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EXISTENCE RESULTS IN THE THEORY OF HYBRID FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. We study in this paper, the existence results for initial value problems for hybrid fractional integro-differential equations. By using fixed point theorems for the sum of three operators are used for proving the main results.An example is also given to demonstrate the applications of our main results.

1. Introduction

Fractional differential equations arise in the mathematical modeling of systems and processes occurring in many engineering and scientific disciplines such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, etc. (see [1]-[4]). Compared with integer order models, the fractional order models describe the underlying processes in a more effective manner by taking into account their past history. This has led to a great interest and considerable attention in the ubject of fractional order differential equations.
For some recent developments on the topic, (see [13]-[16]), and the references therein. Hybrid fractional differential equations have also been studied by several researchers.
This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers [17]-[24].
Hybrid fractional differential equations have also been studied by several researchers. This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. In [27], Surang. Sitho, Sotiris .K. Ntouyas, and Jessada. Tariboon, discussed the following existence results for hybrid fractional integro-differential equations

$$\begin{cases} D^{\alpha}\left(\frac{x(t)-\sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t))}{f(t,x(t))}\right) = g(t,x(t)) \quad t \in J=[0,T], \quad 0<\alpha\leq 1 \\ x(0)=0 \end{cases}$$

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where D^α denotes the Riemann-Liouville fractional derivative of order α, $0 < \alpha \leq 1$, I^ϕ is the Riemann-Liouville fractional integral of order $\phi > 0$, $\phi \in \{\beta_1, \beta_2, \dots, \beta_m\}$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in C(J \times \mathbb{R}, \mathbb{R})$, with $h_i \in C(J \times \mathbb{R}, \mathbb{R})$ and $h_i(0,0) = 0$, $i = 1, 2, \dots, m$.
In [5], K .Hilal and A. Kajouni considered boundary value problems for hybrid differential equations with fractional order (BVPHDEF of short) involving Caputo differential operators of order $0 < \alpha < 1$,

$$\begin{cases} D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right) = g(t, x(t)) \quad a.e. \quad t \in J = [0,T] \\ \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ and a, b, c are real constants with $a + b \neq 0$.
Dhage and Lakshmikantham [23], discussed the following first order hybrid differential equation

$$\begin{cases} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t, x(t)) \quad t \in J = [0,T] \\ \\ x(t_0) = x_0 \in \mathbb{R} \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$. They established the existence, uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and a comparison results.
Zhao, Sun, Han and Li [28], are discussed the following fractional hybrid differential equations involving Riemann-Liouville differential operators

$$\begin{cases} D^q\left[\frac{x(t)}{f(t,x(t))}\right] = g(t, x(t)) \quad t \in J = [0,T] \\ \\ x(0) = 0 \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$. They established the existence theorem for fractional hybrid differential equation, some fundamental differential inequalities are also established and the existence of extremal solutions.
Benchohra and al.[26], we study the following boundary value problems for differential equations with fractional order

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t)), \quad for \quad each \quad t \in J = [0,T], \quad 0 < \alpha < 1 \\ ay(0) + by(T) = c \end{cases}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, is a continuous function, a, b, c are real constants with $a + b \neq 0$.
Motivated by some recent studies on hybrid fractional integro-differential equations

see [5],[27], we consider the following value problem :

$$(1.1)\quad \begin{cases} D^{\alpha}\left(\frac{x(t)-I^{\beta}h(t,x(t),I^{\alpha_1}x(t),\ldots,I^{\alpha_n}x(t))}{f(t,x(t),I^{\alpha_1}x(t),\ldots,I^{\alpha_n}x(t))}\right) = g(t,x(t),I^{\beta_1}x(t),\ldots,I^{\beta_k}x(t)) \\ t\in J=[0,T], \quad 1<\alpha\le 2 \\ \frac{x(0)}{f(0,x(0),\underbrace{0,0\ldots,0}_{n})} = x_0, \qquad \frac{x(T)}{f(T,x(T),I^{\alpha_1}x(T),\ldots,I^{\alpha_n}x(T))} = x_T, \end{cases}$$

Where $\alpha_1,\ldots,\alpha_n>0$, $\beta_1,\ldots,\beta_n>0$, $x\in\mathbb{R}$, D^{α} denotes Caputo fractional derivative of order α. I^{β} is the Riemann-Liouville fractional integral of order $\beta>0$. $f: J\times\mathbb{R}^n \longrightarrow \mathbb{R}\setminus\{0\}$, $h: J\times\mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous with $h(0,x(0),\underbrace{0,0\ldots,0}_{n})=0$, and $g\in\mathcal{C}(J\times\mathbb{R}^k,\mathbb{R})$ is a function via some properties.

The problem 1.1 considered here is general in the sense that it includes the following three well-known classes of initial value problems of fractional differential equations.

Case I: Let $f(t,x(t),I^{\alpha_1}x(t),\ldots,I^{\alpha_n}x(t))=1$ and $h(t,x(t),I^{\alpha_1}x(t),\ldots,I^{\alpha_n}x(t))=0$, for all $t\in J$ and $x\in\mathbb{R}$. Then the problem 1.1 reduces to standard initial value problem of fractional differential equation,

$$\begin{cases} D^{\alpha}(x(t)) = g(t,x(t),I^{\beta_1}x(t),\ldots,I^{\beta_k}x(t)) \quad t\in J=[0,T], \quad 1<\alpha\le 2 \\ x(0)=x_0, \qquad x(T)=x_T, \end{cases}$$

Case II: If $h(t,x(t),I^{\alpha_1}x(t),\ldots,I^{\alpha_n}x(t))=0$ for all $t\in J$ and $x\in\mathbb{R}$ in 1.1. We obtain the following quadratic fractional differential equation,

$$\begin{cases} D^{\alpha}\left(\frac{x(t)}{f(t,x(t),I^{\alpha_1}x(t),\ldots,I^{\alpha_n}x(t))}\right) = g(t,x(t),I^{\beta_1}x(t),\ldots,I^{\beta_k}x(t)) \quad t\in J=[0,T], \quad 1<\alpha\le 2 \\ \frac{x(0)}{f(0,x(0),\underbrace{0,0\ldots,0}_{n})} = x_0, \qquad \frac{x(T)}{f(T,x(T),I^{\alpha_1}x(T),\ldots,I^{\alpha_n}x(T))} = x_T, \end{cases}$$

Case III: If $f(t,x(t),I^{\alpha_1}x(t),\ldots,I^{\alpha_n}x(t))=1$ for all $t\in J$ and $x\in\mathbb{R}$ in 1.1.We obtain the following interesting fractional differential equation,

$$\begin{cases} D^{\alpha}\Big(x(t)-I^{\beta}h(t,x(t),I^{\alpha_1}x(t),\ldots,I^{\alpha_n}x(t))\Big) = g(t,x(t),I^{\beta_1}x(t),\ldots,I^{\beta_k}x(t)) \quad t\in J=[0,T], \\ 1<\alpha\le 2 \\ x(0)=x_0, \qquad x(T)=x_T, \end{cases}$$

Therefore, the main result of this paper also includes the existence the results for the solutions of above mentioned initial value problems of fractional differential equations as special cases.

An existence result is obtained for the initial value problem 1.1. by using a hybrid fixed point theorem for three operators in a Banach algebra due to Dhage [25].

As a second problem we discuss in Section 4 an initial value problem for hybrid fractional sequential integro-differential equations,

$$(1.2)\quad \begin{cases} D^{\alpha}\left(\frac{D^{\omega}x(t)-I^{\beta}h(t,x(t),I^{\alpha_1}x(t),\ldots,I^{\alpha_n}x(t))}{f(t,x(t),I^{\alpha_1}x(t),\ldots,I^{\alpha_n}x(t))}\right) = g(t,x(t),I^{\beta_1}x(t),\ldots,I^{\beta_k}x(t)) \\ \quad t\in J=[0,T], \quad 0<\alpha<1 \\ x(0)=x_0, \quad D^{\omega}x(0)=0, \end{cases}$$

where $0<\alpha,\omega\le 1$, $1<\alpha+\omega\le 2$, functions $f\in C(J\times\mathbb{R}^n,\mathbb{R}\setminus\{0\})$, $h\in C(J\times\mathbb{R}^n,\mathbb{R})$ whith $h(0,x(0),\underbrace{0,0\ldots,0}_{n})=0$ and $g\in\mathcal{C}(J\times\mathbb{R}^k,\mathbb{R})$. I^{β} is the Riemann-Liouville fractional integral of order β .

D^{α}, D^{ω} are denotes Caputo fractional derivative of order α, β respectively.

By using a useful generalization of Krasnoselskii's fixed point theorem due to Dhage , we prove an existence result for the initial value problem 1.2.

This paper is arranged as follows. In Section 2, we recall some concepts and some fractional calculation law and establish preparation results. In Section 3, we study the existence of the initial value problem 1.1, based on the Dhage fixed point theorem,while in Section 4 we deal with the initial value problem 1.2. In Section 5, we give an example to demonstrate the application of our main result.

2. Preliminaries

Next, we review some basic concepts, notations, and technical results that are necessary in our study.
By $E = C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from $J = [0, T]$ into $\mathbb{R}$ with the norm

$$\|y\| = \sup\{|y(t)|, t \in J\}$$

and a multiplication in E by

$$(xy)(t) = x(t)y(t), \forall t \in J$$

.
Clearly E is a Banach algebra with respect to above supremum norm and the multiplication in it. and let $\mathcal{C}(J \times \mathbb{R}^k, \mathbb{R})$ denote the class of functions $g : J \times \mathbb{R}^k \longrightarrow \mathbb{R}$ such that

(*i*) the map $s \longrightarrow g(s, x_1, x_2, \ldots, x_k)$ is mesurable for all $x_1, x_2, \ldots, x_k \in \mathbb{R}$.
(*ii*) $(x_1, x_2, \ldots, x_k) \longrightarrow g(s, x_1, x_2, \ldots, x_k)$ is continuous map for almost all $s \in J$.

The class $\mathcal{C}(J \times \mathbb{R}^k, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R}^k$.
Also, a Carathéodory function $g : J \times \mathbb{R}^k \longrightarrow \mathbb{R}$ is called L^1-Carathéodory whenever for each $\rho > 0$ there exists $\phi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\|g(s, x_1, x_2, \ldots, x_k)\| = \sup\{|v| : v \in g(s, x_1, x_2, \ldots, x_k)\} \leq \phi_\rho(s)$$

for all $|x_1|, |x_2|, \ldots, |x_k| \leq \rho$ and for almost all $s \in J$.
By $L^1(J, \mathbb{R})$ denote the space of Lebesgue integrable real-valued functions on J equipped with the norm $\|.\|_{L^1}$ defined by
$\|x\|_{L^1} = \int_0^T |x(s)|ds$

Definition 2.1. [1] The fractional integral of the function $h \in L^1([a, b], \mathbb{R}^+)$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$

where Γ is the gamma function.

Definition 2.2. [1] For a function h given on the interval $[a, b]$, the Riemann-Liouville fractional-order derivative of h, is defined by

$$({}^cD_{a^+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)}(s)ds$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3. [1] For a function h given on the interval $[a, b]$, the Caputo fractional-order derivative of h, is defined by

$$({}^cD_{a^+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)} h^{(n)}(s)ds$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.4. [1] *Let $\alpha > 0$ and $x \in C(0, T) \cap L(0, T)$. Then the fractional differential equation*

$$D^\alpha x(t) = 0$$

has a unique solution

$$x(t) = k_1 t^{\alpha-1} + k_2 t^{\alpha-2} + \ldots + k_n t^{\alpha-n},$$

where $k_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n - 1 < \alpha < n$.

Lemma 2.5. [1] *Let $\alpha > 0$. Then for $x \in C(0, T) \cap L(0, T)$ we have*

$$I^\alpha D^\alpha x(t) = x(t) + c_0 + c_1 t + \ldots + c_{n-1} t^{n-1},$$

fore some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n - 1$. Where $n = [\alpha] + 1$.

Lemma 2.6. [1] *For $\alpha, \beta > 0$ and f as a suitable function, we have*

(*i*) $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$
(*ii*) $I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t)$
(*iii*) $I^\alpha (f(t) + g(t)) = I^\alpha f(t) + I^\alpha g(t)$

3. Fractional hybrid differential equation

In this section we prove the existence of a solution the initial value problem 1.1 by a fixed point theorem in the Banach algebra due to Dhage [25].

Lemma 3.1. *Let S be a nonempty, closed convex and bounded subset of a Banach algebra E and let $A, C : E \longrightarrow E$ and $B : S \longrightarrow E$ be three operators satisfying:*

- (a_1) *A and C are Lipschitzian with Lipschitz constants δ and ρ, respectively,*
- (b_1) *B is compact and continuous,*
- (c_1) *$x = AxBy + Cx \Longrightarrow x \in S$ for all $y \in S$,*
- (d_1) *$\delta M + \rho < 1$, where $M = \|B(S)\|$.*

Then the operator equation $x = AxBx + Cx$ has a solution.

For brevity let us take,

$$d = \frac{I^{\beta} h(T, x(T), I^{\alpha_1} x(T), \dots, I^{\alpha_n} x(T))}{f(T, x(T), I^{\alpha_1} x(T), \dots, I^{\alpha_n} x(T))}$$

Lemma 3.2. *Suppose that $1 < \alpha \le 2$.*
Then, for any $k \in L^1(J, \mathbb{R})$, the function $x \in C(J, \mathbb{R})$ is a solution of the

$$(3.1)\quad \begin{cases} D^{\alpha}\left(\frac{x(t) - I^{\beta} h(t, x(t), I^{\alpha_1} x(t), \dots, I^{\alpha_n} x(t))}{f(t, x(t), I^{\alpha_1} x(t), \dots, I^{\alpha_n} x(t))}\right) = k(t) \quad t \in J = [0, T] \\ \frac{x(0)}{f(0, x(0), \underbrace{0, 0 \dots, 0}_{n})} = x_0, \qquad \frac{x(T)}{f(T, x(T), I^{\alpha T} x(t), \dots, I^{\alpha_n} x(T))} = x_T, \end{cases}$$

if and only if x satisfies the hybrid integral equation

$$(3.2)\quad \begin{aligned} x(t) &= \Big(f(t, x(t), I^{\alpha_1} x(t), \dots, I^{\alpha_n} x(t))\Big)\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} k(s) ds \\ &+ (1 - \frac{t}{T}) x_0 + \frac{t}{T} x_T - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} k(s) ds - \frac{td}{T}\Big) \\ &+ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s, x(s), I^{\beta_1} x(s), \dots, I^{\alpha_n} x(s)) ds, \quad t \in [0, T] \end{aligned}$$

Proof. Assume that x is a solution of the problem 3.2. By definition, $\frac{x(t)}{f(t, x(t), I^{\alpha_1} x(t), \dots, I^{\alpha_n} x(t))}$ is continuous. Applying the Caputo fractional operator of the order α, we obtain the first equation in 3.1. Again, substituting $t = 0$ and $t = T$ in 3.2 we have

$$\frac{x(0)}{f(0, x(0), \underbrace{0, 0 \dots, 0}_{n})} = x_0, \qquad \frac{x(T)}{f(T, x(T), I^{\alpha_1} x(t), \dots, I^{\alpha_n} x(T))} = x_T,$$

Conversely, $$D^{\alpha}\left(\frac{x(t) - I^{\beta} h(t, x(t), I^{\alpha_1} x(t), \dots, I^{\alpha_n} x(t))}{f(t, x(t), I^{\alpha_1} x(t), \dots, I^{\alpha_n} x(t))}\right) = k(t)$$

so we get

$$\frac{x(t) - I^{\beta} h(t, x(t), I^{\alpha_1} x(t), \dots, I^{\alpha_n} x(t))}{f(t, x(t), I^{\alpha_1} x(t), \dots, I^{\alpha_n} x(t))} = I^{\alpha} k(t) - c_0 - c_1 t$$

$$\frac{x(t)}{f(t, x(t), I^{\alpha_1} x(t), \dots, I^{\alpha_n} x(t))} = I^{\alpha} k(t) - c_0 - c_1 t + \frac{I^{\beta} h(t, x(t), I^{\alpha_1} x(t), \dots, I^{\alpha_n} x(t))}{f(t, x(t), I^{\alpha_1} x(t), \dots, I^{\alpha_n} x(t))}$$

Substituting $t = 0$ we have

$$c_0 = -\frac{x(0)}{f(0, x(0), \underbrace{0, 0 \dots, 0}_{n})} = -x_0$$

And substituting $t = T$ we have

$$\frac{x(T)}{f(T, x(T), I^{\alpha_1}x(T), \ldots, I^{\alpha_n}x(T))} = I^{\alpha}k(T) + x_0 - c_1 T + d$$

Then

$$c_1 = \frac{1}{T}(x_0 + I^{\alpha}k(T) - x_T + d)$$

In consequence, we have

$$\begin{aligned} x(t) &= \Big(f(t, x(t), I^{\alpha_1}x(t), \ldots, I^{\alpha_n}x(t))\Big)\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}k(s)ds + (1-\frac{t}{T})x_0 + \frac{t}{T}x_T \\ &- \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}k(s)ds - \frac{td}{T}\Big) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s, x(s), I^{\beta_1}x(s), \ldots, I^{\alpha_n}x(s))ds, t \in [0,T] \end{aligned}$$

□

In the forthcoming analysis, we need the following assumptions. Assume that :

(H_1) The functions $f : J \times \mathbb{R}^{n+1} \longrightarrow \times\mathbb{R} \setminus \{0\}$, $h : J \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, $g : J \times \mathbb{R}^{k+1} \longrightarrow \mathbb{R}$ are be a Carathéodory function, $h(0, x(0), \underbrace{0, 0 \ldots, 0}_{n})) = 0$ and there exist two positive functions $p, m : J \longrightarrow (0, \infty)$ with bound $\|p\|$ and $\|m\|$ respectively, such that

$$|f(t, y_1, y_2, \ldots, y_{n+1}) - f(t, x_1, x_2, \ldots, x_{n+1})| \leq p(t)\sum_{i=1}^{n+1}|y_i - x_i|$$

and

$$|h(t, y_1, y_2, \ldots, y_{n+1}) - h(t, x_1, x_2, \ldots, x_{n+1})| \leq m(t)\sum_{i=1}^{n+1}|y_i - x_i|$$

for $t \in J$ and $(x_1, x_2, \ldots, x_{n+1}), (y_1, y_2, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}$.

(H_2) There exists a function $h \in L^1(J, \mathbb{R})$ such that .

$$|g(t, x_1, x_2, \ldots, x_k)| \leq h(t) \quad a.e \quad (t, x_1, x_2, \ldots, x_k) \in J \times \mathbb{R}^k$$

(H_3) There exists a real number $r > 0$ such that

$$r \geq \frac{F_0\Big(\frac{2\|h\|_{L^1}T^{\alpha}}{\Gamma(\alpha+1)} + |x_0| + |x_T| + |d|\Big) + \frac{k_0 T^{\beta}}{\Gamma(\beta+1)}}{1 - \Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \ldots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)}\Big)\Big[\|p\|\Big(\frac{2\|h\|_{L^1}T^{\alpha}}{\Gamma(\alpha+1)} + |x_0| + |x_T| + |d|\Big) - \|m\|\frac{T^{\beta}}{\Gamma(\beta+1)}\Big]}$$

where $F_0 = \sup_{t \in J}|f(t, x(0), \underbrace{0, 0 \ldots, 0}_{n})|$ and $K_0 = \sup_{t \in J}|h(t, x(0), \underbrace{0, 0 \ldots, 0}_{n})|$

Theorem 3.3. *Assume that the conditions $(H_1) - (H_3)$ hold. Then the initial value problem 1.1 has at least one solution on J provided that*

$$\Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \ldots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)}\Big)\Big[\|p\|\Big(\frac{2\|h\|_{L^1}T^{\alpha}}{\Gamma(\alpha+1)} + |x_0| + |x_1| + |d|\Big) + \frac{\|m\|T^{\beta}}{\Gamma(\beta+1)}\Big] < 1.$$

Proof. Set $E = C(J, \mathbb{R})$ and define a subset S of E as

$$S = \{x \in E : \|x\| \leq r\},$$

where r satisfies inequality 3.
Clearly S is closed, convex, and bounded subset of the Banach space E. By Lemma 3.2, problem 1.1 is equivalent to the integral equation 3.2. Now we define three operators,
$\mathcal{A} : E \longrightarrow E$ by

$$\mathcal{A}x(t) = f(t, x(t), I^{\alpha_1}x(t), \ldots, I^{\alpha_n}x(t)), \quad t \in J,$$

$\mathcal{B} : S \longrightarrow E$ by

$$\begin{aligned}\mathcal{B}x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x(s), I^{\beta_1}x(s), \dots, I^{\beta_k}x(s))ds + (1-\frac{t}{T})x_0 + \frac{t}{T}x_T \\ &- \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s, x(s), I^{\beta_1}x(s), \dots, I^{\beta_k}x(s))ds - \frac{t}{T}d\Big], \quad t \in J,\end{aligned}$$

and $\mathcal{C} : E \longrightarrow E$ by

$$\mathcal{C}x(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s, x(s), I^{\alpha_1}x(s), \dots, I^{\alpha_n}x(s))ds, \quad t \in J$$

We shall show that the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfy all the conditions of Lemma 3.1. The proof is constructed in several claims.
Claim 1. We will show that $\mathcal{A}$ and $\mathcal{C}$ are lipschitzian on E, that is, the assumption (a_1) of Lemma 3.1 holds.
Let $x, y \in E$. Then by (H_1), for $t \in J$ we have

$$\begin{aligned}|\mathcal{A}x(t) - \mathcal{A}y(t)| &= |f(t, x(t), I^{\alpha_1}x(t), \dots, I^{\alpha_n}x(t)) - f(t, y(t), I^{\alpha_1}y(t), \dots, I^{\alpha_n}y(t))| \\ &\leq \sup_{t\in J}(|p||x(t)-y(t)|)\Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)}\Big) \\ &\leq \|p\|\Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)}\Big)\|x-y\|\end{aligned}$$

for all $t \in J$.
Taking the supremum over the interval J, we obtain
$\|\mathcal{A}x - \mathcal{A}y\| \leq \|p\|\Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)}\Big)\|x-y\|$ for all $x, y \in E$. So $\mathcal{A}$ is a Lipschitz on E with Lipschitz constant $\|p\|\Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)}\Big)$.
Analogously, for any $x, y \in E$, we have

$$\begin{aligned}|\mathcal{C}x(t) - \mathcal{C}y(t)| &= \Big|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}[h(s, x(s), I^{\alpha_1}x(s), \dots, I^{\alpha_n}x(s)) - h(s, y(s), I^{\alpha_1}y(s), \dots, I^{\alpha_n}y(s))]ds\Big| \\ &\leq \sup_{t\in J}(|m||x(t)-y(t)|)\Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)}\Big)\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}ds \\ &\leq \|m\|\Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)}\Big)\frac{T^{\beta}}{\Gamma(\beta+1)}\|x-y\|\end{aligned}$$

for all $t \in J$.
Taking the supremum over the interval J, we obtain

$$\|\mathcal{C}x - \mathcal{C}y\| \leq \|m\|\Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)}\Big)\frac{T^{\beta}}{\Gamma(\beta+1)}\|x-y\|$$

So, $\mathcal{C}$ is a Lipschitzian on E with Lipschitz constant $\|m\|\Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)}\Big)\frac{T^{\beta}}{\Gamma(\beta+1)}$.
Claim 2. The operator $\mathcal{B}$ is completely continuous on S, that is, the assumption (b_1) of Lemma 3.1 holds.
We first show that the operator $\mathcal{B}$ is continuous on E.
Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then by the Lebesgue dominated convergence theorem, for all $t \in J$, we obtain

$$\begin{aligned}\lim_{n\to\infty} &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(t, x_n(s), I^{\beta_1}x_n(s), \dots, I^{\beta_k}x_n(s))ds \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \lim_{n\to\infty} g(s, x_n(s), I^{\beta_1}x_n(s), \dots, I^{\beta_k}x_n(s))ds \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x(s), I^{\beta_1}x(s), \dots, I^{\beta_k}x(s))ds\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\Big[(1 &- \frac{t}{T})x_0+\frac{t}{T}x_T-\frac{t}{T\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}g(s,x_n(s),I^{\beta_1}x_n(s),\ldots,I^{\beta_k}x_n(s))ds-\frac{t}{T}d\Big]\\
&= \lim_{n\to\infty}\Big[(1-\frac{t}{T})x_0+\frac{t}{T}x_T\Big]-\frac{t}{T\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\lim_{n\to\infty}g(t,x_n(t),I^{\beta_1}x_n(s),\ldots,I^{\beta_k}x_n(s))ds\\
&- \frac{t}{T}d\\
&= (1-\frac{t}{T})x_0+\frac{t}{T}x_T-\frac{t}{T\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}g(s,x(s),I^{\beta_1}x(s),\ldots,I^{\beta_k}x(s))ds-\frac{t}{T}d
\end{aligned}$$

In consequence, we have

$$\lim_{n\to\infty}\mathcal{B}x_n=\mathcal{B}x$$

This shows that $\mathcal{B}$ is continuous on S.
It is sufficient to show that the set $\mathcal{B}(S)$ is a uniformly bounded in S. For any $x\in S$, we have

$$\begin{aligned}
|\mathcal{B}x(t)| &= \Big|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),I^{\beta_1}x(s),\ldots,I^{\beta_k}x(s))ds+(1-\frac{t}{T})x_0+\frac{t}{T}x_T\\
&- \frac{t}{T\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}g(t,x(t),I^{\beta_1}x(t),\ldots,I^{\beta_k}x(t))ds-\frac{t}{T}d\Big|\\
&\le \Big(2\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}\Big)+|x_0|+|x_1|+|d|=K_1
\end{aligned}$$

for all $t\in J$. Taking supremum over the interval J, the above inequality becomes, $\|\mathcal{B}x\|\le K_1$ for all $x\in S$. This shows that $\mathcal{B}(S)$ is uniformly bounded on S.
Next we show that $\mathcal{B}(S)$ is an equicontinuous set in E. We take, $\tau_1,\tau_2\in J$ with $\tau_1<\tau_2$ and and $x\in S$.
Then we have

$$\begin{aligned}
|\mathcal{B}x(\tau_2)-\mathcal{B}x(\tau_1)| &= \Big|\int_0^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),I^{\beta_1}x(s),\ldots,I^{\beta_k}x(s))ds\\
&- \int_0^{\tau_1}\frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),I^{\beta_1}x(s),\ldots,I^{\beta_k}x(s))ds+\Big[(1-\frac{\tau_2}{T})-(1-\frac{\tau_1}{T})\Big]x_0\\
&+ (\frac{\tau_1}{T}-\frac{\tau_2}{T})\Big(\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}g(t,x(t),I^{\beta_1}x(t),\ldots,I^{\beta_k}x(t))ds-x_T+d\Big)\Big|\\
&\le \int_0^{\tau_1}\frac{|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}|}{\Gamma(\alpha)}|g(t,x(t),I^{\beta_1}x(t),\ldots,I^{\beta_k}x(t))|ds\\
&+ \int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}|g(t,x(t),I^{\beta_1}x(t),\ldots,I^{\beta_k}x(t))|ds+(\frac{\tau_1-\tau_2}{T})x_0\\
&+ \Big(\frac{\tau_1-\tau_2}{T}\Big)\Big(x_T+\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|g(t,x(t),I^{\beta_1}x(t),\ldots,I^{\beta_k}x(t))|ds+d\Big)\\
&\le \int_0^{\tau_1}\frac{|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}|}{\Gamma(\alpha)}\|h\|_{L^1}ds+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}\|h\|_{L^1}ds\\
&+ (\frac{\tau_1-\tau_2}{T})x_0+\Big(\frac{\tau_1-\tau_2}{T}\Big)\Big(x_T+\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\|h\|_{L^1}ds+d\Big)
\end{aligned}$$

Thus, we have that $|\mathcal{B}x(\tau_2)-\mathcal{B}x(\tau_1)|\longrightarrow 0$ as $\tau_2\longrightarrow\tau_1$
which is independent of $x\in S$.
which is independent of $x\in S$.Thus, $\mathcal{B}(S)$ is equicontinuous. So $\mathcal{B}$ is relatively compact on S. Hence, by the Arzelá-Ascoli theorem, $\mathcal{B}$ is compact on S.
Claim 3. The hypothesis (c_1) of Lemma 3.1 is satisfied.

Let $x \in E$ and $y \in S$ be arbitrary elements such that $x = \mathcal{A}x\mathcal{B}y + \mathcal{C}x$. Then we have

$$\begin{aligned}
|x(t)| &\le |\mathcal{A}x(t)||\mathcal{B}y(t)| + |\mathcal{C}x(t)| \\
&\le |f(t, x(t), I^{\alpha_1}x(t), \dots, I^{\alpha_n}x(t))| \Big[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| g(s, y(s), I^{\beta_1}y(s), \dots, I^{\beta_k}y(s)) \Big| ds \\
&+ (1 - \frac{t}{T})x_0 + \frac{t}{T}x_T + \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \Big| g(s, y(s), I^{\beta_1}y(s), \dots, I^{\beta_k}y(s)) \Big| ds \\
&+ \frac{t|d|}{T} \Big] + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \Big| h(t, x(t), I^{\alpha_1}x(t), \dots, I^{\alpha_n}x(t)) \Big| ds \\
&\le (|f(t, x(t), I^{\alpha_1}x(t), \dots, I^{\alpha_n}x(t)) - f(t, 0, \dots, 0)| + |f(t, 0, \dots, 0)|) \Big[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |h(s)| ds \\
&+ (1 - \frac{t}{T})x_0 + \frac{t}{T}x_T + \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |h(s)| ds + |\frac{t}{T}||d| \Big] \\
&+ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} (|h(t, x(t), I^{\alpha_1}x(t), \dots, I^{\alpha_n}x(t)) - h(s, 0, \dots, 0) + |h(s, 0, \dots, 0)|) \Big| \\
&\le \Big[r\|p\| \Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)} \Big) + F_0 \Big] \Big(\frac{2\|h\|_{L^1} T^\alpha}{\Gamma(\alpha+1)} + |x_0| + |x_T| \\
&+ |d| \Big) + \frac{r\|m\|T^\beta}{\Gamma(\beta+1)} \Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)} \Big) + \frac{T^\beta}{\Gamma(\beta+1)} k_0
\end{aligned}$$

Taking supremum for $t \in J$, we obtain

$$\begin{aligned}
\|x\| &\le \Big[r\|p\| \Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)} \Big) + F_0 \Big] \Big(\frac{2\|h\|_{L^1} T^\alpha}{\Gamma(\alpha+1)} + |x_0| + |x_T| \\
&+ |d| \Big) + \frac{r\|m\|T^\beta}{\Gamma(\beta+1)} \Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)} \Big) + \frac{T^\beta}{\Gamma(\beta+1)} k_0
\end{aligned}$$

that is, $x \in S$.
Claim 4. Finally we show that $\delta M + \rho < 1$, that is, (d_1) of Lemma 3.1 holds.
Since

$$\begin{aligned}
M = \|\mathcal{B}(S)\| &= \sup_{x \in S} \{ \sup_{t \in J} |\mathcal{B}x(t)| \} \\
&\le \frac{2\|h\|_{L^1} T^\alpha}{\Gamma(\alpha+1)} + |x_0| + |x_T| + |d|
\end{aligned}$$

and by theorem 3.3 we have

$$\Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)} \Big) \Big(\|p\| M + \frac{\|m\|T^\beta}{\Gamma(\beta+1)} \Big) < 1$$

with $\delta = \Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)} \Big) \|p\|$ and $\rho = \frac{\|m\|T^\beta}{\Gamma(\beta+1)} \Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \dots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)} \Big)$.
Thus all the conditions of Lemma 3.1 are satisfied and hence the operator equation $x = \mathcal{A}x\mathcal{B}x + \mathcal{C}x$ has a solution in S. In consequence, problem 3.1 has a solution on J. This completes the proof. □

4. Hybrid fractional sequential integro-differential equations

In this section we consider the initial value problem 1.2. An existence result will be proved by using the following fixed point theorem due to Dhage.

Lemma 4.1. *Let M be a nonempty, closed, convex and bounded subset of the Banach space X and let $A : X \longrightarrow X$ and $B : M \longrightarrow X$ be two operators such that*

(i) *A is a contraction,*
(ii) *B is completely continuous, and*
(iii) *$x = Ax + By$ for all $y \in M \Longrightarrow x \in M$.*

Then the operator equation $Ax + Bx = x$ has a solution.

Lemma 4.2. *Suppose that* $0 < \alpha, \omega \leq 1$, $0 < \alpha + \omega \leq 1$, *and the functions* f,g, *and* h *satisfy the problem 1.2. The function* $x \in C(J, \mathbb{R})$ *is a solution of the problem 1.2 if and only if* x *satisfies the hybrid integral equation,*

$$\begin{aligned} x(t) &= \int_0^t \Big(\frac{(t-s)^{\omega-1}}{\Gamma(\omega)} f(s, x(s), I^{\alpha_1}x(s), \ldots, I^{\alpha_n}x(s)) \\ &\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} g(\tau, x(\tau), I^{\beta_1}x(\tau), \ldots, I^{\beta_k}x(\tau)))d\tau \Big) ds \\ &+ \int_0^t \frac{(t-s)^{\beta+\omega-1}}{\Gamma(\beta+\omega)} h(s, x(s), I^{\alpha_1}x(s), \ldots, I^{\alpha_n}x(s))ds + x_0, \quad t \in [0, T] \end{aligned} \tag{4.1}$$

Proof. Assume that x is a solution of the problem 3.2. By definition, $\frac{x(t)}{f(t,x(t),I^{\alpha_1}x(t),\ldots,I^{\alpha_n}x(t))}$ is continuous. Applying the Caputo fractional operator of the order α, we obtain the first equation in 4.1 .
Again, substituting $t = 0$ in 4.1 we have
$x(0) = x_0, \qquad D^{\omega}x(0) = 0$
Conversely,
by lemma 2.5 we have

$$\frac{D^{\omega}x(t) - I^{\beta}h(t, x(t), I^{\alpha_1}x(t), \ldots, I^{\alpha_n}x(t))}{f(t, x(t), I^{\alpha_1}x(t), \ldots, I^{\alpha_n}x(t))} = I^{\alpha}g(t, x(t), I^{\beta_1}x(t), \ldots, I^{\beta_k}x(t))) - c_0$$

By condition $D^{\beta}x(0) = 0$, implies that $c_0 = 0$
Applying the semigroup property, i.e., $I^{\omega}I^{\beta}h = I^{\omega+\beta}h$ proposition 2.6, we obtain the,

$$\begin{aligned} x(t) &= I^{\omega}\Big[f(t, x(t), I^{\alpha_1}x(t), \ldots, I^{\alpha_n}x(t))I^{\alpha}g(t, x(t), I^{\beta_1}x(t), \ldots, I^{\beta_k}x(t))\Big] \\ &+ I^{\omega+\beta}h(t, x(t), I^{\alpha_1}x(t), \ldots, I^{\alpha_n}x(t)) - c_1 \end{aligned}$$

By condition $x(0) = x_0$, implies that $c_1 = -x_0$
Then,

$$\begin{aligned} x(t) &= I^{\omega}\Big[f(t, x(t), I^{\alpha_1}x(t), \ldots, I^{\alpha_n}x(t))I^{\alpha}g(t, x(t), I^{\beta_1}x(t), \ldots, I^{\beta_k}x(t))\Big] \\ &+ I^{\omega+\beta}h(t, x(t), I^{\alpha_1}x(t), \ldots, I^{\alpha_n}x(t)) + x_0 \end{aligned}$$

Consequently,

$$\begin{aligned} x(t) &= \int_0^t \Big[\frac{(t-s)^{\omega-1}}{\Gamma(\omega)} f(s, x(s), I^{\alpha_1}x(s), \ldots, I^{\alpha_n}x(s)) \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} g(\tau, x(\tau), I^{\beta_1}x(\tau), \ldots, I^{\beta_k}x(\tau)))d\tau\Big] ds \\ &+ \int_0^t \frac{(t-s)^{\beta+\omega-1}}{\Gamma(\beta+\omega)} h(s, x(s), I^{\alpha_1}x(s), \ldots, I^{\alpha_n}x(s))ds + x_0 \quad , t \in [0, T] \end{aligned}$$

□

In the forthcoming analysis, we need the following assumptions. Assume that :

(A_1) The functions $f : J \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \setminus \{0\}$ and $g : J \times \mathbb{R}^{k+1} \longrightarrow \mathbb{R}$, are continuous and there exist two positive functions ϕ, χ with bound $\|\phi\|$ and $\|\chi\|$, respectively, such that

$$|f(t, y_1, y_2, \ldots, y_{n+1}) - f(t, x_1, x_2, \ldots, x_{n+1})| \leq \phi(t) \sum_{i=1}^{n+1} |y_i - x_i|$$

for $t \in J$ and $(x_1, x_2, \ldots, x_{n+1}), (y_1, y_2, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}$.
and

$$|g(t, y_1, y_2, \ldots, y_{k+1}) - g(t, x_1, x_2, \ldots, x_{k+1})| \leq \chi(t) \sum_{i=1}^{k+1} |y_i - x_i|$$

for $t \in J$ and $(x_1, x_2, \ldots, x_{k+1}), (y_1, y_2, \ldots, y_{k+1}) \in \mathbb{R}^{k+1}$.

(A_2) $|f(t, x_1, x_2, \ldots, x_{n+1})| \leq \mu(t)$, $\forall (t, x_1, x_2, \ldots, x_{n+1}) \in J \times \mathbb{R}^{n+1}$, $\mu \in C(J, \mathbb{R}^+)$,
$|g(t, x_1, x_2, \ldots, x_{k+1})| \leq \nu(t)$, $\forall (t, x_1, x_2, \ldots, x_{k+1}) \in J \times \mathbb{R}^{k+1}$, $\nu \in C(J, \mathbb{R}^+)$ and
$|h(t, x_1, x_2, \ldots, x_{n+1})| \leq \theta(t)$,$\forall (t, x_1, x_2, \ldots, x_{n+1}) \in J \times \mathbb{R}^{n+1}$, $\theta \in C(J, \mathbb{R}^+)$.

Theorem 4.3. *Assume that the conditions $(A_1)-(A_2)$ hold. Then the initial value problem 1.2 has at least one solution on J provided that*

$$\frac{T^{\alpha+\omega}}{\Gamma(\alpha+1)\Gamma(\omega+1)}\Big\{\Big(1+\frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)}+\ldots+\frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)}\Big)\|\nu\|\|\phi\|+\|\mu\|\|\chi\|\Big(1+\frac{T^{\beta_1}}{\Gamma(\beta_1+1)}+\ldots+\frac{T^{\beta_k}}{\Gamma(\beta_k+1)}\Big)\Big\}<1$$

Proof. Setting $\sup_{t\in J}|\mu(t)|=\|\mu\|$, $\sup_{t\in J}|\nu(t)|=\|\nu\|$, $\sup_{t\in J}|\theta(t)|=\|\theta\|$, and choosing

$$R\geq\frac{T^{\omega+\beta}}{\Gamma(\omega+\beta+1)}\|\theta\|+\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|\mu\|\|\ \nu\|+|x_0|$$

We consider $B_R=\{x\in C(J,\mathbb{R}):\|x\|\leq R\}$. We define the operators $\mathcal{A}:E\longrightarrow E$ as in 3, $\mathcal{D}:B_R\longrightarrow E$ by

$$\mathcal{D}x(t) \quad = \quad \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),I^{\beta_1}x(s),\ldots,I^{\beta_k}x(s))ds,\quad t\in J$$

and

$$\mathcal{Q}x(t)=\int_0^t\frac{(t-s)^{\beta+\omega-1}}{\Gamma(\beta+\omega)}h(s,x(s),I^{\alpha_1}x(s),\ldots,I^{\alpha_n}x(s))ds,\quad t\in J$$

and

$$\mathcal{T}x(t)=\int_0^t\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}\mathcal{A}x(t)\mathcal{D}x(s)ds+x_0,\quad t\in J$$

For any $y\in B_R$, we have

$$\begin{aligned}|x(t)| &= |\mathcal{Q}x(t)+\mathcal{T}y(t)|\\ &\leq \int_0^t\frac{(t-s)^{\omega+\beta-1}}{\Gamma(\omega+\beta)}|h(s,x(s),I^{\alpha_1}x(s),\ldots,I^{\alpha_n}x(s))|ds+\int_0^t\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}|\mathcal{A}y(s)||\mathcal{D}y(s)|ds+|x_0|\\ &\leq \int_0^t\frac{(t-s)^{\omega+\beta-1}}{\Gamma(\omega+\beta)}|\theta(s)|ds+\int_0^t\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}|\mu(t)|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\nu(s)|ds+|x_0|\\ &\leq \frac{T^{\omega+\beta}}{\Gamma(\omega+\beta+1)}\|\theta\|+\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|\mu\|\|\ \nu\|+|x_0|\end{aligned}$$

Taking supremum for $t\in J$, we obtain $\|x\|\leq R$, which means that $x\in B_R$. So, the condition (iii) of Lemma 4.1 holds.
Next we will show that $\mathcal{Q}$ satisfy the condition (ii) of Lemma 4.1. The operator $\mathcal{Q}$ is obviously continuous. Also, $\mathcal{Q}$ is uniformly bounded on B_R as

$$\|\mathcal{Q}x\|\leq\frac{T^{\omega+\beta}}{\Gamma(\omega+\beta+1)}\|\theta\|$$

Let $\tau_1,\tau_2\in J$ with $\tau_1<\tau_2$ and $(x_1,x_2,\ldots,x_{n+1})\in B_R^{n+1}$. We define
$$\sup_{(t,x_1,x_2,\ldots,x_{n+1})\in J\times B_R^{n+1}}|h(t,x_1,x_2,\ldots,x_{n+1})|=\bar{h}<\infty.$$
Then we have

$$\begin{aligned}|\mathcal{Q}x(\tau_2)-\mathcal{Q}x(\tau_1)| &= \Big|\int_0^{\tau_2}\frac{(\tau_2-s)^{\omega+\beta-1}}{\Gamma(\omega+\beta)}h(s,x(s),I^{\alpha_1}x(s),\ldots,I^{\alpha_n}x(s))ds\\ &- \int_0^{\tau_1}\frac{(\tau_1-s)^{\omega+\beta-1}}{\Gamma(\omega+\beta)}h(s,x(s),I^{\alpha_1}x(s),\ldots,I^{\alpha_n}x(s))ds\Big|\\ &\leq \frac{\bar{h}}{\Gamma(\omega+\beta)}\Big|\int_0^{\tau_1}[(\tau_2-s)^{\omega+\beta-1}-(\tau_1-s)^{\omega+\beta-1}]ds+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\omega+\beta-1}ds\Big|\\ &\leq \frac{\bar{h}}{\Gamma(\omega+\beta+1)}|\tau_2^{\omega+\beta}-\tau_1^{\omega+\beta}|\end{aligned}$$

Thus, we have that $|\mathcal{Q}x(\tau_2)-\mathcal{Q}x(\tau_1)|\longrightarrow 0$ as $\tau_2\longrightarrow\tau_1$
which is independent of $x\in S$.Thus, $\mathcal{Q}$ is equicontinuous. So $\mathcal{Q}$ is relatively compact on B_R.

Hence, by the Arzelá-Ascoli theorem, $\mathcal{Q}$ is compact on B_R.
Now we show that $\mathcal{T}$ is a contraction mapping. Let $x, y \in B_R$, then for $t \in J$ we have

$$
\begin{aligned}
&|\mathcal{T}x(t) - \mathcal{T}y(t)| \\
&= \Big| \int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)} [\mathcal{A}x(s)\mathcal{D}x(s)ds - \mathcal{A}y(s)\mathcal{D}y(s)]ds \Big| \\
&= \Big| \int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)} [\mathcal{A}x(s)\mathcal{D}x(s) - \mathcal{A}y(s)\mathcal{D}x(s) + \mathcal{A}y(s)\mathcal{D}x(s) - \mathcal{A}y(s)\mathcal{D}y(s)]ds \Big| \\
&\leq \int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)} \Big\{ |\mathcal{D}x(s)||\mathcal{A}x(s) - \mathcal{A}y(s)| + |\mathcal{A}y(s)||\mathcal{D}x(s) - \mathcal{D}y(s)| \Big\} ds \\
&\leq \int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)} \Big\{ \Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \ldots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)}\Big) \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|\nu\|\|\phi\|\|x-y\| \\
&+ \|\mu\|\|\chi\| \Big(1 + \frac{T^{\beta_1}}{\Gamma(\beta_1+1)} + \ldots + \frac{T^{\alpha_k}}{\Gamma(\beta_k+1)}\Big) \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|x-y\| \Big\} ds \\
&\leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \frac{T^{\omega}}{\Gamma(\omega+1)} \Big\{ \Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \ldots + \frac{T^{\alpha_n}}{\Gamma(\alpha_n+1)}\Big) \|\nu\|\|\phi\| \\
&+ \|\mu\|\|\chi\| \Big(1 + \frac{T^{\beta_1}}{\Gamma(\beta_1+1)} + \ldots + \frac{T^{\alpha_k}}{\Gamma(\beta_k+1)}\Big) \Big\} \|x-y\|
\end{aligned}
$$

So, by theorem 4.3, $\mathcal{T}$ is a contraction mapping, and thus the condition (i) of Lemma 4.3 is satisfied.
Thus all the assumptions of Lemma 4.1 are satisfied. Therefore, the conclusion of Lemma 4.1 implies that problem 1.2 has at least one solution on J. □

Example
In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional boundary value problem:

$$
(4.2)\begin{cases}
D^{\frac{1}{2}} \left(\dfrac{x(t) - I^{\frac{1}{2}} \left[\frac{2te^{-3t}}{15(3+t)} \left(\sin x(t) + \frac{x(t) + 9I^{\sqrt{2}}|x(t)|}{I^{\sqrt{2}}|x(t)|+5} \right) \right]}{\frac{(t+1)^2}{100} \left(\sin x(t) + \frac{|I^{\sqrt{2}}x(t)|}{1+|I^{\sqrt{2}}x(t)|} + 3 \right)} \right) = t^2 \sin x(t) + \cos(I^{\frac{1}{4}} x(t)) + 1 \\
t \in J = [0,1] \\
\frac{x(0)}{f(0,x(0),0)} = \frac{\pi}{2}, \quad \frac{x(1)}{f(1,x(1),I^{\alpha_1}x(1))} = 0,
\end{cases}
$$

Put $\alpha = \frac{1}{2}$, $\alpha_1 = \sqrt{2}$,$\beta = \frac{1}{2}$,$\beta_1 = \frac{1}{4}$,$T = 1$, $n = k = 1$,$f(t,y,x) = \frac{(t+1)^2}{100}\Big(\sin y(t) + \frac{|x|}{1+|x|} + 3\Big)$, $g(t,y,x) = t^2 \sin x(t) + \cos(I^{\frac{1}{4}} x(t)) + 1$, $h(t,y,x) = \frac{2te^{-3t}}{15(3+t)}(\sin y(t) + \frac{x^2(t)+9|x(t)|}{|x(t)|+5})$, $m(t) = \frac{2t}{15(3+t)}$ and $p(t) = \frac{(t+1)^2}{100}$ for $t \in [0,1]$. Note that, $\|g(t,y,x)\| \leq t^2 + 2$, and

$$|f(t,x,y) - f(t,x',y')| \leq \frac{(t+1)^2}{100}(|x-x'| + |y-y'|)$$

and

$$|h(t,x,y) - h(t,x',y')| \leq \Big(\frac{2t}{15(3+t)}\Big)(|x-x'| + |y-y'|)$$

We have

$$\Big(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big)\Big(\frac{2\|p\|\|h\|_{L^1}T^{\alpha}}{\Gamma(\alpha+1)} + |x_0| + |x_1| + |d| + \|m\|\frac{T^{\beta}}{\Gamma(\beta+1)}\Big) \simeq 0.18957628293 < 1$$

.
By using the theorem 3.3, the problem 4.2 has a solution.

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