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Finite Rogers–Ramanujan type continued fractions*

Research Article

Helmut Prodinger

Abstract: New finite continued fractions related to Bressoud and Santos polynomials are established.

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1. Introduction

Define, as it is common today, $(x;q)_n := (1-x)(1-xq)\dots(1-xq^{n-1})$, where we assume that $|q|<1$, and we allow n also to be 0 and infinity. We also need the coefficients $\left[{n \atop k}\right] := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$. These standard notations can be found e. g. in the classic book [1].

The two Rogers-Ramanujan identities [1, 6]

$$\sum_{n\ge0}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty},$$
$$\sum_{n\ge0}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$$

are very popular, influential, useful and historically interesting. Let

$$F(z)=\sum_{n\ge0}\frac{q^{n^2}z^n}{(q;q)_n}\quad\text{and}\quad G(z)=\sum_{n\ge0}\frac{q^{n^2+n}z^n}{(q;q)_n};$$

* *Dedicated to Peter Paule on the occasion of his 60th birthday.*

Helmut Prodinger; Department of Mathematics, University of Stellenbosch 7602, Stellenbosch, South Africa (email: hproding@sun.ac.za).

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the continued fraction (due to Ramanujan)

$$\frac{zG(z)}{F(z)} = \cfrac{z}{1+\cfrac{zq}{1+\cfrac{zq^2}{1+\ddots}}}$$

is also very well known, see [5, Entry 5] and [6, (6.1)].

There are several families of polynomials that approximate $F(z)$ and $G(z)$. Probably the the most well known are

$$f_n(z) = \sum_{j\ge0} q^{j^2} \begin{bmatrix} n+1-j \\ j \end{bmatrix} z^j \to F(z) \quad \text{and} \quad g_n(z) = \sum_{j\ge0} q^{j^2+j} \begin{bmatrix} n-j \\ j \end{bmatrix} z^j \to G(z) \quad \text{for } n\to\infty,$$

because of their link to the Schur polynomials, see [1].

The finite continued fraction

$$\frac{zg_n(z)}{f_n(z)} = \cfrac{z}{1+\cfrac{zq}{1+\cfrac{zq^2}{\ddots\, 1+\cfrac{zq^n}{1}}}}$$

is also known [5, Entry 16].

The polynomials

$$s_n(z) = \sum_{j\ge0} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix} z^j \to F(z) \quad \text{and} \quad t_n(z) = \sum_{j\ge0} q^{j^2+j} \begin{bmatrix} n \\ j \end{bmatrix} z^j \to G(z) \quad \text{for } n\to\infty,$$

due to *Bressoud* [7], are less well known; see also [8].

2. Bressoud polynomials and continued fractions

In this section, we will establish the following attractive finite continued fraction:

Theorem 2.1.

$$\frac{zt_n(z)}{s_n(z)} = \cfrac{z}{1+\cfrac{zq(1-q^n)}{1+\cfrac{zq^2}{1+\cfrac{zq^3(1-q^{n-1})}{1+\cfrac{zq^4}{1+\ddots}}}}}$$

Proof. To prove this statement by induction, define the righthand side by $T_n(z)$. It is plain to see that $T_0(z) = z$, and

$$T_n(z) = \cfrac{z}{1+\cfrac{zq(1-q^n)}{1+T_{n-1}(zq^2)}}.$$

We are left to prove that

$$\frac{zt_n(z)}{s_n(z)} = \frac{z}{1+\cfrac{zq(1-q^n)}{1+\cfrac{zt_{n-1}(zq^2)}{s_{n-1}(zq^2)}}} = \frac{z}{1+\cfrac{zq(1-q^n)s_{n-1}(zq^2)}{s_{n-1}(zq^2)+zt_{n-1}(zq^2)}}$$
$$= \frac{z(s_{n-1}(zq^2)+zt_{n-1}(zq^2))}{s_{n-1}(zq^2)+zt_{n-1}(zq^2)+zq(1-q^n)s_{n-1}(zq^2)},$$

which amounts to prove that

$$t_n(z) = s_{n-1}(zq^2) + zt_{n-1}(zq^2),$$
$$s_n(z) = s_{n-1}(zq^2) + zt_{n-1}(zq^2) + zq(1-q^n)s_{n-1}(zq^2).$$

We will show that the coefficients of z^j coincide, which is trivial for $j=0$, so we assume $j \ge 1$:

$$q^{j^2+j}\begin{bmatrix} n \\ j \end{bmatrix} = q^{j^2+2j}\begin{bmatrix} n-1 \\ j \end{bmatrix} + q^{(j-1)^2+3(j-1)+2}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} n \\ j \end{bmatrix} = q^j\begin{bmatrix} n-1 \\ j \end{bmatrix} + \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}$$

and therefore true. The second one goes like this:

$$q^{j^2}\begin{bmatrix} n \\ j \end{bmatrix} = q^{j^2+2j}\begin{bmatrix} n-1 \\ j \end{bmatrix} + q^{(j-1)^2+3(j-1)+2}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + q(1-q^n)q^{(j-1)^2+2(j-1)}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} n \\ j \end{bmatrix} = q^{2j}\begin{bmatrix} n-1 \\ j \end{bmatrix} + q^j\begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + (1-q^j)\begin{bmatrix} n \\ j \end{bmatrix},$$

and further to

$$\begin{bmatrix} n \\ j \end{bmatrix} = q^j\begin{bmatrix} n-1 \\ j \end{bmatrix} + \begin{bmatrix} n-1 \\ j-1 \end{bmatrix},$$

which finishes the proof. □

3. Identities 39 and 38 from Slater's list

Slater [11] produced a list of Rogers-Ramanujan type series/product identities; Sills [10] in an amazing effort reworked and annotated this list, providing, in particular, finite versions of all of them.

Arguably the second most popular identities in the Rogers-Ramanujan world are Slater's [11] identities (39) and (38)

$$\sum_{n\ge 0}\frac{q^{2n^2}}{(q;q)_{2n}} = \prod_{k\ge 1,\ k\equiv \pm 2,\pm 3,\pm 4,\pm 5 \pmod{16}}\frac{1}{1-q^k},$$
$$\sum_{n\ge 0}\frac{q^{2n^2+2n}}{(q;q)_{2n+1}} = \prod_{k\ge 1,\ k\equiv \pm 1,\pm 4,\pm 6,\pm 7 \pmod{16}}\frac{1}{1-q^k}.$$

Let

$$s_n(z) = \sum_{0\le 2h\le n} q^{2h^2}\begin{bmatrix} n \\ 2h \end{bmatrix} z^h \quad \text{and} \quad t_n(z) = \sum_{0\le 2h\le n} q^{2h^2+2h}\begin{bmatrix} n \\ 2h+1 \end{bmatrix} z^h;$$

these polynomials are called *Santos* polynomials [2–4].

In order to describe the finite continued fraction expansion of $zt_n(z)/s_n(z)$, we define the following numbers and polynomials (power series) which were originally found by guessing:

$$a_{2k} := \frac{(1-q^{4k+1})(q^{n+1-2k};q^2)_{2k}}{q^{2k}(q^{n-2k};q^2)_{2k+1}},$$
$$a_{2k+1} := \frac{(1-q^{4k+3})(q^{n-2k};q^2)_{2k+1}}{q^{2k+2}(q^{n-2k-1};q^2)_{2k+2}};$$

$$S_{2i} := \sum_{j\ge 0} \frac{q^{2(i+j)(i+j+1)}(q^{n-2i-2j};q)_{2j}(q^{n-2i};q^2)_{2i+1}}{(q;q)_{2j+1}(q^{2j+3};q^2)_{2i}} z^j,$$
$$S_{2i+1} := \sum_{j\ge 0} \frac{q^{2(i+j+1)^2}(q^{n-1-2i-2j};q)_{2j}(q^{n-1-2i};q^2)_{2i+2}}{(q;q)_{2j+1}(q^{2j+3};q^2)_{2i+1}} z^j.$$

Theorem 3.1. *The polynomials S_i satisfy the second order recurrence $zS_{i+1} = S_{i-1} - a_iS_i$, $S_{-1} = s_n(z)$, $S_0 = t_n(z)$. Consequently, we get the finite continued fraction expansion*

$$\frac{zt(n)}{s(n)} = \cfrac{z}{a_0 + \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{\ddots}}}},$$

or, more elegantly:

$$\frac{zt(n)}{s(n)} = \cfrac{zb_0}{1 + \cfrac{zb_1}{1 + \cfrac{zb_2}{1+\dots}}}$$

with

$$b_0 = \frac{1}{a_0} = \frac{1-q^n}{1-q} \quad \textit{and} \quad b_i = \frac{1}{a_{i-1}a_i} = \frac{q^{2i}(1-q^{n-i})(1-q^{n+i})}{(1-q^{2i-1})(1-q^{2i+1})} \quad \textit{for } i\ge 1.$$

Proof. The recursion will be shown for even i, the other instance being very similar:

$$\begin{aligned}
S_{2i-1}-a_{2i}S_{2i}&=\sum_{j\ge0}\frac{q^{2(i+j)^2}(q^{n+1-2i-2j};q)_{2j}(q^{n+1-2i};q^2)_{2i}}{(q;q)_{2j+1}(q^{2j+3};q^2)_{2i-1}}z^j\\
&-\frac{(1-q^{4i+1})(q^{n+1-2i};q^2)_{2i}}{q^{2i}(q^{n-2i};q^2)_{2i+1}}\sum_{j\ge0}\frac{q^{2(i+j)(i+j+1)}(q^{n-2i-2j};q)_{2j}(q^{n-2i};q^2)_{2i+1}}{(q;q)_{2j+1}(q^{2j+3};q^2)_{2i}}z^j\\
&=(q^{n+1-2i};q^2)_{2i}\sum_{j\ge0}\frac{q^{2(i+j)^2}(q^{n+1-2i-2j};q)_{2j}}{(q;q)_{2j+1}(q^{2j+3};q^2)_{2i-1}}z^j\\
&-(1-q^{4i+1})(q^{n+1-2i};q^2)_{2i}\sum_{j\ge0}\frac{q^{2(i+j)^2+2j}(q^{n-2i-2j};q)_{2j}}{(q;q)_{2j+1}(q^{2j+3};q^2)_{2i}}z^j\\
&=(q^{n+1-2i};q^2)_{2i}\sum_{j\ge0}\frac{q^{2(i+j)^2}(q^{n+1-2i-2j};q)_{2j-1}(1-q^{n-2i})(1-q^{2j+4i+1})}{(q;q)_{2j+1}(q^{2j+3};q^2)_{2i}}z^j\\
&-(1-q^{4i+1})(q^{n+1-2i};q^2)_{2i}\sum_{j\ge0}\frac{q^{2(i+j)^2+2j}(q^{n+1-2i-2j};q)_{2j-1}(1-q^{n-2i-2j})}{(q;q)_{2j+1}(q^{2j+3};q^2)_{2i}}z^j\\
&=(q^{n+1-2i};q^2)_{2i}\sum_{j\ge0}\frac{q^{2(i+j)^2}(q^{n+1-2i-2j};q)_{2j-1}}{(q;q)_{2j+1}(q^{2j+3};q^2)_{2i}}z^j\\
&\qquad\times\Big[(1-q^{n-2i})(1-q^{2j+4i+1})-q^{2j}(1-q^{4i+1})(1-q^{n-2i-2j})\Big]\\
&=(q^{n+1-2i};q^2)_{2i}\sum_{j\ge0}\frac{q^{2(i+j)^2}(q^{n+1-2i-2j};q)_{2j-1}}{(q;q)_{2j+1}(q^{2j+3};q^2)_{2i}}z^j(1-q^{2j})(1-q^{n+1+2i})\\
&=(q^{n+1-2i};q^2)_{2i+1}\sum_{j\ge1}\frac{q^{2(i+j)^2}(q^{n+1-2i-2j};q)_{2j-1}}{(q;q)_{2j+1}(q^{2j+3};q^2)_{2i}}z^j(1-q^{2j})\\
&=z(q^{n-1-2i};q^2)_{2i+2}\sum_{j\ge0}\frac{q^{2(i+j+1)^2}(q^{n-1-2i-2j};q)_{2j}}{(q;q)_{2j+1}(q^{2j+3};q^2)_{2i+1}}z^j=zS_{2i+1}.
\end{aligned}$$

Further,

$$S_{-1}=\sum_{j\ge0}\frac{q^{2j^2}(q^{n+1-2j};q)_{2j}}{(q;q)_{2j+1}(q^{2j+3};q^2)_{-1}}z^j=\sum_{j\ge0}\frac{q^{2j^2}(q^{n+1-2j};q)_{2j}}{(q;q)_{2j}}z^j=s_0(z)$$

and

$$S_0=\sum_{j\ge0}\frac{q^{2j(j+1)}(q^{n-2j};q)_{2j}(1-q^n)}{(q;q)_{2j+1}}z^j=\sum_{j\ge0}\frac{q^{2j(j+1)}(q^{n-2j};q)_{2j+1}}{(q;q)_{2j+1}}z^j=t_0(z).$$

Now we can iterate this relation in the following form:

$$\frac{zt(n)}{s(n)}=\frac{zS_0}{S_{-1}}=\cfrac{z}{a_0+\cfrac{zS_1}{S_0}}=\cfrac{z}{a_0+\cfrac{z}{a_1+\cfrac{zS_2}{S_1}}}=\dots$$

This is the desired (finite) continued fraction expansion. □

We remark that for $n\to\infty$, the quantities a_i and S_i appear already in [9].

4. Conclusion

We would like to mention that it is more challenging to find the continued fractions and the relevant quantities, as the present proofs (and possibly other ones) consist of routine manipulations.

Since there are many Rogers-Ramanujan type identities and polynomials approximating them are not even unique, there might be additional additional results; compare our previous effort [9] for infinite versions. Most polynomials from Sill's list [10] are, however, not expressable in terms of *one* summation and thus not candidates for the present approach.

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