Hacettepe Journal of Mathematics and Statistics
Volume 46 (3) (2017), 437 – 447

On the category of ultra-groups

B. Tolue*†, Gh. Moghaddasi‡ and P. Zolfaghari§

Abstract

This article has been prepared based on a new concept of an ultra-group ${}_HM$ which is depend on a group G and its subgroup H. Our aim is to introduce the category of ultra-groups and investigate about some properties of this category.

Keywords: Category, transversal, ultra-group

2000 AMS Classification: 08C05, 16B50, 20A05.

Received : 12.04.2016 *Accepted* : 25.07.2016 *Doi* : 10.15672 /HJMS 20174720335

1. Introduction

A pair (A, B) of subsets of a group G is called *transversal* if the equality $ab = a'b'$ implies $a = a'$ and $b = b'$, where $a, a' \in A$, $b, b' \in B$. This notion was introduced by Kurosh in [6] which is the base of the concept of an ultra-group. The definition of the transversal for a pair of subsets can be generalized for a pair of a subgroup and a subset of a group. A pair (H, M) of subgroup H and subset M of a group G is *right transversal*, if $M \cap H = \{e_G\}$ and $M \cap Hg$ contains at most one element, for all $g \in G$. In the other words, the pair (H, M) is a right transversal if and only if a subset M of G obtained by selecting one and only one member from each right coset of H. Moreover, a subset M of a group G is called *right unitary complementary set* with respect to a subgroup H if $G = HM$. Therefore for any elements $m \in M$ and $h \in H$ there exists unique elements $h^{'} \in H$ and $m^{'} \in M$ such that $mh = h^{'}m^{'}$. We denote h' and m' by ${}^mh$ and m^h, respectively. For any elements $m_1, m_2 \in M$ there exist unique elements $[m_1, m_2] \in M$ and ${}^{(m_1,m_2)}h \in H$ such that $m_1m_2 = {}^{(m_1,m_2)}h[m_1, m_2]$. Furthermore for every element $a \in M$, there exists $a^{(-1)} \in H$ and $a^{[-1]} \in M$ such that $a^{-1} = a^{(-1)}a^{[-1]}$. Now we are ready to define right ultra-group.

*Department of Pure Mathematics, Hakim Sabzevari University, Sabzevar, Iran, Email: b.tolue@hsu.ac.ir, b.tolue@gmail.com

†Corresponding Author.

‡Department of Pure Mathematics, Hakim Sabzevari University, Sabzevar, Iran, Email: r.moghadasi@hsu.ac.ir

§Department of Pure Mathematics, Farhangian University, Mashhad, Iran, Email: p.z.math2013@gmail.com

1.1. Definition. Let M be a right transversal set of a subgroup H over the group G. The set M together with a binary operation $\alpha : M\times M \to M$ and a family of unary operations $\beta_h : M \to M$ defined by $\alpha((m_1, m_2)) := [m_1, m_2]$ and $\beta_h(m) := m^h$ for all $h \in H$ is called **right ultra-group**.

In the definition of the right transversal, if we replace the right cosets Hg with the left cosets gH, then M is called *left transversal*, for all $g \in G$. Similarly we can define left ultra-group by use of left transversal set. We use the notation ${}_HM$ (M_H) to represent the right (left) ultra-group M of subgroup H. In this text we concentrate on the right ultra-group. If it is necessary, then we denote the ultra-group by the triple $({}_HM,\ \alpha, \beta_h)$ and note that β_h is a monomorphism.
The aim of this paper is to introduce some of the basic notions in the ultra-groups category. In the next section the category of ultra-groups is studied which is denoted by ***Ulg***. Some elementary facts such as initial and terminal objects in this category is obtained. We define the Cartesian and free product of ultra-groups. The monomorphisms and epimorphisms of ***Ulg*** are one to one and onto homomorphisms, respectively. Especially we show that each set-indexed family of objects in an ultra-groups category has (co)product and each pair of parallel morphisms has (co)equalizer. Therefore ***Ulg*** is (co)complete. The ultra-group category has pullback and pushout. Finally, we discuss about the free object in the category of ultra-groups.

We may hope the category of ultra-groups to be used as a strong tool in proving the theorems and unsolved problems for the other categories such as category of groups and quasi-groups. Moreover, we expect one may consider the properties of the ultra groups in justifying the associate theorems for the groups.

All the notations in this paper is standard, we may refer the reader to see [1, 5, 7] for more details.

2. Preliminaries and notations

In this section we give some preliminaries about the ultra-groups of a subgroup over a group.

2.1. Definition. Let ${}_HM$ be an ultra-group of a subgroup H over a group G. A subset $S \subseteq {}_HM$ which contains the identity element of the group G, is called a **subultra-group** of ${}_HM$, if S is closed under the operations α and β_h in the Definition 1.1.

It is obvious that $\{e\}$ is a trivial subultra-group for all the ultra-groups ${}_HM$. Suppose A, B are two subsets of the ultra-group ${}_HM$. We use the notation $[A, B]$ for the set of all $[a, b]$, where $a \in A$ and $b \in B$. If B is a singleton $\{b\}$, then we denote $[A, B]$ by $[A, b]$. Moreover, if A is a subultra-group of the ultra-group ${}_HM$ and $b \in {}_HM$, then the subset $[A, b]$ is called a *right coset* of A in ${}_HM$. In the following we recall some results which are useful in sequel (see [7] for more details). The next lemma is a direct result of the Definition 2.1.

2.2. Lemma. [7, Lemma 2.1] *Let S be a subultra-group of the ultra-group ${}_HM$ over a group G and $a, b \in {}_HM$. Then the following conditions are equivalent.*
(i) $a \in [S, b]$,
(ii) $[S, a] = [S, b]$,
(iii) $[a^{(b^{(-1)})}, b^{[-1]}] \in S$.

By Lemma 2.2 we deduce $[S, a] = [S, b]$ or $[S, a] \cap [S, b] = \emptyset$, which implies

$${}_HM = \bigcup_{a\in {}_HM}[S, a].$$

2.3. Lemma. [7, Lemma 2.4] *Let S be a subultra-group of the ultra-group ${}_HM$ over group G . Then*
(i) $[a^{b^{(-1)}}, b^{[-1]}] \in S$ *if and only if there exist* $s \in S$ *such that* $a = [s, b]$.
(ii) The relation $\boldsymbol{\theta}$ on ${}_HM$ defined by $a\boldsymbol{\theta}b$ if and only if there exists $s \in S$ such that $a = [s, b]$, is an equivalence relation.

From now on we use the notation $\boldsymbol{\theta}$ for the equivalence relation which is satisfied in the second part of Lemma 2.3.

2.4. Definition. [7, Definition 2.8] A subultra-group N of an ultra-group ${}_HM$ over a group G is called **normal** if $[a, [N, b]] = [N, [a, b]]$, for all $a, b \in {}_HM$.

For instance if we denote the equivalence class of e with respect to the equivalence relation of $\boldsymbol{\theta}$ in the second part of Lemma 2.3 by $[e]_{\boldsymbol{\theta}}$, then $[e]_{\boldsymbol{\theta}} = S$ is a normal subultra-group of ${}_HM$. If it is necessary, then we can switch S and $\boldsymbol{\theta}$ on some situations, in sequel.

2.5. Lemma. [7, Lemma 2.5] *Let N be a normal subultra-group of an ultra-group ${}_HM$ over a group G . Then we have the following properties,*
(i) $[a, N] = [N, a]$, *for all* $a \in {}_HM$.
(ii) $[[N, a], [N, b]] = [N, [a, b]]$, *for all* $a, b \in {}_HM$.
(iii) If $[N, b] = N$, *then* $b \in N$.
(iv) $[N, S]$ *is a subultra-group of ${}_HM$, where S is a subultra-group of ${}_HM$. Moreover, $[N, S]$ is a normal subultra-group of ${}_HM$ if S is also normal subultra-group of ${}_HM$.*

2.6. Definition. Suppose ${}_{H_i}M_i$ is an ultra-group of a subgroup H_i over the group G_i, $i = 1, 2$, and φ is a group homomorphism between two subgroups H_1 and H_2. A function $f : {}_{H_1}M_1 \longrightarrow {}_{H_2}M_2$ is called **ultra-group homomorphism** provided that for all $m, m_1, m_2 \in {}_{H_1}M_1$ and $h \in H_1$:

(i) $f([m_1, m_2]) = [f(m_1), f(m_2)]$,
(ii) $(f(m))^{\varphi(h)} = f(m^h)$.

From the other point of view, an ultra-group homomorphism f is a function such that the following two diagrams commute,

$$\begin{array}{ccccccc} {}_{H_1}M_1 \times_{H_1} M_1 & \overset{\alpha_1}{\rightarrow} & {}_{H_1}M_1 & & {}_{H_1}M_1 \times H_1 & \overset{\beta_1}{\rightarrow} & {}_{H_1}M_1 \\ \downarrow f \times f & & \downarrow\ f & & \downarrow\ f \times \varphi & & \downarrow\ f \\ {}_{H_2}M_2 \times_{H_2} M_2 & \overset{\alpha_2}{\rightarrow} & {}_{H_2}M_2 & & {}_{H_2}M_2 \times H_2 & \overset{\beta_2}{\rightarrow} & {}_{H_2}M_2 \end{array}$$

where α_i is the first binary operation of ${}_{H_i}M_i$, β_i inspired by $\{\beta_{h_i} \mid h_i \in H_i\}$, $i = 1, 2$. If f is a surjective and injective ultra-group homomorphism, then it is called isomorphism and denoted by ${}_{H_1}M_1 \cong {}_{H_2}M_2$. In the sequel φ is a group homomorphism between two subgroups of the group for which the ultra-groups are defined. If S is a subultra-group of ${}_{H_1}M_1$ and φ is onto, then $f(S)$ is a subultra-group of ${}_{H_2}M_2$. Moreover, $\mathrm{Ker}(f)$ is a normal subultra-group of ${}_{H_1}M_1$.

2.7. Remark. By considering the same notations as in the Definition 2.6, we conclude the following identities.

(i) $f(a^{[-1]}) = (f(a))^{[-1]}$,
(ii) $f(a^{b^{(-1)}}) = (f(a))^{(f(b))^{(-1)}}$, where $a, b \in {}_{H_1}M_1$.

It is straightforward that for each ultra-group ${}_HM$, $id_{{}_HM} : {}_HM \longrightarrow {}_HM$ is an ultra-group homomorphism and the composite of two ultra-group homomorphisms is an ultra-group homomorphism. Therefore we have the category ***Ulg***, which the objects are ultra-groups (denoted by ${}_HM,\ {}_KN,\cdots$) and morphisms are ultra-group homomorphisms. Indeed, the set of all ultra-group homomorphisms between two ultra-group ${}_HM$, ${}_KN$ is denoted by $Hom({}_HM,\ {}_KN)$.

2.8. Theorem. *Let $f:\ {}_{H'}M' \longrightarrow {}_HM$ be a ultra-group morphism and let $g:\ {}_{H''}M'' \longrightarrow {}_HM$ be an injective ultra-group morphism. There is an ultra-group morphism $h:\ {}_{H'}M' \longrightarrow {}_{H''}M''$ such that $f = g\circ h$ if and only if $f({}_{H'}M') \subseteq g({}_{H''}M'')$ and $\varphi(H') \subseteq \psi(H'')$, where $\varphi : H' \longrightarrow H$ and $\psi : H'' \longrightarrow H$ are epimorphism and homomorphism in groups respectively.*

Proof. Sufficiently put $h = g^{-1}\circ f$. □

For each subgroup H of a group G, the trivial subultra-group $\{e\}$ is initial and terminal object. Since $\mid Hom(\{e\},\ {}_HM)\mid = \mid Hom({}_HM,\ \{e\})\mid = 1$ for each ultra-group ${}_HM$.

2.9. Definition. Let $({}_{H_i}M_i, \alpha_i, \beta_i)$ be ultra-groups over the groups G_i, $i = 1, 2$. Consider the set ${}_{H_1}M_1 \times {}_{H_2}M_2 = \{(m_1, m_2) : m_i \in {}_{H_i}M_i,\ i = 1, 2\}$, such that α, β_h are defined by $\alpha((m_1, m_2), (m_1', m_2')) = (\alpha_1(m_1, m_1'), \alpha_2(m_2, m_2'))$, and $\beta_h(m_1, m_2) = (\beta_1(m_1), \beta_2(m_2))$, where $h = (h_1, h_2) \in H_1 \times H_2$. By an easy computation we can deduce that ${}_{H_1}M_1 \times {}_{H_2}M_2$ is an ultra-group of subgroup $H_1 \times H_2$ over group $G_1 \times G_2$. We call ${}_{H_1}M_1 \times {}_{H_2}M_2$ the **Cartesian product of two ultra-groups** ${}_{H_1}M_1$ and ${}_{H_2}M_2$.

If N_i are subultra-groups of ultra-groups ${}_{H_i}M_i$, then $N_1 \times N_2$ is a subultra-group of ${}_{H_1}M_1 \times {}_{H_2}M_2$, $i = 1, 2$. The proof of the following lemma is clear so we omit it.

2.10. Lemma. *If $({}_{H_i}M_i, \alpha_i, \beta_i)$ are ultra-groups over the groups G_i, then the surjective map $\pi_i :\ {}_{H_1}M_1 \times {}_{H_2}M_2 \longrightarrow {}_{H_i}M_i$ such that $\pi_i(m_1, m_2) = m_i$ is a ultra-group homomorphism, for $m_i \in {}_{H_i}M_i$, $i = 1, 2$.*

We can extend the Cartesian product of two ultra-groups to the family of ultra-groups $\{{}_{H_i}M_i : i \in I\}$ easily. The following theorem show that this Cartesian is product in the category ***Ulg***.

2.11. Theorem. *Let $\{{}_{H_i}M_i : i \in I\}$ be a family of ultra-groups and $\{f_i :\ {}_HM \longrightarrow {}_{H_i}M_i : i \in I\}$ a family of ultra-group homomorphisms, where ${}_HM$ is an arbitrary ultra-group. Then there is a unique homomorphism $f :\ {}_HM \longrightarrow \prod\ {}_{H_i}M_i$ such that $\pi_i f = f_i$, for all $i \in I$. In other words, $\prod\ {}_{H_i}M_i$ is a product in the category of ultra-groups.*

Proof. We define $f : M \longrightarrow \prod\ {}_{H_i}M_i$ by $f(m) = \{f_i(m)\}_{i\in I} \in \prod_{H_i} M_i$ which is a unique function satisfies $\pi_i f = f_i$ for all $i \in I$. It is easy to prove that f is an ultra-group homomorphism. Hence the assertion follows. □

Suppose $\{(M_i, \alpha_i, \beta_i),: i \in I\}$ is a family of ultra-groups such that $M_i \cap M_j = \emptyset$, for all distinct $i, j \in I$. Similar to the reduced word for free product of groups we can define reduced word here (see [6] for more details). The set of all the reduced words is denoted by $\prod^* M_i$. Assume $G = \prod^* G_i$ and $H = \prod^* H_i$ are known free products in the category of groups, $i \in I$. Define the binary operation α by

$$\begin{array}{lcc} \alpha : \prod^* M_i \times \prod^* M_i & \longrightarrow & \prod^* M_i \\ (a, b) & \mapsto & ab = a_1a_2\cdots a_nb_1b_2\cdots b_m. \end{array}$$

and unary β_h as follows

$$\begin{array}{lcc} \beta_h : \prod^* M_i & \longrightarrow & \prod^* M_i \\ a & \mapsto & \prod_{i=1}^n \beta_j(a_i), \end{array}$$

where $a = a_1a_2\cdots a_n$, $b = b_1b_2\cdots b_m$, $a_i \in M_j$ and $h \in H$. Hence, the triple $(\prod^* M_i, \alpha, \beta_h)$ is an ultra-group of the group G with respect to the subgroup H. Note that, if $a_n, b_1 \in M_j$

then $a_n b_1$ means $\alpha_j(a_n, b_1)$. We call $\prod^* M_i$ the *free product of the family of ultra-groups* $\{M_i : i \in I\}$. We can define an injective ultra-group homomorphism $\tau_j : M_j \longrightarrow \prod^*_{i\in I} M_i$ such that $\tau_j(m) = \{m_i\}_{i\in I}$, where $m_i = e$ for $i \neq j$ and $m_j = m$.

2.12. Notation. Although we have associativity property for the groups, but this property is not valid for the binary operation α of the ultra-groups. Therefore, we convent $\alpha(a, b, c) = \alpha(\alpha(a, b), c)$, where a, b, c are the elements of the ultra-group M and α is its first binary operation over it.

2.13. Theorem. *Let $\{{}_{H_i}M_i : i \in I\}$ be a family of ultra-groups and $\{f_i : {}_{H_i}M_i \longrightarrow M : i \in I\}$ a family of ultra-group homomorphisms. Then there is a unique homomorphism $f : \prod^* {}_{H_i}M_i \longrightarrow M$ such that $f\tau_i = f_i$, for all $i \in I$. In other words, $\prod^* {}_{H_i}M_i$ is a coproduct in the category of ultra-groups.*

Proof. Suppose $m_1 m_2 \cdots m_n$ is a reduced word in $\prod^*_{H_i} M_i$, with $m_k \in M_{i_k}$. It is enough to define $f(m_1 m_2 \cdots m_n) = \alpha(f_{i_1}(m_1), f_{i_2}(m_2), \cdots, f_{i_n}(m_n)) \in M$. □

2.14. Lemma. *Let f be a homomorphism between two ultra groups ${}_{H_1}M_1$ and ${}_{H_2}M_2$ with $K = \mathrm{Ker}(f) = \{m \in M_1 \mid f(m) = e_{M_2}\}$. Then $f(m_1) = f(m_2)$ if and only if $m_1 = [k, m_2]$ for some $k \in K$.*

Proof. Suppose $f(m_1) = f(m_2)$, so we have $[(f(m_1))^{(f(m_2))^{(-1)}}, (f(m_2))^{[-1]}] = f([m_1^{m_2^{(-1)}}, m_2^{[-1]}]) = e$. This means that $[m_1^{m_2^{(-1)}}, m_2^{[-1]}] \in K$, thus $m_1 = [k, m_2]$ for some $k \in K$. Conversely $f(m_1) = f([k, m_2]) = [f(k), f(m_2)] = f(m_2)$. □

2.15. Corollary. *Suppose ${}_{H_i}M_i$ are ultra-groups of H_i over the groups G_i, $i = 1, 2$. An ultra-group homomorphism $f : {}_{H_1}M_1 \longrightarrow {}_{H_2}M_2$ is injective if and only if $\mathrm{Ker}(f) = \{e_{M_1}\}$.*

Proof. Assume $\mathrm{Ker}(f) = \{e_{M_1}\}$ and g, h are ultra-group homomorphisms between ultra-groups ${}_HM$ and ${}_{H_1}M_1$ such that $fg(m) = fh(m)$ for all $m \in_H M$. Therefore, by well-definedness of the unary operation β_h, the fact that f is a homomorphism and identities of Remark 2.7, we deduce the following equivalent equalities.

$$(f(g(m)))^{(f(h(m)))^{(-1)}} = (f(h(m)))^{(f(h(m)))^{(-1)}}$$
$$\Longleftrightarrow [(f(g(m)))^{(f(h(m)))^{(-1)}}, (f(h(m)))^{[-1]}] = [(f(h(m)))^{(f(h(m)))^{(-1)}}, (f(h(m)))^{[-1]}]$$
$$\Longleftrightarrow f[(g(m))^{(h(m))^{(-1)}}, h(m)^{[-1]}] = e_{M_2}.$$

Now, the hypothesis $\mathrm{Ker}(f) = \{e_{M_1}\}$ implies that the $[(g(m))^{(h(m))^{(-1)}}, h(m)^{[-1]}] = e_{M_1}$, but also we know $[(h(m))^{h(m)^{(-1)}}, h(m)^{[-1]}] = e_{M_1}$. By uniqueness of left inverse for every element of right ultra-group and since β_h is a monomorphism, we conclude that $g(m) = h(m)$ for all $m \in M_H$. Conversely, suppose that $m \in \mathrm{Ker}(f)$. Put $g(m) = m$ and $h(m) = e$. Therefore $fg(m) = fh(m)$ and $g = h$. □

2.16. Theorem. *Suppose ${}_{H_i}M_i$ are ultra-groups of H_i over the groups G_i, $i = 1, 2$. An ultra-group homomorphism $f : {}_{H_1}M_1 \longrightarrow {}_{H_2}M_2$ is monomorphism if and only if it is an injective ultra-group homomorphism.*

Proof. Let f be a monomorphism. Define the homomorphisms $g(m) = m$ and $h(m) = [K, m]$ for all $m \in {}_{H_1}M_1$, where $K = \{m \in M_1 \mid f(m) = e_{M_2}\}$. Therefore $fg(m) = f(m) = [e_{M_2}, f(m)] = [f(K), f(m)] = f([K, m]) = fh(m)$ for all $m \in {}_{H_1}M_1$. Since f is a monomorphism, thus $m = [K, m]$ and with the right cancelation for the right ultra-groups conclude that $K = \{e_{M_1}\}$. The converse clear. □

2.17. Theorem. *Suppose ${}_{H_i}M_i$ are ultra-groups of H_i over the groups G_i, $i = 1, 2$ and φ is an onto group homomorphism between H_1 and H_2. An ultra-group homomorphism $f : {}_{H_1}M_1 \longrightarrow {}_{H_2}M_2$ is an epimorphism if and only if f is a surjective ultra-group homomorphism.*

Proof. Suppose that $f : {}_{H_1}M_1 \longrightarrow {}_{H_2}M_2$ is an ultra-group epimorphism and $L = Imf$. Clearly L is a subultra-group of ${}_{H_2}M_2$. If f is not a surjective ultra-group homomorphism, then there exists an $a \in {}_{H_2}M_2 - L$. Put $X = \{[L, m] \mid m \in {}_{H_2}M_2\} \cup \{a\}$, where $[L, m]$ the right coset of ultra-group (see [7] for more details). Define the permutation σ on the set X by

$$\sigma(x) = \begin{cases} a, & x = L \\ L, & x = a \\ [L, m], & x = [L, m], \quad m \in {}_{H_2}M_2 - L. \end{cases}$$

Consider two ultra-group homomorphisms $g, h : {}_{H_2}M_2 \longrightarrow S(X)$ by the rules as follows,

$$g(m)(s) = \begin{cases} [[L, m'], m], & \text{if} \quad s = [L, m'] \quad \text{for some} \quad m' \in {}_{H_2}M_2 \\ a, & \text{if} \quad s = a, \end{cases}$$

and $h(m) = \sigma \circ g(m) \circ \sigma^{-1}$, where $S(X)$ is the permutation group on X, $\circ$ denotes the usual composition and σ^{-1} is the inverse permutation of σ. Therefore, $gf = hf$ and by hypothesis $g = h$. But $a \in ({}_{H_2}M_2 - L)$ and $g(a) = h(a)$ which is a contradiction, because easily one can see if $a \notin L$, then $g(a) \neq h(a)$. Thus, we conclude that such an element a in ${}_{H_2}M_2 - L$ does not exist and the assertion is clear. □

A category is balanced if every monic epic morphism is an isomorphism. Monic epics are sometimes called bimorphisms. In the category ***Ulg*** the monomorphisms and epimorphisms are the homomorphisms which are one to one and onto respectively. Thus we have the following corollary.

2.18. Corollary. *The category **Ulg** is a balanced category.*

We continue with the following useful lemma.

2.19. Lemma. *Let $f, g :_{H_1} M_1 \longrightarrow_{H_2} M_2$ be a pair of morphisms in the category **Ulg**. Then $N = \{m \in_{H_1} M_1 \mid f(m) = g(m)\}$ is a subultra-group of ${}_{H_1}M_1$.*

Proof. Obviously $e \in N$, and $f([m_1, m_2]) = [f(m_1), f(m_2)] = [g(m_1), g(m_2)] = g([m_1, m_2])$ for all $m_1, m_2 \in N$. Moreover, we have $f(m^h) = f(m)^{\varphi(h)} = g(m)^{\varphi(h)} = g(m^h)$ for all $m \in {}_{H_1}M_1$ and $h \in H$ where φ is a group homomorphism between two subgroups H_1 and H_2. □

The equalizer of two arbitrary ultra-group morphisms $f, g : {}_HM \longrightarrow {}_{H'}M^{'}$ is the subultra-group $K = \{m \in {}_HM : f(m) = g(m)\}$ together with the inclusion ultra-group morphism $i : K \longrightarrow {}_HM$. Since by Lemma 2.19 the subset K of M is a subultra-group of ${}_HM$, and $fi = gi$. Moreover, for any other ultra-group $K^{'}$, let $j : K^{'} \longrightarrow {}_HM$ be an ultra-group morphism such that $fj = gj$. Thus $j(K^{'})$ is a subultra-group of K by argument [7, Proposition 2.10]. We can define an ultra-group morphism $h : K^{'} \longrightarrow K$ by $h(k^{'}) = j(k^{'})$. Hence j satisfies the equality $j = ih$ and the uniqueness of i follows by in [1, Proposition 2.4.3].

In particular, the equalizers of the pair (f, f) always exists and is just the identity on ${}_HM$. One can construct an example of equalizer in ***Ulg***. For instance, every normal subultra-group is an equalizer. Let N be a normal subultra-group of ultra-group ${}_HM$. The equivalence relation $\boldsymbol{\theta}$ on ${}_HM$ which is defined by

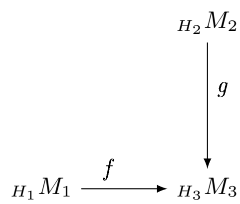
$$a\ \boldsymbol{\theta}\ b \Longleftrightarrow [a^{b^{-1}}, b^{[-1]}] \in N,$$

is a congruence over ultra-group ${}_HM$. Then N can be considered as the equalizer of two ultra-group homomorphisms f_1, f_2 : ${}_HM \longrightarrow {}_HM/\boldsymbol{\theta}$ which is defined by $f_1(m) = [e]_{\boldsymbol{\theta}}$ and $f_2(m) = [m]_{\boldsymbol{\theta}}$ for all $m \in {}_HM$.

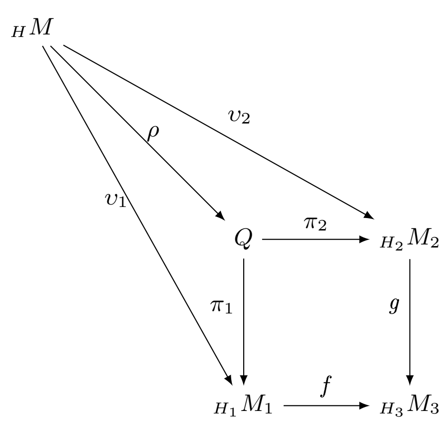
In abstract algebra a complete category is a category in which all small limits exists. It follows from the existence theorem for limits that a category is complete if and only if it has pullbacks and products. In the following we are ready to prove one of our main results.

2.20. Theorem. *The category of* ***Ulg*** *is complete.*

Proof. By the above argument it is enough to present products and pullbacks for ***Ulg***. Clearly by Theorem 2.11, ***Ulg*** has product. Consider the following diagram,



for the arbitrary ultra-groups ${}_{H_1}M_1$, ${}_{H_2}M_2$, ${}_{H_3}M_3$ and the ultra-group homomorphisms f and g. Consider $Q = \{(m_1, m_2) \in {}_{H_1}M_1 \times {}_{H_2}M_2 : f(m_1) = g(m_2)\}$. Similar to the proof of Lemma 2.19, Q is a subultra-group of ${}_{H_1}M_1 \times {}_{H_2}M_2$. Now assume π_1 : $Q \longrightarrow {}_{H_1}M_1$ and π_2 : $Q \longrightarrow {}_{H_2}M_2$ such that $\pi_i(m_1, m_2) = m_i$ for $i = 1, 2$. We show that the triple $(Q, (\pi_1, \pi_2))$ is a pullback of f and g. At first we have $f\pi_1 = g\pi_2$ by the above construction. Let ν_i : ${}_HM \longrightarrow {}_{H_i}M_i$ be ultra-group homorphisms such that $f\nu_1 = g\nu_2$ for an arbitrary ultra-group ${}_HM$, $i = 1, 2$. Define ρ : ${}_HM \longrightarrow Q$ by $\rho(m) = (\upsilon_1(m), \upsilon_2(m))$. Obviously $\pi_i\rho = \upsilon_i$ for $i = 1, 2$. For uniqueness of ρ, let ξ :${}_HM \longrightarrow Q$ be another ultra-group morphism with the properties $\xi_i\rho = \upsilon_i$ for $i = 1, 2$. By the definition of ξ, and the equality $\pi_i\rho = \upsilon_i$, we have the result. Therefore we have the following diagram



□

Although being cocomplete is equivalent to completeness of the category, because of some interesting preliminaries which occur in the proving process, we are going to demonstrate that the category of ***Ulg*** is cocomplete. In the sequel we obtain the smallest congruence relation generated by a set.

2.21. Lemma. *Let* $f_i : {}_HM \longrightarrow {}_{H'}M'$ *be two arbitrary ultra-group morphisms,* $i = 1, 2$ *and* $S = \{(f_1(m), f_2(m)) \mid m \in {}_HM\}$*. Then* $S^* = \bigcup_{i=0}^{\infty} S_i$ *is the smallest congruence relation contain* S*, where*

$$S_0 = S \cup \{(m', m') \mid m' \in f_1({}_HM) \ \ or \ \ m' \in f_2({}_HM)\} \cup S^{-1}$$

$$S_1 = S_0 \circ S_0 = \{(f_1(m), f_2(m')) \mid f_1(m') = f_2(m), m, m' \in_H M\},$$

$$S_n = (S_{n-1} \circ S_{n-1}) \cup \{(\vartheta, \mu)\} \qquad \forall n \geq 2$$

where

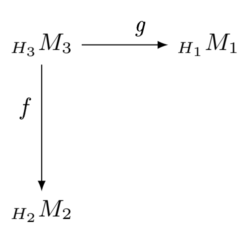
$$(\vartheta, \mu) = ([[\cdots[[f_1(m_1), f_1(m_2)], f_1(m_3)], \cdots], f_1(m_n)], [[\cdots[[f_2(m_1), f_2(m_2)], f_2(m_3)], \cdots], f_2(m_n)]), \quad m_i \in_H M \ \ for \ \ i = 1, 2, \cdots, n.$$

Proof. It is not hard to deduce that S^* is the equivalence relation of ${}_{H'}M' \times {}_{H'}M'$ containing S. By construction of S^* and some basic properties of subultra-group, S^* satisfies the congruence property and it is a congruence. Moreover, let L be another congruence contains S. It is not difficult to obtain that $S^* \subseteq L$. Therefore S^* is the smallest congruence contains S. □

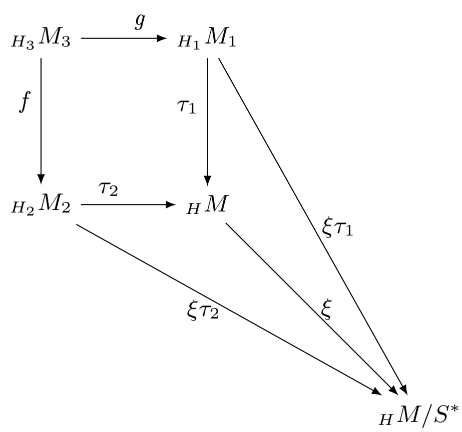
2.22. Theorem. *The category* ***Ulg*** *is cocomplete.*

Proof. Since in Theorem 2.13 we present the coproduct in the category ***Ulg***, it is enough to prove that ***Ulg*** has coequalizers.
Suppose $f, g : {}_HM \longrightarrow {}_{H'}M'$ are two ultra-group homomorphisms. Definition 2.3 implies that $N = \{[(f(m))^{(g(m))^{(-1)}}, g(m)^{[-1]}] \mid m \in {}_HM\}$ is a subultra-group of ${}_{H'}M'$. Consider S which is defined by $(f(m), g(m)) \in S$ if and only if $[f(m)^{g(m)^{-1}}, g(m)^{[-1]}] \in N$. Moreover, note that $[f(m)^{g(m)^{-1}}, g(m)^{[-1]}] \in N$ is equivalent to $[f(m), N] = [g(m), N]$ by Lemma 2.2. Further, let S^* be the smallest equivalence relation on ${}_HM$ such that $S \subseteq S^*$ (see Lemma 2.21). Also suppose ${}_{H'}M'/S^*$ is the factor ultra-group together with the natural projection ultra-group morphism $h : {}_{H'}M' \longrightarrow {}_{H'}M'/S^*$, which is known to be surjective. We just verify that the coequalizer (f, g) is $({}_{H'}M'/S^*, h)$. Due to the definition of S, $(hf)(m) = [f(m), N] = [g(m), N] = (hg)(m)$ for every $m \in {}_HM$. Let $i : {}_{H'}M' \longrightarrow {}_{H''}M''$ be another ultra-group morphism such that $if = ig$. Since $if(m) = ig(m)$ we deduce $Kerh = N \subseteq Keri$. Now according to Theorem 2.8, there exists $j : {}_{H'}M'/N \longrightarrow {}_{H''}M''$ such that $i = jh$. □

However, we know the dual of pullbacks is pushouts, in order to emphasis on the importance of this notion, we intend to discuss about it in the end briefly. Suppose the following diagram.



By Theorem 2.13 we know the coproduct of ${}_{H_1}M_1$ and ${}_{H_2}M_2$ is the free product ${}_HM$ $={}_{H_1}M_1 * {}_{H_2}M_2$. Now, it is enough to construct the coequalizer of $({}_HM, (\tau_1 g, \tau_2 f))$. Theorem 2.22 implies that the coequlizer is ${}_HM/S^*$, where S^* was introduced in Theorem 2.22. Hence we can complete the above diagram as the following.

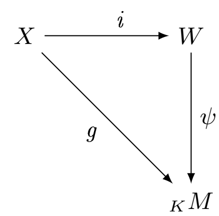


3. Free ultra-group

Let F be a free group on the non-empty set X (see [3] for more details). The Nielsen-Schreier theorem states that every subgroup of a free group is itself a free group. Choose H one of the subgroups of F. Constructing all the ultra-groups of a subgroup over a group has been vastly discussed in [7]. Suppose $(W(X), \alpha, \beta_h)$ is the ultra-group of the subgroup H over the free group F, where α and β_h are binary and unary operations, for all $h \in H$. Let $w_1, w_2 \in W(X)$. Since $W(X) \subseteq F$ elements of $W(X)$ are all reduced words. The binary operation on the free group F is just juxtaposition of two reduced words. Therefore, since $w_1w_2 \in F$ and $F = HW(X)$ we deduce $w_1w_2 = {}^{(w_1,w_2)}h[w_1, w_2]$, where ${}^{(w_1,w_2)}h \in H$ and $[w_1, w_2] \in W(X)$. It is not hard to see that $\alpha(w_1, w_2) = [w_1, w_2]$ by an ultra-group definition. Furthermore, since $W(X)H \subseteq F = HW(X)$ we have $wh = {}^whw^h$. Thus $\beta_h(w) = w^h$, for $w \in W(X)$ and all $h \in H$. We call $W(X)$ the free ultra-group on the non-empty set $Y \subseteq X$, where Y is the set of all one letter word such that the words of $W(X)$ is obtained. In the rest of the article without making causing any problems with the overall content, we consider the free ultra-group W on the non-empty set X.

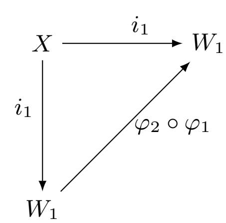
3.1. Theorem. *The ultra-group W which is described in the above argument is a free object in **Ulg**.*

Proof. It is enough to show that ultra-group W satisfies the universal property. Let $i : X \longrightarrow W$ be the inclusion map, ${}_KM$ be any ultra-group of a subgroup K over the group G and $g : X \longrightarrow {}_KM$ a function. We claim that there exists a unique ultra-group homomorphism $\psi : W \longrightarrow {}_KM$ such that the following diagram commutes.



Since F is a free group on the set X, for the group G and the functions $g' : X \longrightarrow G$ and $i' : X \longrightarrow F$ such that $g' \mid_X = g$ and $i' \mid_X = i$, there exists a unique group homomorphism $\varphi : F \longrightarrow G$ such that $\varphi \circ i' = g'$. Note that, since $F = HW$ we have $\varphi(w) = \varphi(h_1h_2 \ldots h_n w_1 w_2 \ldots w_m) = \varphi(h_1h_2 \ldots h_n)\varphi(w_1w_2 \ldots w_m)$ for $w \in F$, $h_i \in H$, $1 \leq i \leq n$ and $w_j \in W$, $1 \leq j \leq m$. Moreover, $\varphi(h_1h_2 \ldots h_n) \in \varphi(H) \leq G$. Hence it is enough to define the ultra-group homomorphism $\psi(w_1w_2 \ldots w_m) = \varphi(w_1w_2 \ldots w_m) = \varphi(w_1)\varphi(w_2) \ldots \varphi(w_m)$. □

Finally, the free ultra-group on X is unique. In other words, if two free ultra-groups on X are given, then there exists an ultra-groups isomorphism between them. Suppose W_1 and W_2 are free ultra-groups on X and $i_j : X \longrightarrow W_j$, $j = 1, 2$ are inclusion maps. If we consider W_1 as a free ultra-group on X and W_2 as an arbitrary ultra-group, then by Theorem 3.1 which demonstrate the universal property of free object we deduce that the ultra-group homomorphism $\varphi_1 : W_1 \longrightarrow W_2$ such that $\varphi_1 \circ i_1 = i_2$. By changing the role of W_1 and W_2, there exists a unique ultra-group homomorphism $\varphi_2 : W_2 \longrightarrow W_1$ such that $\varphi_2 \circ i_2 = i_1$. Now by substituting i_2 from the first equation we obtain $\varphi_2 \circ \varphi_1 \circ i_1 = i_1$. Thus the following diagram commutes.



Since W_1 is a free ultra-group on X, the universal property implies that ultra-group homomorphism $\varphi_2 \circ \varphi_1$ is unique, so $\varphi_2 \circ \varphi_1 = id_{W_1}$. Similarly, $\varphi_1 \circ \varphi_2 = id_{W_2}$. Thus W_1 and W_2 are isomorphic. Hence we have the following result.

3.2. Corollary. *The free ultra-group on a set is unique up to isomorphism.*

Now, we are ready to describe the free functor for the ultra-groups, and subsequently conclude adjoint being of free and forgetful functor. Recall that the forgetful functor $\mathcal{U}$ is a functor from ***Ulg*** to the category Set which maps each ultra-group $_HM$ to its underlying set and each ultra-group homomorphism to the corresponding set function.

3.3. Remark. The free functor from the category of sets to ***Ulg*** is denoted by $\mathcal{F}$, which maps each set X to the free ultra-group W that was discussed previously. Moreover, it maps every function to an ultra-group homomorphism. Suppose $f : X_1 \to X_2$ is a function in the category of sets, and consider the inclusion maps $i_1 : X_1 \longrightarrow W_1$, $i_2 : X_2 \longrightarrow W_2$. Clearly, $i_2 \circ f : X_1 \to W_2$. By the universal property of a free object W_1, there exists a unique ultra-group homomorphism $\varphi : W_1 \longrightarrow W_2$ such that $\varphi \circ i_1 = i_2 \circ f$. Hence $\mathcal{F}(f) = \varphi$.

We complete this paper by the following corollary which is a direct result of the above remark.

3.4. Corollary. *The (free) functor* $\mathcal{F} : Set \to Ulg$ *given by* $\mathcal{F}(X) = W$ *is a left adjoint to the forgetful functor* $\mathcal{U} : Ulg \to Set$.

Acknowledgements: The authors would like to thank the referee for her/his helpful comments.

References

[1] Borceux F., *Hand book Of Categorical Algebra 1 Basic category Theory*, Cambridge university, 1994.
[2] Burris S., Sankapanavar H.P., *A course in universal algebra*, Springer, 1981.
[3] Hungerford, T. W. *Algebra*, Springer, New York, 1974.
[4] Kilp M., Knauer U. and Mikhalev A. *Monoids, acts and categories*, Walter de Gruyter, Berlin, New York 2000.
[5] Knighten R.L., *Notes on Category Theory*, November 9, 2007.
[6] Kurosh A. G., *The theory of groups*, American Mathematical Society, 1960.
[7] Moghaddasi Gh., Tolue B. and Zolfaghari P., *On the structure of the ultra-groups over a finite group*, U.P.B. Sci. Bull. **2**, 173-184, 2016.
[8] Rotman J., *An introduction to the theory of groups*, Fourth edition. Graduate Texts in Mathematics 148, Springer-Verlag, New York, 1995.
[9] Turi D. *Category Theory Lecture Notes Laboratory for Foundations of Computer Science* University of Edinburgh September 1996.