COMMON FIXED POINT THEOREMS IN b-MENGER SPACES

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ABSTRACT. In this work, we prove a common fixed point theorem in b-Menger spaces for nonlinear contractions. An example is provided to illustrate this result.

1. INTRODUCTION

Probabilistic metric spaces are introduced in 1942 by Menger [7]. In such spaces, the notion of distance between two points p and q is replaced by a distribution function $F_{pq}(x)$. Thus one thinks of the distance between points as being probabilistic with $F_{pq}(x)$ representing the probability that the distance between p and q is less than x.

Fixed point theory plays one of the important roles in nonlinear analysis. It has been applied in physical sciences, computing sciences and engineering. The first result from the fixed point theory in probabilistic metric spaces was obtained by Sehgal and Bharucha-Reid [9] in 1972 and their fixed point theorem is further generalized by many authors, for example see [4, 6, 8].

In 1989, Bakhtin [1] defined b-metric spaces as a generalization of metric spaces, and an extension of Banach's contraction [2] in these spaces was showed by Czerwik [3].

Recently, Mbarki et al [6] introduced a probabilistic b-metric spaces (b-Menger spaces) as a generalization of probabilistic metric spaces (Menger spaces) and they studied topological structures and properties and showed the fixed point property for nonlinear contractions in these spaces.

In this paper, we prove the existence and uniqueness of the common fixed point for nonlinear contraction in b-Menger spaces.

2. PRELIMINARIES

Throughout this paper, the standard notations and terminologies of probabilistic b-metric theory are used. For more details, we refer the reader to [6].

A nonnegative real function f defined on $\mathbb{R}^+ \cup \{\infty\}$ is called a distance distribution function (briefly, a d.d.f) if it is nondecreasing, left continuous on $(0, \infty)$, with

$f(0) = 0$ and $f(\infty) = 1$. The set of all d.d.f's will be noted by Δ^+; and the set of all F in Δ^+ for which $\lim\limits_{t \to \infty} f(t) = 1$ by D^+.

A simple example of distribution function is Heavyside function in D^+

$$H(t) = \begin{cases} 0 & \text{if} \quad t \leq 0, \\ 1 & \text{if} \quad t > 0. \end{cases}$$

A commutative, associative and nondecreasing mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if and only if

(1) $T(a,1) = a$, for all $a \in [0,1]$,
(2) $T(0,0) = 0$.

As examples we mention the three typical examples of continuous t-norms as follows:
$T_p(a,b) = ab$, $T_M(a,b) = Min(a,b)$ and $T_L(a,b) = Max\{a + b - 1, 0\}$.

Definition 2.1. [6] A b-Menger space is a quadruple (X, F, T, s) where X is a nonempty set, F is a function from $X \times X$ into Δ^+, T is a t-norm, $s \geq 1$ is a real number, and the following conditions are satisfied: for all $p, q, r \in X$ and $x, y > 0$,

(1) $F_{pp} = H$,
(2) $F_{pq} = H \Rightarrow p = q$,
(3) $F_{pq} = F_{qp}$,
(4) $F_{pq}(s(x + y)) \geq T(F_{pr}(x), F_{rq}(y))$.

Definition 2.2. Let (X, F) be a probabilistic semimetric space (i.e., (1), (2) and (3) of Definition 2.1 are satisfied). For p in X and $t > 0$, the strong t-neighborhood of p is the set

$$N_p(t) = \{q \in M : \ F_{pq}(t) > 1 - t\}.$$

The strong neighborhood system at p is the collection $\wp_p = \{N_p(t) : \ t > 0\}$, and the strong neighborhood system for X is the union $\wp = \bigcup_{p \in X} \wp_p$.

Let (X, F, T, s) be a b-Menger space. The family $\Im$ consisting of $\emptyset$ and all unions of elements of the strong neighborhood system for X determines a topology for X. Moreover, Mbarki et al [6] proved that if T is continuous, (X, F, T, s) endowed with this topology is a Hausdorff space and the function F is in general not continuous.

Example 2.3. Let $X = \mathbb{R}$. Define $F : X \times X \to \Delta^+$ by

$$F_{pq}(t) = H(t - |p - q|^2).$$

By Lemma 3.1 in [6], $(X, F, T_M, 2)$ is a b-Menger space with continuous probabilistic b-metric F.

Example 2.4. [6] Let $X = \mathbb{N} \cup \{\infty\}$. Define $F : X \times X \to \Delta^+$ as follow:

$$F_{pq}(t) = \begin{cases} H(t) & \text{if } p = q, \\ H(t - 7) & \text{if } p \text{ and } q \text{ are odd and } p \neq q, \\ H(t - |\frac{1}{p} - \frac{1}{q}|) & \text{if } p \text{ and } q \text{ are even or } pq = \infty, \\ H(t - 3) & \text{otherwise.} \end{cases}$$

It easy to show that $(X, F, T_M, 4)$ is a b-Menger space with T_M is continuous. Consider the sequence $x_n = 2n$, $n \in \mathbb{N}$. Then $F_{2n\infty}(t) = H(t - \frac{1}{2n})$. Therefore $x_n \to \infty$, but $F_{2n1}(t) = H(t - 3) \neq H(t - 1) = F_{1\infty}(t)$. Hence F is not continuous at ∞.

In b-Menger space, the convergence of sequence is defined as follows

Definition 2.5. Let $\{p_n\}$ be a sequence in a b-Menger space (X, F, T, s).

(1) A sequence $\{p_n\}$ in X is said to be convergent to p in M if, for every $\epsilon > 0$ and $\delta \in (0, 1)$, there exists a positive integer $N(\epsilon, \delta)$ such that $F_{p_n p}(\epsilon) > 1 - \delta$, whenever $n \geq N(\epsilon)$.

(2) A sequence $\{p_n\}$ in X is called a Cauchy sequence if, for every $\epsilon > 0$ and $\delta \in (0, 1)$, there exists a positive integer $N(\epsilon, \delta)$ such that $F_{p_n p_m}(\epsilon) > 1 - \delta$, whenever $n, m \geq N(\varepsilon, \delta)$.

(3) (X, F, T, s) is said to be complete if every Cauchy sequence has a limit.

The letter Ψ denotes the set of all function $\varphi : [0, \infty) \to [0, \infty)$ such that

$$0 < \varphi(t) < t \ \ and \ \ lim_{n \to \infty} \varphi^n(t) = 0 \ \ for \ \ each \ \ t > 0.$$

Definition 2.6. [5] We say that a t-norm T is of H-type if the family $\{T^n(t)\}$ is equicontinuous at $t = 1$, that is,

$\forall \epsilon \in (0, 1), \exists \lambda \in (0, 1) : t > 1 - \lambda \Rightarrow T^n(t) > 1 - \epsilon, \ \ \forall n \geq 1,$
Where $T^1(x) = T(x, x)$, $T^n(x) = T(x, T^{n-1}(x))$, for every $n \geq 2$.

The t-norm T_M is a trivial example of t-norm of H-type.

Lemma 2.7. [6] *Let (X, F, T, s) be a complete b-Menger space under a t-norm T of H-type and $RanF \subset D^+$. Let $\{x_n\}$ be a sequence in X. If there exists a function $\varphi \in \Psi$ such that*

$$F_{x_{m+1} x_{n+1}}(\varphi(t)) \geq F_{x_m x_n}(st) \quad (n, m \geq 0, t > 0),$$

then $\{x_n\}$ is a Cauchy sequence.

3. Main results

Before stating the main common fixed point theorem, we need the following concept.

Definition 3.1. Let f and g be two selmaps on a complete b-Menger space (X, F, T, s). f and g are said to be compatible if, whenever $\{x_n\}$ is a sequence of point in X such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = x$ then $fx = gx$.

Theorem 3.2. *Let (X, F, T, s) be a complete b-Menger space under a continuous t-norm T of H-type and $RanF \subset D^+$, and let $f, g : X \to X$ be maps that satisfy the following conditions:*

(1) *$f(X) \subseteq g(X)$,*

(2) *f and g are compatible, g is continuous,*

(3) *There exists $\varphi \in \Psi$ such that $F_{fxfy}(\varphi(t)) \geq F_{gxgy}(st)$, for all $x, y \in X$ and $t > 0$.*

Then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$. By condition (1), we can find x_1 such that $g(x_1) = f(x_0)$. By induction, we can define a sequence $\{x_n\}$ such that $g(x_n) = f(x_{n-1})$. We put $y_n = g(x_n) = f(x_{n-1})$. By condition (3), we have

$$F_{f(x_m)f(x_n)}(\varphi(t)) \geq F_{g(x_m)g(x_n)}(st).$$

So,
$$F_{y_{m+1}y_{n+1}}(\varphi(t)) \geq F_{y_m y_n}(st).$$
Then the sequence $\{y_n\}$ satisfies the Lemma 2.7, hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there is some $y \in X$ such that

$$y_n \to y \quad as\ n \to \infty. \tag{3.1}$$

Now we will show that y is a common fixed point of f and g. Let $\varepsilon > 0$ and $\delta \in (0,1)$, since $\varphi(\varepsilon) < \varepsilon$, by the monotonicity of F and the condition (3), we get

$$\begin{aligned}
F_{y_{n+1}f(y)}(\varepsilon) &\geq& F_{y_{n+1}f(y)}(\varphi(\varepsilon)) \\
&=& F_{f(y_n)f(y)}(\varphi(\varepsilon)) \\
&\geq& F_{g(y_n)g(y)}(s\varepsilon).
\end{aligned}$$

By condition (3) and since g is continuous, then f is also continuous. Then $\{y_n\}$ converges to y implies that $\{g(y_n)\}$ converges to $g(y)$, then there exists $n_0 \in \mathbb{N}$ such that
$$F_{g(y_n)g(y)}(s\varepsilon) > 1 - \delta \quad for\ each\ \ n \geq n_0.$$
So,
$$F_{y_{n+1}f(y)}(\varepsilon) > 1 - \delta \quad for\ each\ \ n \geq n_0.$$
Then
$$\lim_{n \to \infty} y_{n+1} = f(y). \tag{3.2}$$
Since X is a Hausdorff space, then
$$f(y) = y.$$
So by (3.1) and (3.2) we have
$$y_{n+1} = f(x_n) \to y \ \ and\ \ g(x_n) \to y.$$
And since f and g are compatible, then
$$f(y) = g(y).$$
Finally,
$$y = f(y) = g(y).$$
Then y is a common fixed point for f and g.

Suppose, to the contrary, that there exists another common fixed point z in X of f and g. Then let $t > 0$, from condition (3), since $\varphi(t) < t$ and the fact that the distance distribution functions are nondecreasing, we have

$$\begin{aligned}
F_{yz}(\varphi(t)) &=& F_{f(y)f(z)}(\varphi(t)) \\
&\geq& F_{g(y)g(z)}(st) \\
&\geq& F_{g(y)g(z)}(t) \\
&=& F_{yz}(t) \\
&\geq& F_{yz}(\varphi(t)).
\end{aligned}$$

Thus
$$F_{yz}(\varphi(t)) = F_{yz}(t).$$
By the same arguments, we get
$$F_{yz}(\varphi^n(t)) = F_{yz}(t), \quad for\ all\ \ n \geq 1.$$

Suppose that there exists $\alpha > 0$ such that $F_{yz}(\alpha) < 1$. Since $F_{yz} \in D^+$, then $\lim_{t \to \infty} F_{yz}(t) = 1$, so there exists $\beta > \alpha$ such that

$$F_{yz}(\beta) > F_{yz}(\alpha).$$

There exists a positive integer $n > 1$ such that $\varphi^n(\beta) < \alpha$ because $\lim_{n \to \infty} \varphi^n(\beta) = 0$. Then, the fact that F_{yz} is nondecreasing function, we get

$$F_{yz}(\varphi^n(\beta)) \leq F_{yz}(\alpha).$$

Hence

$$F_{yz}(\beta) = F_{yz}(\varphi^n(\beta)) \leq F_{yz}(\alpha),$$

a contradiction. Therefore $F_{yz}(t) = 1$ for all $t > 0$, since $F_{yz} \in D^+$. Hence $F_{yz} = H$.
Then $y = z$. □

Example 3.3. Let $X = [0, \infty)$. Define $F : X \times X \to \Delta^+$ as follows

$$F_{xy}(t) = H(t - |x - y|^2).$$

By Examples 3.2, 3.3 and Lemma 3.1 in [6], $(X, F, T_M, 2)$ is a complete b-Menger and is not a standard Menger space.
Define $f(x) = \frac{x}{12}$ and $g(x) = \frac{x}{4}$ in X. It is evident that $f(X) \subseteq g(X)$. Also, for $\varphi(t) = \frac{16t}{77}$,

$$\begin{aligned}
F_{f(x)f(y)}(\frac{16t}{77}) &= H(\frac{16t}{77} - |\frac{x}{12} - \frac{y}{12}|^2) \\
&= H(\frac{\frac{16t}{77}}{\frac{1}{144}} - |x - y|^2) \\
&= H(\frac{2t}{\frac{1}{16}} - |x - y|^2) \\
&= H(2t - |\frac{x}{4} - \frac{y}{4}|^2) \\
&= F_{g(x)g(y)}(2t).
\end{aligned}$$

Thus all the conditions of Theorem 3.2 are satisfied and f and g have the common fixed point 0.

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