), \quad t \in \Omega_{t_0,t_1},$$

with $x_{\alpha_1}(t) := \dfrac{\partial x}{\partial t^{\alpha_1}}(t), ..., x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t) := \dfrac{\partial^{s-1} x}{\partial t^{\alpha_1}\partial t^{\alpha_2}\ldots\partial t^{\alpha_{s-1}}}(t)$, and

$$\alpha_j \in \{1,2,...,m\},\ j = \overline{1,s-1},\ x = (x^1,...,x^n) = \left(x^i\right),\ i = \overline{1,n};$$

• $t = (t^\beta),\ \beta = \overline{1,m}$, and $x = (x^i),\ i = \overline{1,n}$, are the local coordinates on the Riemannian manifolds $(T, \mathbf{h})$ and $(M, \mathbf{g})$, respectively; in addition, M is a complete manifold;

• Γ_{t_0,t_1} represents a piecewise C^{s-1}-class curve joining the diagonally opposite points $t_0 = \left(t^1_0, ..., t^m_0\right)$ and $t_1 = \left(t^1_1, ..., t^m_1\right)$ of the hyper-parallelepiped $\Omega_{t_0,t_1} \subset R^m$;

• throughout this work, there are used the customary relations between two vectors of the same dimension;

• the set $\mathcal{F}(\Omega_{t_0,t_1})$ of all feasible solutions in (MFP) is

$$x \in C^\infty(\Omega_{t_0,t_1}, M),\ g\left(\chi_{x_{\alpha_1 \cdots \alpha_{s-1}}}(t)\right) \le 0,\ h\left(\chi_{x_{\alpha_1 \cdots \alpha_{s-1}}}(t)\right) = 0,\ t \in \Omega_{t_0,t_1}$$

$$x(t_\xi) = x_\xi,\ x_{\alpha_1\ldots\alpha_j}(t_\xi) = \tilde{x}_{\alpha_1\ldots\alpha_j\xi},\ \alpha_\zeta \in \{1,...,m\},\ \zeta, j = \overline{1,s-2},\ \xi \in \{0,1\},$$

where

$$g\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) \leq 0, \quad h\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) = 0, \quad t \in \Omega_{t_0,t_1},$$

are partial differential inequations (PDIs), respectively partial differential equations (PDEs) of evolution, generated by the C^∞-class Lagrange matrix densities

$$g = \left(g_a^b\right) : J^{s-1}(T, M) \to R^{pq}, \quad a = \overline{1,q},\ b = \overline{1,p},\ p < n,$$

$$h = \left(h_a^b\right) : J^{s-1}(T, M) \to R^{de}, \quad a = \overline{1,e},\ b = \overline{1,d},\ d < n,$$

and

$$C^\infty\left(\Omega_{t_0,t_1}, M\right) := \{x\colon \Omega_{t_0,t_1} \to M;\ x\ \ of\ \ C^\infty - class\}$$

is equipped with the distance

$$d\left(x, x^0\right) = d\left(x(\cdot), x^0(\cdot)\right) = \sup_{t\in\Omega} d_{\mathbf{g}}\left(x(t), x^0(t)\right),$$

where $d_{\mathbf{g}}\left(x(t), x^0(t)\right)$ is geodesic distance in $(M, \mathbf{g})$.

Also, in this paper, we shall use the *multi-index notation* (see Saunders [17]). Saunders defines a *multi-index* as an m-tuple I of natural numbers. Its components are denoted $I(\alpha)$, where α is an ordinary index, $1 \leq \alpha \leq m$. For instance, the multi-index 1_α is defined as follows: $1_\alpha(\alpha) = 1,\ 1_\alpha(\beta) = 0$ for $\alpha \neq \beta$. Define on components the addition and the substraction of the multi-indexes (although the result of a substraction might not be a multi-index): $(I \pm J)(\alpha) = I(\alpha) \pm J(\alpha)$. We call the *length* of a multi-index the following number $\mid I \mid = \sum_{\alpha=1}^{m} I(\alpha)$, and its *factorial* is $I! = \prod_{\alpha=1}^{m}(I(\alpha))!$. The number of distinct indices represented by $\{\alpha_1, \alpha_2, ..., \alpha_k\},\ \alpha_j \in \{1, 2, ..., m\}$, $j = \overline{1,k}$, is

$$n(\alpha_1, \alpha_2, ..., \alpha_k) := \frac{\mid 1_{\alpha_1} + 1_{\alpha_2} + ... + 1_{\alpha_k} \mid !}{(1_{\alpha_1} + 1_{\alpha_2} + ... + 1_{\alpha_k})!}.$$

2 Preliminaries

To make complete our presentation, we recall and introduce some definitions and preliminary results.

Definition 1 *A feasible solution $x^0(\cdot) \in \mathcal{F}\left(\Omega_{t_0,t_1}\right)$ of the problem (MFP) is called efficient solution if there exists no other feasible solution $x(\cdot) \in \mathcal{F}\left(\Omega_{t_0,t_1}\right)$ such that $K\left(x(\cdot)\right) \preceq K\left(x^0(\cdot)\right)$, where*

$$K\left(x(\cdot)\right) := \left(\frac{\displaystyle\int_{\Gamma_{t_0,t_1}} f_\beta^1\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta}{\displaystyle\int_{\Gamma_{t_0,t_1}} w_\beta^1\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta}, ..., \frac{\displaystyle\int_{\Gamma_{t_0,t_1}} f_\beta^r\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta}{\displaystyle\int_{\Gamma_{t_0,t_1}} w_\beta^r\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta} \right).$$

In Treanţă [23], the following result is proved: if $x^0(\cdot) \in F(\Omega_{t_0,t_1})$ is [normal] efficient solution of the problem (MFP), then there exist the multipliers $\lambda \in R^r$, μ and ν such that the following conditions are fulfilled:

$$\sum_{c=1}^{r} \lambda_c \left[\frac{\partial f_\beta^c}{\partial x} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t) \right) - R_0^c \frac{\partial w_\beta^c}{\partial x} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t) \right) \right] \tag{1}$$

$$+\mu_\beta(t) \frac{\partial g}{\partial x} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t) \right) + \nu_\beta(t) \frac{\partial h}{\partial x} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t) \right)$$

$$- D_{\alpha_1} \left\{ \sum_{c=1}^{r} \lambda_c \left[\frac{\partial f_\beta^c}{\partial x_{\alpha_1}} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t) \right) - R_0^c \frac{\partial w_\beta^c}{\partial x_{\alpha_1}} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t) \right) \right] \right\}$$

$$- D_{\alpha_1} \left\{ \mu_\beta(t) \frac{\partial g}{\partial x_{\alpha_1}} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t) \right) + \nu_\beta(t) \frac{\partial h}{\partial x_{\alpha_1}} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t) \right) \right\}$$

$$+ \ldots + (-1)^{s-1} \frac{1}{n(\alpha_1, \ldots, \alpha_{s-1})} D^{s-1}_{\alpha_1 \ldots \alpha_{s-1}} \left\{ \sum_{c=1}^{r} \lambda_c \frac{\partial f_\beta^c}{\partial x_{\alpha_1 \ldots \alpha_{s-1}}} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t) \right) \right\}$$

$$+(-1)^{s} \frac{1}{n(\alpha_1, \ldots, \alpha_{s-1})} D^{s-1}_{\alpha_1 \ldots \alpha_{s-1}} \left\{ \sum_{c=1}^{r} \lambda_c R_0^c \frac{\partial w_\beta^c}{\partial x_{\alpha_1 \ldots \alpha_{s-1}}} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t) \right) \right\}$$

$$+(-1)^{s-1} \frac{1}{n(\alpha_1, \ldots, \alpha_{s-1})} D^{s-1}_{\alpha_1 \ldots \alpha_{s-1}} \left\{ \mu_\beta(t) \frac{\partial g}{\partial x_{\alpha_1 \ldots \alpha_{s-1}}} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t) \right) \right\}$$

$$+(-1)^{s-1} \frac{1}{n(\alpha_1, \ldots, \alpha_{s-1})} D^{s-1}_{\alpha_1 \ldots \alpha_{s-1}} \left\{ \nu_\beta(t) \frac{\partial h}{\partial x_{\alpha_1 \ldots \alpha_{s-1}}} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t) \right) \right\} = 0$$

$$(higher - order\ Euler - Lagrange\ PDEs), \quad \beta = \overline{1,m}$$

$$\mu_\beta(t) g \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t) \right) = 0, \quad \mu_\beta(t) \geq 0, \quad t \in \Omega_{t_0,t_1}, \quad \beta = \overline{1,m}$$

$$\lambda \geq 0, \quad e^t \lambda = 1, \quad e^t := (1, 1, \ldots, 1) \in R^r.$$

Further, we shall introduce a *generalized* (ρ, b)*-quasiinvexity* associated with the aforementioned optimization problem involving path-independent curvilinear integral functionals. The concept of (ρ, b)*-quasiinvexity*, associated with simple integral functionals, was also used in recent works for the study of some multiobjective variational problems (see Treanţă [20, 22]).

Let ρ be a real number and b: $\left[C^\infty(\Omega_{t_0,t_1}, M)\right]^{2s} \to [0, \infty)$ a functional. In the following, we consider the notations:

$$b\left(x(\cdot), x_{\alpha_1}(\cdot), \ldots, x_{\alpha_1 \ldots \alpha_{s-1}}(\cdot), x^0(\cdot), x^0_{\alpha_1}(\cdot), \ldots, x^0_{\alpha_1 \ldots \alpha_{s-1}}(\cdot)\right) := b_{xx^0}$$

$$\eta\left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1 \ldots \alpha_{s-1}}(t), x^0(t), x^0_{\alpha_1}(t), \ldots, x^0_{\alpha_1 \ldots \alpha_{s-1}}(t)\right) := \eta_{txx^0}, \quad t \in \Omega_{t_0,t_1}.$$

Also, let $a = (a_\beta) : J^{s-1}(T, M) \to R^m$ be a closed Lagrange 1-form that determines the following path-independent curvilinear integral functional

$$A(x(t)) = \int_{\Gamma_{t_0,t_1}} a_\beta \left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}(t) \right) dt^\beta.$$

Definition 2 *The functional $A(x)$ is [strictly] (ρ, b)-quasiinvex at x^0 if there exist the vector functions $\eta = (\eta_1, ..., \eta_n)$, with the property*

$$\eta_{tx^0x^0} = 0,\ D_{\alpha_1}\eta_{tx^0x^0} = 0,\ \cdots, D^{s-2}_{\alpha_1 \ldots \alpha_{s-2}}\eta_{tx^0x^0} = 0$$

$$\alpha_\zeta \in \{1, ..., m\}, \quad \zeta = \overline{1, s-2}, \quad t \in \Omega_{t_0,t_1},$$

and θ: $[C^\infty(\Omega_{t_0,t_1}, M)]^{2s} \to R^n$ such that, for any x $[x \neq x^0]$, we have the following implication:

$$\left[A(x) \leq A(x^0)\right]$$

$$\Longrightarrow [b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left[\eta_{txx^0}\frac{\partial a_\beta}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + (D_{\alpha_1}\eta_{txx^0})\frac{\partial a_\beta}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$+ ... + b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left[\left(\frac{1}{n(\alpha_1, ..., \alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\eta_{txx^0}\right)\frac{\partial a_\beta}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$[<] \leq -\rho b_{xx^0} \parallel \theta_{xx^0} \parallel^2].$$

Example 1 *Consider*

$$x : [0,1] \to \mathfrak{M} \subseteq R^2, \quad x(t) = \left(x^1(t), x^2(t)\right),$$

a C^2-class function defined on the real interval $[0,1]$. Let $h : [0,1] \times \mathfrak{M} \to \mathbb{R}$ be a continuously differentiable function. The following functional of curvilinear integral type

$$H(x(\cdot)) = \int_0^1 h(t, \ddot{x}(t))\, dt$$

is, as it can be verified, $(\rho, 1)$-quasiinvex, for $\rho \leq 0$ and any θ, at $x^0(\cdot)$ with respect to

$$\eta\left(t, x(t), \dot{x}(t), \ddot{x}(t), x^0(t), \dot{x}^0(t), \ddot{x}^0(t)\right)$$

$$= \left(\eta_1(t, x(t), \dot{x}(t), \ddot{x}(t), x^0(t), \dot{x}^0(t), \ddot{x}^0(t)), \eta_2(t, x(t), \dot{x}(t), \ddot{x}(t), x^0(t), \dot{x}^0(t), \ddot{x}^0(t))\right)$$

$$= \left(H(x(\cdot)) - H\left(x^0(\cdot)\right)\right)\left(D^2\frac{\partial h}{\partial \ddot{x}^1}\left(t, \ddot{x}^0(t)\right), D^2\frac{\partial h}{\partial \ddot{x}^2}\left(t, \ddot{x}^0(t)\right)\right).$$

The previous example can be easily extended to n-dimensional vector valued functions and, by using normal coordinates, to the multidimensional case.

3 Main result

The next theorem is the main result of this paper.

Theorem 3.1 *(Sufficient efficiency conditions for (MFP)). Let $x^0(\cdot) \in F(\Omega_{t_0,t_1})$, $\lambda \in R^r$, μ and ν satisfying (1). As well, assume that the following hypotheses are fulfilled:*

a) the functionals $\int_{\Gamma_{t_0,t_1}}\left[f^l_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) - R^l_0 w^l_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$, $l = \overline{1,r}$, $\beta = \overline{1,m}$, *are (ρ^l_1, b)-quasiinvex at $x^0(\cdot)$ with respect to η and θ;*

b) $\int_{\Gamma_{t_0,t_1}} \mu_\beta(t) g\left(\chi_{x_{\alpha_1\cdots\alpha_{s-1}}}(t)\right) dt^\beta$ *is* (ρ_2, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

c) $\int_{\Gamma_{t_0,t_1}} \nu_\beta(t) h\left(\chi_{x_{\alpha_1\cdots\alpha_{s-1}}}(t)\right) dt^\beta$ *is* (ρ_3, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

d) at least one of the integrals of a) - c) is strictly (ρ, b)*-quasiinvex at the point* $x^0(\cdot)$ *with respect to* η *and* θ; $\quad$ *(see* $\rho = \rho_1^l, \rho_2$ *or* ρ_3*)*

e) $\sum_{l=1}^{r} \lambda_l \rho_1^l + \rho_2 + \rho_3 \geq 0 \quad (\rho_1^l, \rho_2, \rho_3 \in R)$.

Then the point $x^0(\cdot)$ *is an efficient solution of the problem (MFP).*

Proof. Consider, by reductio ad absurdum, that $x^0(\cdot)$ is not an efficient solution of (MFP). Then, for $l = \overline{1, r}$, there exists $x(\cdot) \in F(\Omega_{t_0,t_1})$ such that

$$\int_{\Gamma_{t_0,t_1}} \left[f_\beta^l \left(\chi_{x_{\alpha_1\cdots\alpha_{s-1}}}(t)\right) - R_0^l w_\beta^l \left(\chi_{x_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

$$\leq \int_{\Gamma_{t_0,t_1}} \left[f_\beta^l \left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right) - R_0^l w_\beta^l \left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

and there exists at least $k \in \{1, 2, ..., r\}$ with

$$\int_{\Gamma_{t_0,t_1}} \left[f_\beta^k \left(\chi_{x_{\alpha_1\cdots\alpha_{s-1}}}(t)\right) - R_0^k w_\beta^k \left(\chi_{x_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

$$< \int_{\Gamma_{t_0,t_1}} \left[f_\beta^k \left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right) - R_0^k w_\beta^k \left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right] dt^\beta.$$

Using the hypothesis a) and setting $X_\beta^l := f_\beta^l - R_0^l w_\beta^l$, we have

$$b_{xx^0} \int_{\Gamma_{t_0,t_1}} \left[\eta_{txx^0} \frac{\partial X_\beta^l}{\partial x} \left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right) + \left(D_{\alpha_1}\eta_{txx^0}\right) \frac{\partial X_\beta^l}{\partial x_{\alpha_1}} \left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

$$+ ... + b_{xx^0} \int_{\Gamma_{t_0,t_1}} \left[\left(\frac{1}{n\left(\alpha_1, ..., \alpha_{s-1}\right)} D^{s-1}_{\alpha_1\ldots\alpha_{s-1}} \eta_{txx^0}\right) \frac{\partial X_\beta^l}{\partial x_{\alpha_1\ldots\alpha_{s-1}}} \left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

$$\leq -\rho_1^l b_{xx^0} \parallel \theta_{xx^0} \parallel^2 .$$

Multiplying by $\lambda_l \geq 0$ and making the sum from $l = 1$ to $l = r$, we obtain

$$b_{xx^0} \int_{\Gamma_{t_0,t_1}} \left[\eta_{txx^0} \lambda \frac{\partial X_\beta}{\partial x} \left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right] dt^\beta \tag{2}$$

$$+ b_{xx^0} \int_{\Gamma_{t_0,t_1}} \left[\left(D_{\alpha_1}\eta_{txx^0}\right) \lambda \frac{\partial X_\beta}{\partial x_{\alpha_1}} \left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

$$+...+b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left[\left(\frac{1}{n(\alpha_1,...,\alpha_{s-1})}D^{s-1}_{\alpha_1...\alpha_{s-1}}\eta_{txx^0}\right)\lambda\frac{\partial X_\beta}{\partial x_{\alpha_1...\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$\leq -\left(\sum_{l=1}^{r}\lambda_l\rho_1^l\right)b_{xx^0}\parallel\theta_{xx^0}\parallel^2 .$$

The following inequality

$$\int_{\Gamma_{t_0,t_1}}\mu_\beta(t)g\left(\chi_{x_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)dt^\beta \leq \int_{\Gamma_{t_0,t_1}}\mu_\beta(t)g\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)dt^\beta,$$

according to b), leads us to

$$b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left[\eta_{txx^0}\mu_\beta(t)\frac{\partial g}{\partial x}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta \tag{3}$$
$$+b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left[\left(D_{\alpha_1}\eta_{txx^0}\right)\mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$+...+b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left[\left(\frac{1}{n(\alpha_1,...,\alpha_{s-1})}D^{s-1}_{\alpha_1...\alpha_{s-1}}\eta_{txx^0}\right)\mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1...\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$\leq -\rho_2 b_{xx^0}\parallel\theta_{xx^0}\parallel^2 .$$

Also, the equality (see c))

$$\int_{\Gamma_{t_0,t_1}}\nu_\beta(t)h\left(\chi_{x_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)dt^\beta = \int_{\Gamma_{t_0,t_1}}\nu_\beta(t)h\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)dt^\beta$$

gives

$$b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left[\eta_{txx^0}\nu_\beta(t)\frac{\partial h}{\partial x}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta \tag{4}$$
$$+b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left[\left(D_{\alpha_1}\eta_{txx^0}\right)\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$+...+b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left[\left(\frac{1}{n(\alpha_1,...,\alpha_{s-1})}D^{s-1}_{\alpha_1...\alpha_{s-1}}\eta_{txx^0}\right)\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1...\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$\leq -\rho_3 b_{xx^0}\parallel\theta_{xx^0}\parallel^2 .$$

Making the sum $(2)+(3)+(4)$, side by side, and taking into account d), we get

$$b_{xx^0}\int_{\Gamma_{t_0,t_1}}\eta_{txx^0}\left[\lambda\frac{\partial X_\beta}{\partial x}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)+\mu_\beta(t)\frac{\partial g}{\partial x}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$+b_{xx^0}\int_{\Gamma_{t_0,t_1}}\eta_{txx^0}\left[\nu_\beta(t)\frac{\partial h}{\partial x}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$+b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left(D_{\alpha_1}\eta_{txx^0}\right)\left[\lambda\frac{\partial X_\beta}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)+\mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$+b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left(D_{\alpha_1}\eta_{txx^0}\right)\left[\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$+...+b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left(\frac{1}{n\left(\alpha_1,...,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\eta_{txx^0}\right)\left[\lambda\frac{\partial X_\beta}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$+b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left(\frac{1}{n\left(\alpha_1,...,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\eta_{txx^0}\right)\left[\mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$+b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left(\frac{1}{n\left(\alpha_1,...,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\eta_{txx^0}\right)\left[\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$<-\left(\sum_{l=1}^{r}\lambda_l\rho_1^l+\rho_2+\rho_3\right)b_{xx^0}\parallel\theta_{xx^0}\parallel^2 .$$

This implies that $b_{xx^0}>0$ and the foregoing inequality can be rewritten as

$$\int_{\Gamma_{t_0,t_1}}\eta_{txx^0}\left[\lambda\frac{\partial X_\beta}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)+\mu_\beta(t)\frac{\partial g}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$+\int_{\Gamma_{t_0,t_1}}\eta_{txx^0}\left[\nu_\beta(t)\frac{\partial h}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$+\int_{\Gamma_{t_0,t_1}}\left(D_{\alpha_1}\eta_{txx^0}\right)\left[\lambda\frac{\partial X_\beta}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)+\mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$+\int_{\Gamma_{t_0,t_1}}\left(D_{\alpha_1}\eta_{txx^0}\right)\left[\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$+...+\int_{\Gamma_{t_0,t_1}}\left(\frac{1}{n\left(\alpha_1,...,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\eta_{txx^0}\right)\left[\lambda\frac{\partial X_\beta}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$+\int_{\Gamma_{t_0,t_1}}\left(\frac{1}{n\left(\alpha_1,...,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\eta_{txx^0}\right)\left[\mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$+\int_{\Gamma_{t_0,t_1}}\left(\frac{1}{n\left(\alpha_1,...,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\eta_{txx^0}\right)\left[\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$<-\left(\sum_{l=1}^{r}\lambda_l\rho_1^l+\rho_2+\rho_3\right)\parallel\theta_{xx^0}\parallel^2,$$

or, after integrating by parts, we get

$$\int_{\Gamma_{t_0,t_1}}\eta_{txx^0}\left[\lambda\frac{\partial X_\beta}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)+\mu_\beta(t)\frac{\partial g}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$+\int_{\Gamma_{t_0,t_1}}\eta_{txx^0}\left[\nu_\beta(t)\frac{\partial h}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$

$$-\int_{\Gamma_{t_0,t_1}} \eta_{txx^0} D_{\alpha_1}\left[\lambda \frac{\partial X_\beta}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + \mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

$$-\int_{\Gamma_{t_0,t_1}} \eta_{txx^0} D_{\alpha_1}\left[\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

$$+ ... + (-1)^{s-1}\int_{\Gamma_{t_0,t_1}} \eta_{txx^0} \frac{1}{n(\alpha_1, ..., \alpha_{s-1})} D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left[\lambda \frac{\partial X_\beta}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

$$(-1)^{s-1}\int_{\Gamma_{t_0,t_1}} \eta_{txx^0} \frac{1}{n(\alpha_1, ..., \alpha_{s-1})} D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left[\mu_\beta(t) \frac{\partial g}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

$$(-1)^{s-1}\int_{\Gamma_{t_0,t_1}} \eta_{txx^0} \frac{1}{n(\alpha_1, ..., \alpha_{s-1})} D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left[\nu_\beta(t) \frac{\partial h}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

$$< -\left(\sum_{l=1}^{r} \lambda_l \rho_1^l + \rho_2 + \rho_3\right) \parallel \theta_{xx^0} \parallel^2 .$$

The above given computation is obtained by using the boundary conditions

$$x(t_\xi) = x_\xi,\ x_{\alpha_1\ldots\alpha_j}(t_\xi) = \tilde{x}_{\alpha_1\ldots\alpha_j\xi},\ \alpha_\zeta \in \{1, ..., m\},\ \zeta, j = \overline{1, s-2},\ \xi \in \{0, 1\},$$

(see $x(t_\xi) = x_\xi = x^0(t_\xi),\ x_{\alpha_1\ldots\alpha_j}(t_\xi) = \tilde{x}_{\alpha_1\ldots\alpha_j\xi} = x^0_{\alpha_1\ldots\alpha_j}(t_\xi)$), and the following conditions (see Definition 2),

$$\eta_{tx^0x^0} = 0,\ D_{\alpha_1}\eta_{tx^0x^0} = 0,\ \cdots, D^{s-2}_{\alpha_1\ldots\alpha_{s-2}}\eta_{tx^0x^0} = 0$$

$$\alpha_\zeta \in \{1, ..., m\}, \quad \zeta = \overline{1, s-2}, \quad t \in \Omega_{t_0,t_1}.$$

The conditions (1) lead us to

$$0 < -\left(\sum_{l=1}^{r} \lambda_l \rho_1^l + \rho_2 + \rho_3\right) \parallel \theta_{xx^0} \parallel^2 .$$

Applying the hypothesis e) and $\parallel \theta_{xx^0} \parallel^2 \geq 0$, we get a contradiction. Thus, the point x^0 is an efficient solution for (MFP) and the proof is complete.

Corollary 1 *(Sufficient efficiency conditions for (MFP)). Let $x^0(\cdot)$ be a feasible solution of the problem (MFP) and assume that (1) is fulfilled. Also, consider the following properties hold:*

a) the functionals $\int_{\Gamma_{t_0,t_1}} \left[f^l_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) - R^l_0 w^l_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right] dt^\beta$, $l = \overline{1, r},\ \beta = \overline{1, m}$, are (ρ^l_1, b)-quasiinvex at the point $x^0(\cdot)$ with respect to η and θ;

b') the functional $\int_{\Gamma_{t_0,t_1}} \left[\mu_\beta(t) g\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + \nu_\beta(t) h\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right] dt^\beta$ is (ρ_2, b)-quasiinvex at the point $x^0(\cdot)$ with respect to η and θ;

d') at least one of the integrals $\int_{\Gamma_{t_0,t_1}} \left[f_\beta^l \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) - R_0^l w_\beta^l \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta$, $l = \overline{1,r}$, $\beta = \overline{1,m}$, $\int_{\Gamma_{t_0,t_1}} \left[\mu_\beta(t) g \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) + \nu_\beta(t) h \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta$, *is strictly* (ρ_1^l, b) *or* (ρ_2, b)*-quasiinvex at the point* $x^0(\cdot)$ *with respect to* η *and* θ*;*

e') $\sum_{l=1}^{r} \lambda_l \rho_1^l + \rho_2 \geq 0 \quad (\rho_1^l, \rho_2 \in R)$.

Then the point $x^0(\cdot)$ *is an efficient solution of the problem (MFP).*

Proof. The proof follows in the same manner as in Theorem 3.1. The functionals

$$\int_{\Gamma_{t_0,t_1}} \left[\mu_\beta(t) g \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta, \quad \int_{\Gamma_{t_0,t_1}} \left[\nu_\beta(t) h \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta$$

are replaced by $\int_{\Gamma_{t_0,t_1}} \left[\mu_\beta(t) g \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) + \nu_\beta(t) h \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta$.

Corollary 2 *(Sufficient efficiency conditions for (MFP)). Consider that* (1) *and the following hypotheses are fulfilled:*

a') the functionals

$$\int_{\Gamma_{t_0,t_1}} \left[W^l \left(x^0(t) \right) f_\beta^l \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) - F^l \left(x^0(t) \right) w_\beta^l \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta,$$

$l = \overline{1,r}$, $\beta = \overline{1,m}$, *are* (ρ_1^l, b)*-quasiinvex at the point* $x^0(\cdot)$ *with respect to* η *and* θ*;*

b') the functional $\int_{\Gamma_{t_0,t_1}} \left[\mu_\beta(t) g \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) + \nu_\beta(t) h \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta$ *is* (ρ_2, b)*-quasiinvex at the point* $x^0(\cdot)$ *with respect to* η *and* θ*;*

d") at least one of the integrals

$$\int_{\Gamma_{t_0,t_1}} \left[W^l \left(x^0(t) \right) f_\beta^l \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) - F^l \left(x^0(t) \right) w_\beta^l \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta,$$

$l = \overline{1,r}$, $\beta = \overline{1,m}$, $\int_{\Gamma_{t_0,t_1}} \left[\mu_\beta(t) g \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) + \nu_\beta(t) h \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta$ *is strictly* (ρ_1^l, b) *or* (ρ_2, b)*-quasiinvex at the point* $x^0(\cdot)$ *with respect to* η *and* θ*;*

e") $\sum_{l=1}^{r} \lambda_l \rho_1^l + \rho_2 \geq 0 \quad (\rho_1^l, \rho_2 \in R)$.

Then the point $x^0(\cdot)$ *is an efficient solution of the problem (MFP).*

Proof. Taking into account the definition of R_0^l and redefining μ_β and ν_β, the functional

$$\int_{\Gamma_{t_0,t_1}} \left[f_\beta^l \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) - R_0^l w_\beta^l \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta$$

is changed into

$$\int_{\Gamma_{t_0,t_1}} \left[W^l\left(x^0(t)\right) f^l_\beta\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right) - F^l\left(x^0(t)\right) w^l_\beta\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

and the integrals

$$\int_{\Gamma_{t_0,t_1}} \left[\mu_\beta(t) g\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right)\right] dt^\beta, \quad \int_{\Gamma_{t_0,t_1}} \left[\nu_\beta(t) h\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

are replaced by $\displaystyle\int_{\Gamma_{t_0,t_1}} \left[\mu_\beta(t) g\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right) + \nu_\beta(t) h\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right)\right] dt^\beta.$

Corollary 3 *(Sufficient efficiency conditions for (MFP)). If the conditions* (1) *are fulfilled and the following properties hold:*

a') the functionals

$$\int_{\Gamma_{t_0,t_1}} \left[W^l\left(x^0(t)\right) f^l_\beta\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right) - F^l\left(x^0(t)\right) w^l_\beta\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right)\right] dt^\beta,$$

$l = \overline{1,r},\ \beta = \overline{1,m}$, *are* (ρ_1^l, b)*-quasiinvex at the point* $x^0(\cdot)$ *with respect to* η *and* θ*;*

b) $\displaystyle\int_{\Gamma_{t_0,t_1}} \mu_\beta(t) g\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right) dt^\beta$ *is* (ρ_2, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

c) $\displaystyle\int_{\Gamma_{t_0,t_1}} \nu_\beta(t) h\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right) dt^\beta$ *is* (ρ_3, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

d) at least one of the integrals of a'), b), c) is strictly* (ρ, b)*-quasiinvex at the point* $x^0(\cdot)$ *with respect to* η *and* θ*;* (*see* $\rho = \rho_1^l, \rho_2$ *or* ρ_3)

e)* $\displaystyle\sum_{l=1}^{r} \lambda_l \rho_1^l + \rho_2 + \rho_3 \geq 0 \quad (\rho_1^l, \rho_2, \rho_3 \in R)$,

then the point $x^0(\cdot)$ *is an efficient solution of the problem (MFP).*

Proof. Taking into account the definition of R_0^l and redefining μ_β and ν_β, the functional

$$\int_{\Gamma_{t_0,t_1}} \left[f^l_\beta\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right) - R_0^l w^l_\beta\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

is replaced by

$$\int_{\Gamma_{t_0,t_1}} \left[W^l\left(x^0(t)\right) f^l_\beta\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right) - F^l\left(x^0(t)\right) w^l_\beta\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

and the proof follows in the same manner as in Theorem 3.1.

4 Conclusion

In this paper, motivated by the ongoing research in this area, by using the extended notion of (ρ, b)-*quasiinvexity*, we have formulated and proved sufficient efficiency conditions for a class of multidimensional vector ratio optimization problems (MFP) of minimizing a vector of path-independent curvilinear integral functionals (mechanical work) quotients subject to PDE and/or PDI constraints involving higher-order partial derivatives. Due to physical meaning of the objective functionals, the importance of this research paper has been supported both from theoretical and practical considerations.

References

[1] R.P. Agarwal, I. Ahmad, A. Iqbal, S. Ali, Generalized invex sets and preinvex functions on Riemannian manifolds, Taiwanese Journal of Mathematics, 16(5) 2012, 1719-1732.

[2] A. Barani, M.R. Pouryayevali, Invex sets and preinvex functions on Riemannian manifolds, J. Math. Anal. Appl., 328(2) 2007, 767-779.

[3] T. Antczak, G-pre-invex functions in mathematical programming, J. Comput. Appl. Math., **217**, 1, (2008), 212-226.

[4] M. Arana-Jiménez, R. Osuna-Gómez, G. Ruiz-Garzón, M. Rojas-Medar, On variational problems: Characterization of solutions and duality, J. Math. Anal. Appl., 311(1) 2005, 1-12.

[5] A. Chinchuluun, P.M. Pardalos, A survey of recent developments in multiobjective optimization, Ann. Oper. Res., 154(1) 2007, 29-50.

[6] M.A. Hanson, On sufficiency of Kuhn-Tucker conditions, J. Math. Anal. Appl., 80(2) 1981, 545-550.

[7] R. Jagannathan, Duality for nonlinear fractional programming, Z. Oper. Res., 17 1973, 1-3.

[8] V. Jeyakumar, Strong and weak invexity in mathematical programming, Math. Oper. Res., 55 1985, 109-125.

[9] Z.A. Liang, H.X. Huang, P.M. Pardalos, Efficiency conditions and duality for a class of multiobjective fractional programming problems, J. Glob. Optim., 27(4) 2003, 447-471.

[10] Şt. Mititelu, S. Treanţă, Efficiency conditions in vector control problems governed by multiple integrals, J. Appl. Math. Comput., 57 (1-2) 2018, 647-665.

[11] B. Mond and I. Husain, Sufficient optimality criteria and duality for variational problems with generalised invexity, J. Aust. Math. Soc. (B), **31**, 1, (1989), 108-121.

[12] M.A. Noor, K.I. Noor, Some characterizations of strongly preinvex functions, J. Math. Anal. Appl., 316(2) 2006, 697-706.

[13] R. Pini, Convexity along curves and indunvexity, Optimization, 29, 4 1994, 301-309.

[14] V. Preda, On efficiency and duality for multiobjective programs, J. Math. Anal. Appl., 166(2) 1992, 365-377.

[15] V. Preda, Optimality and duality in fractional multiple objective programming involving semilocally preinvex and related functions, J. Math. Anal. Appl., 288(2) 2003, 365-382.

[16] T. Rapcsák, Smooth Nonlinear Optimization in R^n, Nonconvex Optimization and Its Applications, Kluwer Academic, 1997.

[17] D.J. Saunders, The Geometry of Jet Bundles, Cambridge Univ. Press, 1989.

[18] W. Tang, X. Yang, The sufficiency and necessity conditions of strongly preinvex functions, OR Transactions, 10(3) 2006, 50-58.

[19] T. Tanino and Y. Sawaragi, Duality theory in multiobjective programming, J. Optim. Theory Appl., 27(4) 1979, 509-529.

[20] S. Treanţă and C. Udrişte, On Efficiency Conditions for Multiobjective Variational Problems Involving Higher Order Derivatives, Proceedings of the 15th International Conference on Automatic Control, Modelling & Simulation (ACMOS-13), June 1-3, Braşov, Romania, (2013), 157-164.

[21] S. Treanţă, PDEs of Hamilton-Pfaff type via multi-time optimization problems, U.P.B. Sci. Bull., Series A: Appl. Math. Phys.,76(1) 2014, 163-168.

[22] S. Treanţă, On a vector optimization problem involving higher order derivatives, U.P.B. Sci. Bull., Series A: Appl. Math. Phys., 77(1) 2015, 115-128.

[23] S. Treanţă, Multiobjective fractional variational problem on higher order jet bundles, Commun. Math. Stat., 4(3) 2016, 323-340.

[24] S. Treanţă, Higher-order Hamilton dynamics and Hamilton-Jacobi divergence PDE, Computers and Mathematics with Applications, 75(2) 2018, 547-560.

[25] S. Treanţă, M. Arana-Jiménez, KT-pseudoinvex multidimensional control problem, Optim. Control Appl. Meth, https://doi.org/10.1002/oca.2410, 2018, 1-10.

[26] S. Treanţă, Efficiency in generalised V-KT-pseudoinvex control problems, International Journal of Control, DOI: 10.1080/00207179.2018.1483082, 2018, 1-8.

[27] S. Treanţă, On a new class of vector variational control problems, Numerical Functional Analysis and Optimization, DOI: 10.1080/01630563.2018.1488142, 2018, 1-7.

[28] C. Udrişte, Convex Functions and Optimization Methods on Riemannian Manifolds, Mathematics and Its Applications, Kluwer Academic, 297, 1994.

[29] T. Weir, B. Mond, Pre-invex functions in multiple objective optimization, J. Math. Anal. Appl., 136(1) 1988, 29-38.

[30] T. Weir and B. Mond, Generalized convexity and duality in multiple objective programming, Bulletin of the Australian Mathematical Society, 39(2) 1989, 287-299.