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Incomplete k-Pell, k-Pell-Lucas and modified k-Pell numbers

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Abstract

In this paper, it is defined the incomplete k-Pell, k-Pell-Lucas and Modified k-Pell numbers, it is studied the recurrence relations, some properties of these sequences of integers and their generating functions.

Keywords: Incomplete Pell numbers, Incomplete Pell-Lucas numbers, Incomplete Modified Pell numbers, k-Pell, k-Pell-Lucas and Modified k-Pell numbers.

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1. Introduction

The Fibonacci and Lucas sequences are the sequences of positive integers that have been studied over several years. Many authors have investigated about these sequence [18], [21], [22], [23], [32], [35] and also some generalizations of them have been a great topic of research [1], [2], [3], [8], [12], [13], [14], [16]. These sequences are examples of a sequences defined by a recurrence relation of second order. It is well known that the Fibonacci sequence $\{F_n\}_n$ is defined by the following recurrence relation

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2$$

with $F_0 = 0$ and $F_1 = 1$. In the case of the Lucas sequence $\{L_n\}_n$ we have

$$L_n = L_{n-1} + L_{n-2},\ n \geq 2$$

with $L_0 = 2$ and $L_1 = 1$.

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As we have mentioned above, many generalizations of Fibonacci (and Lucas) sequence has been investigated. One of these generalizations is the $k-$Fibonacci sequence $\{F_{k,n}\}_n$ with k any integer. This sequence is defined by

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2},\ n \geq 2$$

with $F_{k,0} = 0$ and $F_{k,1} = 1$.

The sequence of Pell numbers is a particular case of this sequence, considering $k = 2$. Hence the Pell sequence, denoted by $\{P_n\}_n$, is defined by the recursive sequence given by

$$P_n = 2P_{n-1} + P_{n-2},\ n \geq 2 \tag{1.1}$$

with initial terms $P_0 = 0$ and $P_1 = 1$.

If we modify the values of the initial conditions of this sequence, we obtain two more sequences called by Pell-Lucas and the Modified Pell sequences.

The sequence of Pell-Lucas numbers is defined by the recursive sequence given by

$$Q_n = 2Q_{n-1} + Q_{n-2},\ n \geq 2 \tag{1.2}$$

with initial terms $Q_0 = Q_1 = 2$, and the Modified Pell sequence $\{q_n\}_n$ is given by the following recursive relation

$$q_n = 2q_{n-1} + q_{n-2},\ n \geq 2 \tag{1.3}$$

with initial terms $q_0 = q_1 = 1$. Some identities related with these sequences involving sums formulae for products involving its terms have been studied by several authors (see, for example, [17], among others).

Catarino in [5] studied a generalization of Pell sequence, as well as in [6] and in [7], where Catarino and Vasco studied generalizations of Pell-Lucas and Modified Pell sequences, respectively. Also in [30], the authors studied some identities and norms of Hankel matrices whose entries are elements of these sequences, as well as in [31], we can find some properties involving sums of products with its terms. Such generalizations are defined by a recurrence relations of second order given, for any positive real number k, by:

$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2},\ n \geq 2 \tag{1.4}$$

with initial terms $P_{k,0} = 0$ and $P_{k,1} = 1$, for k-Pell sequence;

$$Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2},\ n \geq 2 \tag{1.5}$$

with initial terms $Q_{k,0} = Q_{k,1} = 2$, for k-Pell-Lucas sequence; and

$$q_{k,n} = 2q_{k,n-1} + kq_{k,n-2},\ n \geq 2 \tag{1.6}$$

with initial terms $q_{k,0} = q_{k,1} = 1$, for Modified k-Pell sequence. Note that in the particular case where $k = 1$, (1.4)-(1.5)-(1.6) reduces to (1.1)-(1.2)-(1.3), respectively.

Also, Djordjević in [9] considers the convolution of the generalized Pell and Pell-Lucas numbers $P^{(s)}_{n,m}$ and $Q^{(s)}_{n,m}$ respectively, for a nonnegative integers n, s and a natural number m. For $s = 0$, the sequence $P^{(0)}_{n,m}$ represents the generalized Pell numbers $P_{n,m}$, and the sequence $Q^{(0)}_{n,m}$ represents the generalized Pell-Lucas numbers $Q_{n,m}$. In addition, for $s = 0$ and $m = 2$ the numbers $P^{(0)}_{n,2}$ and $Q^{(0)}_{n,2}$ are Pell and Pell-Lucas numbers, respectively. In this paper the author presents the recurrence relation of the sequences $\{P^{(s)}_{n,m}\}$ and $\{Q^{(s)}_{n,m}\}$ and some properties of these sequences are stated.

There are many directions in order to develop our investigation through to these type of special numbers. For example, the mathematical term *incomplete* on Fibonacci, Lucas and Tribonacci numbers and polynomials has been considered. For studies about the incomplete Fibonacci and Lucas numbers and their generating functions and properties, see, for example, [15] and [24], and for the incomplete Tribonacci numbers, see, for example, [26] and [33]. The incomplete generalized Fibonacci and Lucas numbers are presented in [10] and the incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers in [11]. We may also refer to [25], [28], [29] and [34], among others.

Motivated essentially by these works, we aim to introduce and study here the analogously *incomplete* version of each one of these three classes of numbers: k-Pell, k-Pell-Lucas and Modified k-Pell numbers. We begin recalling some properties involving these sequences of numbers and then, for each one, we introduce and present some properties and the generating function.

2. A summary of some properties of k-Pell, k-Pell-Lucas and Modified k-Pell numbers

The characteristic equation associated with the recurrence relations (1.4), (1.5) and (1.6) is $r^2 - 2r - k = 0$ whose roots are $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$. Clearly, $r_1 + r_2 = 2,\ r_1 - r_2 = 2\sqrt{1+k}$ and $r_1 r_2 = -k$.

The k-Pell, k-Pell-Lucas and Modified k-Pell numbers verify the following properties (see, [5], [6] and [7] for the proofs), respectively:

- Binet's formula: $P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$; $Q_{k,n} = r_1^n + r_2^n$; $q_{k,n} = \frac{r_1^n + r_2^n}{2}$.
- Generating function: $f(t) = \frac{-t}{1-2t-kt^2}$; $g(t) = \frac{2-2t}{1-2t-kt^2}$; $h(t) = \frac{1-t}{1-2t-kt^2}$.

The Pell, Pell-Lucas and Modified Pell numbers verify the corresponding following properties (see, [19] and [20] for the proofs), respectively:

- Binet's formula: $P_n = \frac{r_1^n - r_2^n}{2\sqrt{2}}$; $Q_n = r_1^n + r_2^n$; $q_n = \frac{r_1^n + r_2^n}{2}$.
- Generating function: $f(t) = \frac{-t}{1-2t-t^2}$; $g(t) = \frac{2-2t}{1-2t-t^2}$; $h(t) = \frac{1-t}{1-2t-t^2}$.

and

- Explicit Formula: $P_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} 2^{n-1-2j}$ (see, for example, Theorem 9.1 in [20], p.173, or, see also (1.1) for $s = 0$ and $m = 2$ in [9]); $Q_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} 2^{n-2j}$; and since $2q_n = Q_n,\ n \geq 0$, then the explicit formula to Modified Pell number is given by $q_n = \frac{1}{2} Q_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} 2^{n-2j-1}$.

3. The incomplete k-Pell numbers and some properties

3.1. Definition. With k any integer, the incomplete k-Pell numbers are defined by

$$P_{k,n}^{l} := \sum_{j=0}^{l} \binom{n-1-j}{j} 2^{n-1-2j} k^j, \quad (0 \leq l \leq \lfloor \frac{n-1}{2} \rfloor;\ n \in \mathbb{N}). \tag{3.1}$$

Note that

$$P_{k,n}^{\lfloor \frac{n-1}{2} \rfloor} = P_{k,n}$$

and some special cases of (3.1) are:

$$P_{k,n}^{0} = 2^{n-1},\ (n \geq 1);$$
$$P_{k,n}^{1} = 2^{n-1} + k(n-2)2^{n-3},\ (n \geq 3);$$
$$P_{k,n}^{2} = 2^{n-1} + k(n-2)2^{n-3} + \frac{(n-4)(n-3)}{2} 2^{n-5} k^2,\ (n \geq 5);$$

$$P_{k,n}^{\lfloor\frac{n-3}{2}\rfloor} = \begin{cases} P_{k,n} - nk^{\frac{n-2}{2}}, & (n \text{ even}) \\ P_{k,n} - k^{\frac{n-1}{2}}, & (n \text{ odd}) \end{cases} \quad (n \geq 3).$$

By Definition 3.1, in the Table 1 we present a few incomplete k-Pell numbers.

$n\backslash j$	$j=0$	$j=1$	$j=2$	$j=3$
$n=1$	1			
$n=2$	2			
$n=3$	4	$4+k$		
$n=4$	8	$8+4k$		
$n=5$	16	$16+12k$	$k^2+12k+16$	
$n=6$	32	$32+32k$	$6k^2+32k+32$	
$n=7$	64	$64+80k$	$24k^2+80k+64$	$k^3+24k^2+80k+64$
$n=8$	128	$128+192k$	$80k^2+192k+128$	$8k^3+80k^2+192k+128$

Table 1. The incomplete $P_{k,n}^l$ for $1 \leq n \leq 8$

In the next result we present the recurrence relation verified by these numbers followed by other results which reveals some properties of the numbers $P_{k,n}^l$.

3.2. Proposition. *With k any integer, the recurrence relation of the incomplete k-Pell numbers $P_{k,n}^l$ is*

$$P_{k,n+2}^{l+1} = 2P_{k,n+1}^{l+1} + kP_{k,n}^l, \quad (0 \leq l \leq \frac{n-2}{2};\ n \in \mathbb{N}). \tag{3.2}$$

Proof. By Definition 3.1 we get

$$\begin{aligned} 2P_{k,n+1}^{l+1} + kP_{k,n}^l &= 2\sum_{j=0}^{l+1}\binom{n-j}{j}2^{n-2j}k^j + k\sum_{j=0}^{l}\binom{n-1-j}{j}2^{n-1-2j}k^j \\ &= \sum_{j=0}^{l+1}\binom{n-j}{j}2^{n-2j+1}k^j + \sum_{j=0}^{l}\binom{n-1-j}{j}2^{n-1-2j}k^{j+1} \\ &= \sum_{j=0}^{l+1}\binom{n-j}{j}2^{n-2j+1}k^j + \sum_{j=1}^{l+1}\binom{n-j}{j-1}2^{n-2j+1}k^j \\ &= \sum_{j=0}^{l+1}\left[\left(\binom{n-j}{j} + \binom{n-j}{j-1}\right)2^{n+1-2j}k^j\right] \\ &= \sum_{j=0}^{l+1}\binom{n-j+1}{j}2^{n-2j+1}k^j \\ &= P_{k,n+2}^{l+1}, \end{aligned}$$

as required. □

3.3. Proposition. *With k any integer and n natural number, the relation of the incomplete k-Pell numbers $P_{k,n}^l$ given in (3.2) can be transformed into the non-homogeneous recurrence relation*

$$P_{k,n+2}^l = 2P_{k,n+1}^l + kP_{k,n}^l - \binom{n-1-l}{l}2^{n-1-2l}k^{l+1}, \quad (0 \leq l \leq \frac{n-2}{2}). \tag{3.3}$$

Proof. By Proposition 3.2 and Definition 3.1 we get

$$P^l_{k,n+2} - 2P^l_{k,n+1} - kP^l_{k,n} = \left(2P^l_{k,n+1} + kP^{l-1}_{k,n}\right) - 2P^l_{k,n+1} - kP^l_{k,n}$$
$$= kP^{l-1}_{k,n} - kP^l_{k,n} = k\left(P^{l-1}_{k,n} - P^l_{k,n}\right)$$
$$= k\left(\sum_{j=0}^{l-1}\binom{n-1-j}{j}2^{n-1-2j}k^j - \sum_{j=0}^{l}\binom{n-1-j}{j}2^{n-1-2j}k^j\right)$$

$$= -\binom{n-1-l}{l}2^{n-1-2l}k^{l+1},$$

as required. □

More properties involving the incomplete k-Pell numbers are given in the next results.

3.4. Proposition. *With k any integer, and n, s natural numbers,*

$$(3.4)\qquad P^{l+s}_{k,n+2s} = \sum_{i=0}^{s}\binom{s}{i}P^{l+i}_{k,n+i}2^ik^{s-i}, \quad 0 \le l \le \frac{n-s-1}{2}.$$

Proof. We prove by induction on s. It is clear that for $s = 0$ and $s = 1$, the equation (3.4) holds. Now suppose that the result is true for all $j < s+1$ and we shall prove it for $s+1$. Then using some combinatorial properties and Proposition 3.2, we have:

$$\sum_{i=0}^{s+1}\binom{s+1}{i}P^{l+i}_{k,n+i}2^ik^{s+1-i} = \sum_{i=0}^{s+1}\left[\binom{s}{i}+\binom{s}{i-1}\right]P^{l+i}_{k,n+i}2^ik^{s+1-i}$$
$$= \sum_{i=0}^{s+1}\binom{s}{i}P^{l+i}_{k,n+i}2^ik^{s+1-i} + \sum_{i=0}^{s+1}\binom{s}{i-1}P^{l+i}_{k,n+i}2^ik^{s+1-i}$$
$$= kP^{l+s}_{k,n+2s} + \binom{s}{s+1}P^{l+s+1}_{k,n+s+1}2^{s+1} + \sum_{i=-1}^{s}\binom{s}{i}P^{l+i+1}_{k,n+i+1}2^{i+1}k^{s-i}$$
$$= kP^{l+s}_{k,n+2s} + \sum_{i=0}^{s}\binom{s}{i}P^{l+i+1}_{k,n+i+1}2^{i+1}k^{s-i}$$
$$= kP^{l+s}_{k,n+2s} + 2P^{l+s+1}_{k,n+2s+1}$$

$$= P^{l+(s+1)}_{k,n+2(s+1)},$$

as required. □

Now let us consider the sum of s consecutive elements of the lth column of the array shown in Table 1.

3.5. Proposition. *With k any integer, n, s natural numbers and once l has been chosen, for $n \ge 2l+2$, we have*

$$(3.5)\qquad \sum_{i=0}^{s-1}P^l_{k,n+i}2^{s-i-1}k = P^{l+1}_{k,n+s+1} - 2^sP^{l+1}_{k,n+1}.$$

Proof. We proceed by induction on s. The sum (3.5) is true for $s = 1$ (see Proposition 3.2). Now suppose that the result is valid for all $j < s$ and we shall prove it for s. Using Proposition 3.2, we have:

$$\begin{aligned} P^{l+1}_{k,n+s+2} - 2^{s+1}P^{l+1}_{k,n+1} &= \left(2P^{l+1}_{k,n+s+1} + kP^{l}_{k,n+s}\right) - 2^{s+1}P^{l+1}_{k,n+1} \\ &= 2\left(P^{l+1}_{k,n+s+1} - 2^{s}P^{l+1}_{k,n+1}\right) + kP^{l}_{k,n+s} \\ &= 2\sum_{i=0}^{s-1} P^{l}_{k,n+i}2^{s-i-1}k + kP^{l}_{k,n+s} \\ &= \sum_{i=0}^{s-1} P^{l}_{k,n+i}2^{s-i}k + kP^{l}_{k,n+s} \\ &= \sum_{i=0}^{s} P^{l}_{k,n+i}2^{s-i}k. \end{aligned}$$

□

4. The incomplete k-Pell-Lucas numbers and some properties

4.1. Definition. With k any integer, the incomplete k-Pell-Lucas numbers are defined by

$$(4.1) \qquad Q^{l}_{k,n} := \sum_{j=0}^{l} \frac{n}{n-j}\binom{n-j}{j} 2^{n-2j}k^{j}, \quad (0 \le l \le \lfloor \frac{n}{2} \rfloor;\ n \in \mathbb{N}).$$

Note that

$$Q^{\lfloor \frac{n}{2} \rfloor}_{k,n} = Q_{k,n}$$

and some special cases of (4.1) are:

$$Q^{0}_{k,n} = 2^{n},\ (n \ge 1);$$

$$Q^{1}_{k,n} = 2^{n-2}(4 + nk),\ (n \ge 2);$$

$$Q^{2}_{k,n} = 2^{n}(1 + nk2^{-2} + n(n-3)k^{2}2^{-5}),\ (n \ge 4);$$

$$Q^{\lfloor \frac{n-2}{2} \rfloor}_{k,n} = \begin{cases} Q_{k,n} - 2k^{\frac{n}{2}}, & (n \text{ even}) \\ Q_{k,n} - 2nk^{n+1}, & (n \text{ odd}) \end{cases} \quad (n \ge 2).$$

By Definition 4.1, we present in Table 2 a few incomplete k-Pell-Lucas numbers.

$n\backslash j$	$j=0$	$j=1$	$j=2$	$j=3$
$n=1$	2			
$n=2$	4	$4+2k$		
$n=3$	8	$8+6k$		
$n=4$	16	$16+16k$	$2k^2+16k+16$	
$n=5$	32	$32+40k$	$10k^2+40k+32$	
$n=6$	64	$64+96k$	$36k^2+96k+64$	$2k^3+36k^2+96k+64$
$n=7$	128	$128+224k$	$112k^2+224k+128$	$14k^3+112k^2+224k+128$

Table 2. The incomplete $Q^{l}_{k,n}$ for $1 \le n \le 7$

In the following result we present the recurrence relation verified by the incomplete k-Pell-Lucas numbers.

4.2. Proposition. *With k any integer, the recurrence relation of the incomplete k-Pell-Lucas numbers $Q_{k,n}^{l}$ is*

$$Q_{k,n+2}^{l} = 2Q_{k,n+1}^{l} + kQ_{k,n}^{l-1}, \ (0 \le l \le \frac{n}{2}; \ n \in \mathbb{N}). \tag{4.2}$$

Proof. Using Definition 4.1 we obtain

$$\begin{aligned}
2Q_{k,n+1}^{l}+kQ_{k,n}^{l-1} &= 2\sum_{j=0}^{l} \frac{n+1}{n+1-j} \binom{n+1-j}{j} 2^{n+1-2j}k^{j}+k\sum_{j=0}^{l-1} \frac{n}{n-j} \binom{n-j}{j} 2^{n-2j}k^{j} \\
&= \sum_{j=0}^{l} \frac{n+1}{n+1-j} \binom{n+1-j}{j} 2^{n-2j+2}k^{j} + \sum_{j=0}^{l-1} \frac{n}{n-j} \binom{n-j}{j} 2^{n-2j}k^{j+1} \\
&= \sum_{j=0}^{l} \frac{n+1}{n+1-j} \binom{n+1-j}{j} 2^{n-2j+2}k^{j} + \sum_{j=1}^{l} \frac{n}{n+1-j} \binom{n-j+1}{j-1} 2^{n-2j+2}k^{j} \\
&= \sum_{j=0}^{l} \left[\frac{n+1}{n+1-j} \binom{n+1-j}{j} + \frac{n}{n+1-j} \binom{n+1-j}{j-1}\right] 2^{n-2j+2}k^{j} \\
&= \sum_{j=0}^{l} \left[\frac{n}{n+1-j} \binom{n+2-j}{j} + \frac{1}{n+1-j} \binom{n+1-j}{j}\right] 2^{n-2j+2}k^{j} \\
&= \sum_{j=0}^{l} \left[\frac{n}{n+1-j} \binom{n+2-j}{j} + \frac{n+2-2j}{(n+1-j)(n+2-j)} \binom{n+2-j}{j}\right] 2^{n-2j+2}k^{j} \\
&= \sum_{j=0}^{l} \left[\frac{n+2}{n+2-j} \binom{n+2-j}{j}\right] 2^{n-2j+2}k^{j} \\
&= Q_{k,n+2}^{l},
\end{aligned}$$

as required. □

4.3. Proposition. *The relation of the incomplete k-Pell-Lucas numbers $Q_{k,n}^{l}$ given in (4.2) can be transformed into the non-homogeneous recurrence relation*

$$Q_{k,n+2}^{l} = 2Q_{k,n+1}^{l} + kQ_{k,n}^{l} - \frac{n}{n-l} \binom{n-l}{l} 2^{n-2l}k^{l+1}. \tag{4.3}$$

Proof. By Proposition 4.2 and Definition 4.1 we get

$$\begin{aligned}
&Q_{k,n+2}^{l} - 2Q_{k,n+1}^{l} - kQ_{k,n}^{l} = \left(2Q_{k,n+1}^{l} + kQ_{k,n}^{l-1}\right) - 2Q_{k,n+1}^{l} - kQ_{k,n}^{l} \\
&= kQ_{k,n}^{l-1} - kQ_{k,n}^{l} = k\left(Q_{k,n}^{l-1} - Q_{k,n}^{l}\right) \\
&= k\left(\sum_{j=0}^{l-1} \frac{n}{n-j} \binom{n-j}{j} 2^{n-2j}k^{j} - \sum_{j=0}^{l} \frac{n}{n-j} \binom{n-j}{j} 2^{n-2j}k^{j}\right) \\
&= -\frac{n}{n-l} \binom{n-l}{l} 2^{n-2l}k^{l+1},
\end{aligned}$$

as required. □

More properties involving the incomplete k-Pell-Lucas numbers are given in the next results.

4.4. Proposition. *With k any integer, the incomplete k-Pell-Lucas numbers verify*

$$(4.4)\qquad Q^{l+s}_{k,n+2s}=\sum_{i=0}^{s}\left(\begin{array}{c}s\\ i\end{array}\right)Q^{l+i}_{k,n+i}2^ik^{s-i},\ (0\le l\le\frac{n-s}{2};\ n\in\mathbb{N}).$$

Proof. We prove by induction on s. It is clear that for $s=0$ and $s=1$, the equation (4.4) holds. Now suppose that the result is true for all $j<s+1$ and we shall prove it for $s+1$. Then using some combinatorial properties and Proposition 4.2, we have:

$$\begin{aligned}
\sum_{i=0}^{s+1}\left(\begin{array}{c}s+1\\ i\end{array}\right)Q^{l+i}_{k,n+i}2^ik^{s+1-i}&=\sum_{i=0}^{s+1}\left[\left(\begin{array}{c}s\\ i\end{array}\right)+\left(\begin{array}{c}s\\ i-1\end{array}\right)\right]Q^{l+i}_{k,n+i}2^ik^{s+1-i}\\
&=\sum_{i=0}^{s+1}\left(\begin{array}{c}s\\ i\end{array}\right)Q^{l+i}_{k,n+i}2^ik^{s+1-i}+\sum_{i=0}^{s+1}\left(\begin{array}{c}s\\ i-1\end{array}\right)Q^{l+i}_{k,n+i}2^ik^{s+1-i}\\
&=kQ^{l+s}_{k,n+2s}+\left(\begin{array}{c}s\\ s+1\end{array}\right)Q^{l+s+1}_{k,n+s+1}2^{s+1}+\sum_{i=0}^{s+1}\left(\begin{array}{c}s\\ s-i\end{array}\right)Q^{l+i}_{k,n+i}2^ik^{s+1-i}\\
&=kQ^{l+s}_{k,n+2s}+\sum_{i=-1}^{s}\left(\begin{array}{c}s\\ i\end{array}\right)Q^{l+i+1}_{k,n+i+1}2^{i+1}k^{s-i}\\
&=kQ^{l+s}_{k,n+2s}+\sum_{i=0}^{s}\left(\begin{array}{c}s\\ i\end{array}\right)Q^{l+i+1}_{k,n+i-1}2^{i+1}k^{s-i}\\
&=kQ^{l+s}_{k,n+2s}+2Q^{l+s+1}_{k,n+2s+1}\\
&=Q^{l+(s+1)}_{k,n+2(s+1)},
\end{aligned}$$

as required. □

Now let us consider the sum of s consecutive elements of the lth column of the array shown in Table 2.

4.5. Proposition. *Once l has been chosen, for $n\ge 2l+1$, we have*

$$(4.5)\qquad \sum_{i=0}^{s-1}Q^{l}_{k,n+i}2^{s-i-1}k=Q^{l+1}_{k,n+s+1}-2^sQ^{l+1}_{k,n+1},\ (n,\ s\in\mathbb{N}).$$

Proof. The proof of the following result is similar to Proposition 3.5. □

Catarino in [4] presents two relations between the k-Pell numbers and the k-Pell-Lucas numbers. In the next result we present some relationship between the incomplete numbers $P^l_{k,n}$ and $Q^l_{k,n}$ which are similar with the relations stated in [4].

4.6. Proposition. *Let l be a nonnegative integer and n be a positive integer. Then the following equalities hold:*

(1) *For $l=0$, $Q^0_{k,n}=2P^0_{k,n}$, $(n\ge 1)$;*
(2) *If $n=2l$ then $Q^l_{k,2l}=2\left(P^{l-1}_{k,2l}+kP^{l-1}_{k,2l-1}\right)$, $(n\ge 2)$;*
(3) $Q^l_{k,n}=2\left(P^l_{k,n}+kP^{l-1}_{k,n-1}\right)$, $(0<l<\lfloor\frac{n}{2}\rfloor)$;
(4) *If $n=2l$ then $Q^l_{k,2l}=2\left(P^l_{k,2l+1}-P^{l-1}_{k,2l}\right)$, $(n\ge 2)$;*
(5) $Q^l_{k,n}=2\left(P^l_{k,n+1}-P^l_{k,n}\right)$, $(0<l<\lfloor\frac{n}{2}\rfloor)$.

Proof. The proof of (1) is easy, being sufficient to use the definitions 3.1 and 4.1 in the case where $l=0$.

Proof of (2): From the definitions 3.1 and 4.1 and the use of some combinatorial properties we have:

$$\begin{aligned}
2\left(P_{k,2l}^{l-1}+kP_{k,2l-1}^{l-1}\right) &= 2\left[\sum_{j=0}^{l-1}\binom{2l-1-j}{j}2^{2l-1-2j}k^j + k\sum_{j=0}^{l-1}\binom{2l-2-j}{j}2^{2l-2-2j}k^j\right]\\
&= \sum_{j=0}^{l-1}\binom{2l-1-j}{j}2^{2l-2j}k^j + \sum_{j=0}^{l-1}\binom{2l-2-j}{j}2^{2l-1-2j}k^{j+1}\\
&= \sum_{j=0}^{l-1}\binom{2l-1-j}{j}2^{2l-2j}k^j + \sum_{j=1}^{l}\binom{2l-1-j}{j-1}2^{2l+1-2j}k^j\\
&= \sum_{j=0}^{l}\left[\binom{2l-1-j}{j}+2\binom{2l-1-j}{j-1}\right]2^{2l-2j}k^j\\
&= \sum_{j=0}^{l}\left[\binom{2l-j}{j}+\binom{2l-1-j}{j-1}\right]2^{2l-2j}k^j\\
&= \sum_{j=0}^{l}\left[\binom{2l-j}{j}+\frac{j}{2l-j}\binom{2l-j}{j}\right]2^{2l-2j}k^j\\
&= \sum_{j=0}^{l}\left[\binom{2l-j}{j}\frac{2l}{2l-j}\right]2^{2l-2j}k^j\\
&= Q_{k,2l}^{l}.
\end{aligned}$$

In a similar way used in the proof of statement (2) we can easily prove the statements (3). Also for the proofs of (4) and (5) we can use the Definitions 3.1 and 4.1 and some combinatorial properties and we easily obtain the result required. □

Now if we denote by $q_{k,n}^l$ the term of order n of a sequence $\{q_{k,n}^l\}$ such that

$$2q_{k,n}^l = Q_{k,n}^l,\ \ (0\le l\le \lfloor \frac{n}{2}\rfloor;\ n\in\mathbb{N})$$

we get what we call by the incomplete Modified k-Pell numbers sequence. From this relation all the results, which include these numbers, can be proved by a similar way used in case of the incomplete k-Pell-Lucas numbers. Note that, it is well known this equality for the case of non incomplete version of these sequences.

5. Generating functions of the incomplete k-Pell, k-Pell-Lucas and Modified k-Pell numbers

In this section we give the generating functions

$$GF_P=\sum_{j=0}^{\infty}P_{k,j}^l t^j,\ GF_Q=\sum_{j=0}^{\infty}Q_{k,j}^l t^j,\ GF_q=\sum_{j=0}^{\infty}q_{k,j}^l t^j$$

of the incomplete k-Pell, k-Pell-Lucas and Modified k-Pell numbers, respectively. First, we need to recall a lemma stated in [24], and then we will use it in the proof of the statement of each one of the generating function. Then we have the following result (see [24], p. 592):

5.1. Lemma. *Let $\{s_n\}_{n=0}^{\infty}$ be a complex sequence satisfying the following nonhomogeneous recurrence relation:*

$$s_n = as_{n-1} + bs_{n-2} + r_n \quad (n > 1),$$

where a and b are complex numbers and $\{r_n\}$ is a given complex sequence. Then the generating function $U(t)$ of the sequence $\{s_n\}$ is

$$U(t) = \frac{G(t) + s_0 - r_0 + (s_1 - s_0 a - r_1)t}{1 - at - bt^2},$$

where $G(t)$ denotes the generating function of $\{r_n\}$.

Now we shall use this result in the proof of the next three results.

5.2. Theorem. *The generating function of the incomplete k-Pell numbers $P_{k,n}^{l}$ is given by*

$$GF_P = t^{2l+1}\left[P_{k,2l+1}^{l} + \left(P_{k,2l+2}^{l} - 2P_{k,2l+1}^{l}\right)t - \frac{t^2}{(1-2kt)^{l+1}}\right]\left[1 - 2t - kt^2\right]^{-1}.$$

Proof. Let l be a positive integer. From Definition 3.1 and Proposition 3.3, we get that $P_{k,n}^{l} = 0$ for $0 \le n < 2l+1$, $P_{k,2l+1}^{l} = P_{k,2l+1}$, $P_{k,2l+2}^{l} = P_{k,2l+2}$, and that

$$P_{k,n}^{l} = 2P_{k,n-1}^{l} + kP_{k,n-2}^{l} - \binom{n-3-l}{l} 2^{n-3-2l} k^{l+1}.$$

Suppose that $s_0 = P_{k,2l+1}^{l}$, $s_1 = P_{k,2l+2}^{l}$ and $s_n = P_{k,n+2l+1}^{l}$. Suppose also that $r_0 = r_1 = 0$ and $r_n = \binom{n+l-1}{n-2} 2^{n-2} k^{n-2}$. Then, using [27], (p. 355), for the generating function $G(t)$ of the sequence $\{r_n\}$, we have

$$G(t) = \frac{t^2}{(1-2kt)^{l+1}}.$$

Thus, using Lemma 5.1, the generating function $S_{k,l}^{l}(t)$ of the sequence $\{s_n\}$ satisfies the following relationship:

$$S_{k,l}^{l}(t)(1 - 2t - kt^2) + \frac{t^2}{(1-2kt)^{l+1}} = P_{k,2l+1}^{l} + \left(P_{k,2l+2}^{l} - 2P_{k,2l+1}^{l}\right)t.$$

Hence $GF_P = \sum_{j=0}^{\infty} P_{k,j}^{l} t^j = t^{2l+1} S_{k,l}^{l}(t)$ and the result follows. □

5.3. Theorem. *The generating function of the incomplete k-Pell-Lucas numbers $Q_{k,n}^{l}$ is given by*

$$GF_Q = t^{2l}\left[Q_{k,2l}^{l} + \left(Q_{k,2l+1}^{l} - 2Q_{k,2l}^{l}\right)t - \frac{t^2(2-t)}{(1-2kt)^{l+1}}\right]\left[1 - 2t - kt^2\right]^{-1}.$$

Proof. Let l be a positive integer. From Definition 4.1 and Proposition 4.3, we get that $Q_{k,n}^{l} = 0$ for $0 \le n < 2l$, $Q_{k,2l}^{l} = Q_{k,2l}$, $Q_{k,2l+1}^{l} = Q_{k,2l+1}$, and that

$$Q_{k,n}^{l} = 2Q_{k,n-1}^{l} + kQ_{k,n-2}^{l} - \frac{n-2}{n-2-l}\binom{n-2-l}{l} 2^{n-2-2l} k^{l+1}.$$

Now suppose that $s_0 = Q_{k,2l}^{l}$, $s_1 = Q_{k,2l+1}^{l}$ and $s_n = Q_{k,n+2l}^{l}$. Consider $r_0 = r_1 = 0$ and $r_n = \binom{n+2l-2}{n+l-2} 2^{n+2l-2} k^{n+2l-2}$. Then, using [27], (p. 355), for the generating function $G(t)$ of the sequence $\{r_n\}$, we have

$$G(t) = \frac{t^2(2-t)}{(1-2kt)^{l+1}}.$$

Thus, using Lemma 5.1, the generating function $S^l_{k,l}(t)$ of the sequence $\{s_n\}$ satisfies the following relationship:

$$S^l_{k,l}(t)(1-2t-kt^2)+\frac{t^2(2-t)}{(1-2kt)^{l+1}}=Q^l_{k,2l}+\left(Q^l_{k,2l+1}-2Q^l_{k,2l}\right)t.$$

Hence $GF_Q=\sum_{j=0}^{\infty}Q^l_{k,j}t^j=t^{2l}S^l_{k,l}(t)$ and the result follows. □

Once again using the relationship between the incomplete k-Pell-Lucas numbers and the incomplete Modified k-Pell numbers we have that $GF_Q=2GF_q$ and then we obtain

5.4. Theorem. *The generating function of the incomplete Modified k-Pell numbers $q^l_{k,n}$ is given by*

$$GF_q=t^{2l}\left[q^l_{k,2l}+\left(q^l_{k,2l+1}-2q^l_{k,2l}\right)t-\frac{t^2(2-t)}{2(1-2kt)^{l+1}}\right]\left[1-2t-kt^2\right]^{-1}.$$

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