Results in Nonlinear Analysis 1 (2018) No. 2, 74–87
Available online at www.nonlinear-analysis.com
Results in Nonlinear Analysis
ISSN 2636-7556
Peer Reviewed Scientific Journal

Existence results for Navier problems with degenerated (p,q)-Laplacian and (p,q)-Biharmonic operators

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Abstract

In this article, we prove the existence and uniqueness of solutions for the Navier problem

$$(P)\begin{cases} \Delta\big[\omega(x)(\,|\Delta u|^{p-2}\Delta u+|\Delta u|^{q-2}\Delta u)\big] \\ -\operatorname{div}\big[\omega(x)(\,|\nabla u|^{p-2}\nabla u+|\nabla u|^{q-2}\nabla u)\big] \\ = f(x)-\operatorname{div}(G(x)), \quad \text{in} \quad \Omega, \\ u(x)=\Delta u=0, \quad \text{in} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N\geq 2)$, $\dfrac{f}{\omega}\in L^{p'}(\Omega,\omega)$ and $\dfrac{G}{\omega}\in [L^{q'}(\Omega,\omega)]^N$.

Keywords: Degenerate nonlinear elliptic equations; weighted Sobolev space.
2010 MSC: 35J60, 35J70.

1. Introduction and Preliminaries

The main purpose of this paper (see Theorem 3.3) is to establish the existence and uniqueness of solutions for the Navier problem

$$(P)\begin{cases} Lu(x)=f(x)-\operatorname{div}(G(x)), \quad \text{in} \quad \Omega, \\ u(x)=\Delta u(x)=0, \quad \text{in} \quad \partial\Omega, \end{cases}$$

where

$$\begin{aligned} Lu(x) &= \Delta\big[\omega(x)(\,|\Delta u|^{p-2}\Delta u+|\Delta u|^{q-2}\Delta u)\big] \\ &- \operatorname{div}\big[\omega(x)(\,|\nabla u|^{p-2}\nabla u+|\nabla u|^{q-2}\nabla u)\big], \end{aligned}$$

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Received June 04, 2018, Accepted August 08, 2018, Online August 09, 2018.

$\Omega \subset \mathbb{R}^N$ is a bounded open set, $\dfrac{f}{\omega} \in L^{p'}(\Omega,\omega)$, $\dfrac{G}{\omega} \in [L^{q'}(\Omega,\omega)]^N$, ω is a weight function (i.e., a locally integrable function on $\mathbb{R}^N$ such that $0 < \omega(x) < \infty$ a.e. $x \in \mathbb{R}^N$), Δ is the Laplacian operator, $1 < q < p < \infty$, $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1], [4], [5], [7], [8] and [11]). The type of a weight depends on the equation type.

A class of weights, which is particularly well understood, is the class of A_p weights that was introduced by B.Muckenhoupt in the early 1970's (see [8]). These classes have found many useful applications in harmonic analysis (see [9] and [10]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^N$ often belong to A_p (see [3] and [11]). There are, in fact, many interesting examples of weights (see [7] for p-admissible weights).

In the non-degenerate case (i.e. with $\omega(x) \equiv 1$), for all $f \in L^p(\Omega)$ the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x), \text{ in } \Omega \\ u(x) = 0, \text{ in } \partial\Omega \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [6]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x), \text{ in } \Omega \\ u(x) = 0, \text{ in } \partial\Omega \end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [2]), where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. In the degenerate case, the degenerated p-Laplacian has been studied in [3].

The paper is organized as follow. In Section 2 we present the definitions and basic results. In Section 3 we prove our main result about existence and uniqueness of solutions for problem (P).

2. Definitions and Basic Results

By a weight we shall mean a locally integrable function ω on $\mathbb{R}^N$ such that $0 < \omega(x) < \infty$ for a.e. $x \in \mathbb{R}^N$. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^N$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \displaystyle\int_E \omega(x)\,dx$ for measurable sets $E \subset \mathbb{R}^N$.

Definition 2.1. Let $1 \le p < \infty$. A weight ω is said to be an A_p-weight, if there is a positive constant C such that, for every ball $B \subset \mathbb{R}^N$

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)\,dx\right)^{p-1} \le C, \quad \text{if} \quad p > 1,$$

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\operatorname*{ess\,sup}_{x \in B} \frac{1}{\omega(x)}\right) \le C, \quad \text{if} \quad p = 1,$$

where $|.|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$. The infimum over all such constants C is called the A_p - constant of ω and is denotaded by $C_{p,\omega}$.

If $1 < q \le p$, then $A_q \subset A_p$ (see [5], [7] or [11] for more information about A_p- weights). As an example of an A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, is in A_p if and only if $-N < \alpha < N(p-1)$ (see [11], Chapter IX, Corollary 4.4). If $\varphi \in BMO(\mathbb{R}^N)$, then $\omega(x) = \mathrm{e}^{\alpha\,\varphi(x)} \in A_2$ for some $\alpha > 0$ (see [9]).

Remark 2.2. If $\omega \in A_p$, $1 < p < \infty$, then

$$\left(\frac{|E|}{|B|}\right)^p \le C_{p,\omega}\,\frac{\mu(E)}{\mu(B)}$$

for all measurable subsets E of B (see *15.5 strong doubling property* in [7]). Therefore, $\mu(E) = 0$ if and only if $|E| = 0$; so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e.. □

Definition 2.3. Let ω be a weight. We shall denote by $L^p(\Omega,\omega)$ $(1 \le p < \infty)$ the Banach space of all measurable functions f defined in Ω for which

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)\, dx \right)^{1/p} < \infty.$$

We denote $[L^p(\Omega,\omega)]^N = L^p(\Omega,\omega) \times ... \times L^p(\Omega,\omega)$.

Remark 2.4. If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega,\omega) \subset L^1_{\text{loc}}(\Omega)$ (see [11], Remark 1.2.4). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.5. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $1 < p < \infty$, k be a nonnegative integer and $\omega \in A_p$. We shall denote by $W^{k,p}(\Omega,\omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$, $1 \le |\alpha| \le k$. The norm in the space $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)\, dx + \sum_{1 \le |\alpha| \le k} \int_\Omega |D^\alpha u(x)|^p \omega(x)\, dx \right)^{1/p}. \tag{2.1}$$

We also define the space $W^{k,p}_0(\Omega,\omega)$ as the closure of $C^\infty_0(\Omega)$ with respect to the norm (2.1). We have that the spaces $W^{k,p}(\Omega,\omega)$ and $W^{k,p}_0(\Omega,\omega)$ are Banach spaces (see Proposition 2.1.2 in [11]). The dual space of $W^{1,p}_0(\Omega,\omega)$ is the space $[W^{1,p}_0(\Omega,\omega)]^* = W^{-1,p'}(\Omega,\omega)$,

$$W^{-1,p'}(\Omega,\omega) = \{T = f - \text{div}(G) : G = (g_1, ..., g_N), \frac{f}{\omega}, \frac{g_j}{\omega} \in L^{p'}(\Omega,\omega)\}.$$

It is evident that a weight function ω which satisfies $0 < C_1 \le \omega(x) \le C_2$, for a.e. $x \in \Omega$, gives nothing new (the space $\mathrm{W}^{k,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $\mathrm{W}^{k,p}(\Omega)$). Consequently, we shall be interested in all above such weight functions ω which either vanish somewhere in $\Omega \cup \partial\Omega$ or increase to infinity (or both).

We need the following basics results.

Theorem 2.6. *(The weighted Sobolev inequality) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let ω be an A_p-weight, $1 < p < \infty$. Then there exists positive constants C_Ω and δ such that for all $f \in C^\infty_0(\Omega)$ and $1 \le \eta \le N/(N-1) + \delta$*

$$\|f\|_{L^{\eta p}(\Omega,\omega)} \le C_\Omega \||\nabla f|\|_{L^p(\Omega,\omega)}, \tag{2.2}$$

where C_Ω may be taken to depend only on N, the A_p - constant of ω, p and the diameter of Ω.

Proof. See [4], Theorem 1.3. □

Lemma 2.7. *(a) Let $1 < p < \infty$, then exists a constant C_p such that for all $\xi, \eta \in \mathbb{R}^N$*

$$|\, |\xi|^{p-2}\xi - |\eta\,|^{p-2}\eta| \le C_p\, |\xi - \eta|\, (|\xi| + |\eta|)^{p-2}.$$

(b) Let $1 < p < \infty$. There exist two positive constants α_p and β_p such that for every $\xi, \eta \in \mathbb{R}^N$ $(N \ge 1)$

$$\alpha_p(\,|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \le \langle\, |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\rangle \le \beta_p(\,|\xi| + |\eta|)^{p-2}|\xi - \eta|^2,$$

where $\langle .,. \rangle$ denotes here the Euclidean scalar product in $\mathbb{R}^N$.

Proof. See Proposition 17.2 and Proposition 17.3 in [2]. □

3. Main Results

We denote by $X = W^{2,p}(\Omega,\omega) \cap W_0^{1,p}(\Omega,\omega)$ with the norm

$$\|u\|_X = \left(\int_\Omega |\nabla u|^p \,\omega\, dx + \int_\Omega |\Delta u|^p \,\omega\, dx \right)^{1/p}.$$

In this section we prove the existence and uniqueness of weak solutions $u \in X$ to the Navier problem

$$(P) \begin{cases} Lu(x) = f(x) - \operatorname{div}(G(x)), & \text{in } \Omega, \\ u(x) = \Delta u = 0, & \text{in } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, $\dfrac{f}{\omega} \in L^{p'}(\Omega,\omega)$ and $\dfrac{G}{\omega} \in [L^{q'}(\Omega,\omega)]^N$, $G = (g_1, ..., g_N)$.

Definition 3.1. We say that $u \in X$ is a weak solution for problem (P) if

$$\begin{aligned} &\int_\Omega |\Delta u|^{p-2} \Delta u \, \Delta\varphi \,\omega\, dx + \int_\Omega |\Delta u|^{q-2} \Delta u \, \Delta\varphi \,\omega\, dx \\ &+ \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla\varphi\rangle \,\omega\, dx + \int_\Omega |\nabla u|^{q-2} \langle \nabla u, \nabla\varphi\rangle \,\omega\, dx \\ &= \int_\Omega f\,\varphi\, dx + \int_\Omega \langle G, \nabla\varphi\rangle\, dx, \end{aligned} \tag{3.1}$$

for all $\varphi \in X$, with $f/\omega \in L^{p'}(\Omega,\omega)$ and $G/\omega \in [L^{q'}(\Omega,\omega)]^N$, where $\langle .,. \rangle$ denotes here the Euclidean scalar product in $\mathbb{R}^N$.

Remark 3.2. (i) Since $1 < q < p < \infty$, there exists a constant $C_{p,q} = [\mu(\Omega)]^{(p-q)/pq}$ such that $\|u\|_{L^q(\Omega,\omega)} \leq C_{p,q} \|u\|_{L^p(\Omega,\omega)}$, where $\mu(\Omega) = \int_\Omega \omega(x)\, dx$.
(ii) Since $1 < q < p < \infty$, then $1 < p' < q' < \infty$, and there exists a constant $\tilde{C}_{p,q} = [\mu(\Omega)]^{(q'-p')/q'p'}$ such that $\|u\|_{L^{p'}(\Omega,\omega)} \leq \tilde{C}_{p,q} \|u\|_{L^{q'}(\Omega,\omega)}$.
(iii) By (ii), if $G/\omega \in [L^{q'}(\Omega,\omega)]^N$, then $G/\omega \in [L^{p'}(\Omega,\omega)]^N$.
Hence, $T = f - \operatorname{div}(G) \in [W_0^{1,p}(\Omega,\omega)]^*$.

Theorem 3.3. *Let $\omega \in A_p$, $1 < q < p < \infty$, $\dfrac{f}{\omega} \in L^{p'}(\Omega,\omega)$ and $\dfrac{G}{\omega} \in [L^{q'}(\Omega,\omega)]^N$. Then the problem (P) has a unique solution $u \in X$ and*

$$\|u\|_X \leq \left[C_\Omega \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega,\omega)} \right]^{1/(p-1)},$$

where C_Ω is the constant in Theorem 2.6 and $C_{p,q}$ is the constant in Remark 3.2 (i).

Proof. (I) *Existence.* By Theorem 2.6 (with $\eta = 1$), we have that

$$\begin{aligned} \left| \int_\Omega f\,\varphi\, dx \right| &\leq \left(\int_\Omega \left| \frac{f}{\omega} \right|^{p'} \omega\, dx \right)^{1/p'} \left(\int_\Omega |\varphi|^p \,\omega\, dx \right)^{1/p} \\ &\leq C_\Omega \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega,\omega)} \| \, |\nabla\varphi| \, \|_{L^p(\Omega,\omega)} \\ &\leq C_\Omega \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega,\omega)} \|\varphi\|_X, \end{aligned} \tag{3.2}$$

and by Remark 3.2(i)

$$\begin{aligned}\left|\int_\Omega \langle G, \nabla\varphi\rangle\, dx\right| &\leq \int_\Omega |\langle G, \nabla\varphi\rangle|\, dx \\ &\leq \int_\Omega |G||\nabla\varphi|\, dx \\ &= \int_\Omega \frac{|G|}{\omega}|\nabla\varphi|\,\omega\, dx \\ &\leq \left\|\frac{|G|}{\omega}\right\|_{L^{q'}(\Omega,\omega)} \|\,|\nabla\varphi|\,\|_{L^q(\Omega,\omega)} \\ &\leq C_{p,q}\left\|\frac{|G|}{\omega}\right\|_{L^{q'}(\Omega,\omega)} \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} \\ &\leq C_{p,q}\left\|\frac{|G|}{\omega}\right\|_{L^{q'}(\Omega,\omega)} \|\varphi\|_X. \end{aligned} \tag{3.3}$$

Define the functional $J : X \to \mathbb{R}$ by

$$\begin{aligned} J(\varphi) &= \frac{1}{p}\int_\Omega |\Delta\varphi|^p\,\omega\, dx + \frac{1}{q}\int_\Omega |\Delta\varphi|^q\,\omega\, dx \\ &+ \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\, dx + \frac{1}{q}\int_\Omega |\nabla\varphi|^q\,\omega\, dx - \int_\Omega f\,\varphi\, dx - \int_\Omega \langle G, \nabla\varphi\rangle\, dx. \end{aligned}$$

Using (3.2), (3.3), Remark 3.2(i) and Young's inequality, we have that

$$\begin{aligned} J(\varphi) &\geq \frac{1}{p}\int_\Omega |\Delta\varphi|^p\,\omega\, dx + \frac{1}{q}\int_\Omega |\Delta\varphi|^q\,\omega\, dx \\ &+ \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\, dx + \frac{1}{q}\int_\Omega |\nabla\varphi|^q\,\omega\, dx \\ &- \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\varphi\|_{L^p(\Omega,\omega)} - \left\|\frac{|G|}{\omega}\right\|_{L^{q'}(\Omega,\omega)} \|\,|\nabla\varphi|\,\|_{L^q(\Omega,\omega)} \\ &\geq \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\, dx + \frac{1}{q}\int_\Omega |\nabla\varphi|^q\,\omega\, dx \\ &- C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} \\ &- \left\|\frac{|G|}{\omega}\right\|_{L^{q'}(\Omega,\omega)} \|\,|\nabla\varphi|\,\|_{L^q(\Omega,\omega)} \\ &\geq \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\, dx + \frac{1}{q}\int_\Omega |\nabla\varphi|^q\,\omega\, dx \\ &- \frac{C_\Omega^{p'}}{p'}\left\|\frac{f}{\omega}\right\|^{p'}_{L^{p'}(\Omega,\omega)} - \frac{1}{p}\|\,|\nabla\varphi|\,\|^p_{L^p(\Omega,\omega)} - \frac{1}{q'}\left\|\frac{|G|}{\omega}\right\|^{q'}_{L^{p'}(\Omega,\omega)} \\ &- \frac{1}{q}\|\,|\nabla\varphi|\,\|^q_{L^q(\Omega,\omega)} \\ &= -\frac{C_\Omega^{p'}}{p'}\left\|\frac{f}{\omega}\right\|^{p'}_{L^{p'}(\Omega,\omega)} - \frac{1}{q'}\left\|\frac{|G|}{\omega}\right\|^{q'}_{L^{q'}(\Omega,\omega)} \end{aligned}$$

that is, J is bounded from below. Let $\{u_n\}$ be a minimizing sequence, that is, a sequence such that

$$J(u_n) \to \inf_{\varphi \in X} J(\varphi).$$

Then for n large enough, we obtain

$$\begin{aligned} 0 \geq J(u_n) &= \frac{1}{p}\int_\Omega |\Delta u_n|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\Delta u_n|^q\,\omega\,dx \\ &+ \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\nabla u_n|^q\,\omega\,dx \\ &- \int_\Omega f\,u_n\,dx - \int_\Omega \langle G, \nabla u_n\rangle\,dx, \end{aligned}$$

and we have

$$\begin{aligned} &\frac{1}{p}\int_\Omega |\Delta u_n|^p\,\omega\,dx + \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\omega\,dx \\ &\leq \frac{1}{p}\int_\Omega |\Delta u_n|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\Delta u_n|^q\,\omega\,dx + \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\omega\,dx \\ &+ \frac{1}{q}\int_\Omega |\nabla u_n|^q\,\omega\,dx \\ &\leq \int_\Omega f\,u_n\,dx + \int_\Omega \langle G, u_n\rangle\,dx. \end{aligned}$$

Hence, by Theorem 2.6 (with $\eta = 1$) and Remark **??**(i), we obtain

$$\begin{aligned} \|u_n\|_X^p &= \int_\Omega |\Delta u_n|^p\,\omega\,dx + \int_\Omega |\nabla u_n|^p\,\omega\,dx \\ &\leq p\left(\int_\Omega f\,u_n\,dx + \int_\Omega \langle G, \nabla u_n\rangle\,dx\right) \\ &\leq p\left(\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|u_n\|_{L^p(\Omega,\omega)} + \left\|\frac{|G|}{\omega}\right\|_{L^{q'}(\Omega,\omega)} \|\,|\nabla u_n|\,\|_{L^q(\Omega,\omega)}\right) \\ &\leq p\left(C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\,|\nabla u_n|\,\|_{L^p(\Omega,\omega)} + C_{p,q}\left\|\frac{|G|}{\omega}\right\|_{L^{q'}(\Omega,\omega)} \|\,|\nabla u_n|\,\|_{L^p(\Omega,\omega)}\right) \\ &\leq p\left(C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q}\left\|\frac{|G|}{\omega}\right\|_{L^{q'}(\Omega,\omega)}\right)\|u_n\|_X. \end{aligned}$$

Hence,

$$\|u_n\|_X \leq \left[p\left(C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q}\left\|\frac{|G|}{\omega}\right\|_{L^{q'}(\Omega,\omega)}\right)\right]^{1/(p-1)}.$$

Therefore $\{u_n\}$ is bounded in X. Since X is reflexive, there exists a subsequence, still denoted by $\{u_n\}$, and a function $u \in X$ such that $u_n \rightharpoonup u$ in X. Since,

$$X \ni \varphi \mapsto \int_\Omega f\,\varphi\,dx + \int_\Omega \langle G, \nabla\varphi\rangle\,dx,$$

and

$$X \ni \varphi \mapsto \|\Delta\varphi\|_{L^p(\Omega,\omega)} + \|\Delta\varphi\|_{L^q(\Omega,\omega)} + \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} + \|\,|\nabla\varphi|\,\|_{L^q(\Omega,\omega)},$$

are continuous then J is continuous. Moreover since $1 < q < p < \infty$ we have that J is convex and thus lower semi-continuous for the weak convergence. It follows that

$$J(u) \leq \liminf_n J(u_n) = \inf_{\varphi \in X} J(\varphi),$$

and thus u is a minimizer of J on X (see Theorem 25.C and Corollary 25.15 in [12]). For any $\varphi\in X$ the function

$$\begin{aligned}\lambda \mapsto \quad & \frac{1}{p}\int_\Omega |\Delta(u+\lambda\varphi)|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\Delta(u+\lambda\varphi)|^q\,\omega\,dx \\ & +\frac{1}{p}\int_\Omega |\nabla(u+\lambda\,\varphi)|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\nabla(u+\lambda\varphi)|^q\,\omega\,dx \\ & -\int_\Omega (u+\lambda\,\varphi)\,f\,dx - \int_\Omega \langle G, \nabla(u+\lambda\,\varphi)\rangle\,dx\end{aligned}$$

has a minimum at $\lambda=0$. Hence,

$$\frac{d}{d\lambda}\Big(J(u+\lambda\,\varphi)\Big)\Big|_{\lambda=0} = 0,\ \forall\,\varphi\in X.$$

We have

$$\frac{d}{d\,\lambda}\Big(|\,\nabla(u+\lambda\,\varphi)|^p\,\omega\Big) = p\,\{|\nabla(u+\lambda\,\varphi)|^{p-2}(\langle\nabla u,\nabla\varphi\rangle+\lambda\,|\nabla\varphi|^2)\}\,\omega,$$

and

$$\frac{d}{d\,\lambda}\Big(|\,\Delta(u+\lambda\,\varphi)|^p\,\omega\Big) = p\,|\Delta u+\lambda\Delta\varphi|^{p-2}(\,\Delta u+\lambda\Delta\varphi)\,\Delta\varphi\,\omega.$$

Then we obtain

$$\begin{aligned} 0 &= \frac{d}{d\lambda}\Big(J(u+\lambda\,\varphi)\Big)\Big|_{\lambda=0} \\ &= \Big[\frac{1}{p}\Big(p\int_\Omega |\nabla(u+\lambda\,\varphi)|^{p-2}(\langle\nabla u,\nabla\varphi\rangle+\lambda\,|\nabla\varphi|^2)\,\omega\,dx \\ &+ p\int_\Omega |\Delta u+\lambda\Delta\varphi|^{p-2}(\,\Delta u+\lambda\Delta\varphi)\,\Delta\varphi\,\omega\,dx\Big) \\ &+ \frac{1}{q}\Big(q\int_\Omega |\nabla(u+\lambda\,\varphi)|^{q-2}(\langle\nabla u,\nabla\varphi\rangle+\lambda\,|\nabla\varphi|^2)\,\omega\,dx \\ &+ q\int_\Omega |\Delta u+\lambda\Delta\varphi|^{q-2}(\,\Delta u+\lambda\Delta\varphi)\,\Delta\varphi\,\omega\,dx\Big) \\ &- \int_\Omega \varphi\,f\,dx - \int_\Omega \langle G,\nabla\varphi\rangle\,dx\Big]\Big|_{\lambda=0} \\ &= \int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx + \int_\Omega |\nabla u|^{p-2}\,\langle\nabla u,\nabla\varphi\rangle\,\omega\,dx \\ &+ \int_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi\,\omega\,dx + \int_\Omega |\nabla u|^{q-2}\,\langle\nabla u,\nabla\varphi\rangle\,\omega\,dx \\ &- \int_\Omega f\,\varphi\,dx - \int_\Omega \langle G,\nabla\varphi\rangle\,dx. \end{aligned}$$

Therefore

$$\begin{aligned} &\int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx + \int_\Omega |\nabla u|^{p-2}\langle\nabla u\,\nabla\varphi\rangle\,\omega\,dx \\ &+\int_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi\,\omega\,dx + \int_\Omega |\nabla u|^{q-2}\,\langle\nabla u,\nabla\varphi\rangle\,\omega\,dx \\ &=\int_\Omega f\,\varphi\,dx + \int_\Omega \langle G,\nabla\varphi\rangle\,dx, \end{aligned}$$

for all $\varphi\in X$, that is, $u\in X$ is a solution of problem (P).

(II) *Uniqueness.* If $u_1, u_2 \in X$ are two weak solutions of problem (P), we have for all $\varphi \in X$,

$$\int_\Omega |\Delta u_1|^{p-2}\Delta u_1\,\Delta\varphi\,\omega\,dx + \int_\Omega |\Delta u_1|^{q-2}\Delta u_1\,\Delta\varphi\,\omega\,dx + \int_\Omega |\nabla u_1|^{p-2}\,\langle\nabla u_1,\nabla\varphi\rangle\,\omega\,dx + \int_\Omega |\nabla u_1|^{q-2}\,\langle\nabla u_1,\nabla\varphi\rangle\,\omega\,dx = \int_\Omega f\,\varphi\,dx + \int_\Omega \langle G,\nabla\varphi\rangle\,dx,$$

and

$$\int_\Omega |\Delta u_2|^{p-2}\Delta u_2\,\Delta\varphi\,\omega\,dx + \int_\Omega |\Delta u_2|^{q-2}\Delta u_2\,\Delta\varphi\,\omega\,dx + \int_\Omega |\nabla u_2|^{p-2}\,\langle\nabla u_2,\nabla\varphi\rangle\,\omega\,dx + \int_\Omega |\nabla u_2|^{q-2}\,\langle\nabla u_2,\nabla\varphi\rangle\,\omega\,dx = \int_\Omega f\,\varphi\,dx + \int_\Omega \langle G,\nabla\varphi\rangle\,dx.$$

Hence

$$\begin{aligned}
&\int_\Omega \Big(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta u_2|^{p-2}\Delta u_2\Big)\Delta\varphi\,\omega\,dx \\
&+ \int_\Omega \Big(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2\Big)\Delta\varphi\,\omega\,dx \\
&+ \int_\Omega \Big(|\nabla u_1|^{p-2}\langle\nabla u_1,\nabla\varphi\rangle - |\nabla u_2|^{p-2}\langle\nabla u_2,\nabla\varphi\rangle\Big)\,\omega\,dx \\
&+ \int_\Omega \Big(|\nabla u_1|^{q-2}\langle\nabla u_1,\nabla\varphi\rangle - |\nabla u_2|^{q-2}\langle\nabla u_2,\nabla\varphi\rangle\Big)\,\omega\,dx = 0.
\end{aligned}$$

Taking $\varphi = u_1 - u_2$, and using Lemma 2.7(b) there exist positive constants $\alpha_p, \tilde\alpha_p, \alpha_q, \tilde\alpha_q$ such that

$$\begin{aligned}
0 &= \int_\Omega \Big(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta u_2|^{p-2}\Delta u_2\Big)(\,\Delta u_1 - \Delta u_2)\,\omega\,dx \\
&+ \int_\Omega \Big(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2\Big)(\,\Delta u_1 - \Delta u_2)\,\omega\,dx \\
&+ \int_\Omega \Big(|\nabla u_1|^{p-2}\langle\nabla u_1,\nabla u_1-\nabla u_2\rangle - |\nabla u_2|^{p-2}\langle\nabla u_2,\nabla u_1-\nabla u_2\rangle\Big)\,\omega\,dx \\
&+ \int_\Omega \Big(|\nabla u_1|^{q-2}\langle\nabla u_1,\nabla u_1-\nabla u_2\rangle - |\nabla u_2|^{q-2}\langle\nabla u_2,\nabla u_1-\nabla u_2\rangle\Big)\,\omega\,dx \\
&= \int_\Omega \Big(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta u_2|^{p-2}\Delta u_2\Big)(\,\Delta u_1 - \Delta u_2)\,\omega\,dx \\
&+ \int_\Omega \Big(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2\Big)(\,\Delta u_1 - \Delta u_2)\,\omega\,dx \\
&+ \int_\Omega \langle\,|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla u_1 - \nabla u_2\rangle\,\omega\,dx \\
&+ \int_\Omega \langle\,|\nabla u_1|^{q-2}\nabla u_1 - |\nabla u_2|^{q-2}\nabla u_2, \nabla u_1 - \nabla u_2\rangle\,\omega\,dx \\
&\ge \alpha_p\int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{p-2}|\Delta u_1 - \Delta u_2|^2\,\omega\,dx \\
&+ \tilde\alpha_p\int_\Omega \Big(|\nabla u_1| + |\nabla u_2|\Big)^{p-2}|\nabla u_1 - \nabla u_2|^2\,\omega\,dx \\
&+ \alpha_q\int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{q-2}|\Delta u_1 - \Delta u_2|^2\,\omega\,dx \\
&+ \tilde\alpha_q\int_\Omega \Big(|\nabla u_1| + |\nabla u_2|\Big)^{q-2}|\nabla u_1 - \nabla u_2|^2\,\omega\,dx.
\end{aligned}$$

Therefore $\Delta u_1 = \Delta u_2$ and $\nabla u_1 = \nabla u_2$ a.e. and since $u_1, u_2 \in X$, then $u_1 = u_2$ a.e. (by Remark 2.2).
(III) *Estimate for* $\|u\|_X$.
In particular, for $\varphi = u \in X$ in Definition 3.1 we have

$$\int_\Omega |\Delta u|^p \,\omega\, dx + \int_\Omega |\Delta u|^q \,\omega\, dx + \int_\Omega |\nabla u|^p \,\omega\, dx + \int_\Omega |\nabla u|^q \,\omega\, dx$$
$$= \int_\Omega f\, u\, dx + \int_\Omega \langle G, \nabla u\rangle\, dx.$$

Then, by Theorem 2.6 and Remark 3.2(i), we obtain

$$\begin{aligned}
\|u\|_X^p &= \int_\Omega |\Delta u|^p \,\omega\, dx + \int_\Omega |\nabla u|^p \,\omega\, dx \\
&\leq \int_\Omega |\Delta u|^p \,\omega\, dx + \int_\Omega |\Delta u|^q \,\omega\, dx + \int_\Omega |\nabla u|^p \,\omega\, dx + \int_\Omega |\nabla u|^q \,\omega\, dx \\
&= \int_\Omega f\, u\, dx + \int_\Omega \langle G, \nabla u\rangle\, dx \\
&\leq \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|u\|_{L^p(\Omega,\omega)} + \left\|\frac{|G|}{\omega}\right\|_{L^{q'}(\Omega,\omega)} \|\, |\nabla u|\, \|_{L^q(\Omega,\omega)} \\
&\leq C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\nabla u\|_{L^p(\Omega,\omega)} + C_{p,q} \left\|\frac{|G|}{\omega}\right\|_{L^{q'}(\Omega,\omega)} \|\, |\nabla u|\, \|_{L^p(\Omega,\omega)} \\
&\leq \left(C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\|\frac{|G|}{\omega}\right\|_{L^{q'}(\Omega,\omega)} \right) \|u\|_X.
\end{aligned}$$

Therefore,

$$\|u\|_X \leq \left(C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\|\frac{|G|}{\omega}\right\|_{L^{q'}(\Omega,\omega)} \right)^{1/(p-1)}.$$

□

Corollary 3.4. *Under the assumptions of Theorem 3.3 with* $2 \leq q < p < \infty$. *If* $u_1, u_2 \in X$ *are solutions of*

$$(P_1) \begin{cases} Lu_1(x) = f(x) - \mathrm{div}(G(x)), & \text{in } \Omega, \\ u_1(x) = \Delta u_1(x) = 0, & \text{in } \partial\Omega, \end{cases}$$

and

$$(P_2) \begin{cases} Lu_2(x) = \tilde{f}(x) - \mathrm{div}(\tilde{G}(x)), & \text{in } \Omega, \\ u_2(x) = \Delta u_2(x) = 0, & \text{in } \partial\Omega, \end{cases}$$

then

$$\|u_1 - u_2\|_X \leq \frac{1}{\gamma^{1/(p-1)}} \left(C_\Omega \left\|\frac{f - \tilde{f}}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\|\frac{|G - \tilde{G}|}{\omega}\right\|_{L^{q'}(\Omega,\omega)} \right)^{1/(p-1)},$$

where γ *is a positive constant,* C_Ω *and* $C_{p,q}$ *are the same constants of Theorem 3.3.*

Proof. If u_1 and u_2 are solutions of $(P1)$ and $(P2)$ then for all $\varphi \in X$ we have

$$\begin{aligned}
&\int_\Omega |\Delta u_1|^{p-2}\Delta u_1\, \Delta\varphi\, \omega\, dx + \int_\Omega |\Delta u_1|^{q-2}\Delta u_1\, \Delta\varphi\, \omega\, dx \\
&+ \int_\Omega |\nabla u_1|^{p-2}\langle \nabla u_1, \nabla\varphi\rangle\, \omega\, dx + \int_\Omega |\nabla u_1|^{q-2}\, \langle \nabla u_1, \nabla\varphi\rangle\, \omega\, dx \\
&-\Big(\int_\Omega |\Delta u_2|^{p-2}\Delta u_2\, \Delta\varphi\, \omega\, dx + \int_\Omega |\Delta u_2|^{q-2}\Delta u_2\, \Delta\varphi\, \omega\, dx \\
&+ \int_\Omega |\nabla u_1|^{p-2}\langle \nabla u_2, \nabla\varphi\rangle\, \omega\, dx + \int_\Omega |\nabla u_2|^{q-2}\, \langle \nabla u_2, \nabla\varphi\rangle\, \omega\, dx \Big) \\
= \ &\int_\Omega (f-\tilde f)\, \varphi\, dx + \int_\Omega \langle G-\tilde G, \nabla\varphi\rangle\, dx. \qquad (3.4)
\end{aligned}$$

In particular, for $\varphi = u_1 - u_2$, we obtain
(i) Since $2 \le q < p < \infty$ and by Lemma 2.7(b), there exist two positive constants α_p and α_q such that

$$\begin{aligned}
&\int_\Omega \Big(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta_2|^{p-2}\Delta u_2 \Big)\, \Delta(u_1-u_2)\, \omega\, dx \\
&\ge \alpha_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2| \Big)^{p-2} |\Delta u_1 - \Delta u_2|^2\, \omega\, dx \\
&\ge \alpha_p \int_\Omega |\Delta u_1 - \Delta u_2|^{p-2} |\Delta u_1 - \Delta u_2|^2\, \omega\, dx = \alpha_p \int_\Omega |\Delta(u_1-u_2)|^p\, \omega\, dx,
\end{aligned}$$

and analogously

$$\int_\Omega \Big(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2 \Big)\, \Delta(u_1-u_2)\, \omega\, dx \ge \alpha_q \int_\Omega |\Delta(u_1-u_2)|^q\, \omega\, dx \ge 0.$$

(ii) Since $2 \le q < p < \infty$ and by Lemma 2.7(b), there exit two positive constants $\tilde\alpha_p$ and $\tilde\alpha_q$ such that

$$\begin{aligned}
&\int_\Omega \Big(|\nabla u_1|^{p-2}\langle \nabla u_1, \nabla(u_1-u_2)\rangle - |\nabla u_2|^{p-2}\langle \nabla u_2, \nabla(u_1-u_2)\rangle \Big)\, \omega\, dx \\
&= \int_\Omega \langle |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla(u_1-u_2)\rangle\, \omega\, dx \\
&\ge \tilde\alpha_p \int_\Omega (|\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla u_1 - \nabla u_2|^2\, \omega\, dx \\
&\ge \tilde\alpha_p \int_\Omega |\nabla u_1 - \nabla u_2|^{p-2}\, |\nabla u_1 - \nabla u_2|^2\, \omega\, dx = \tilde\alpha_p \int_\Omega |\nabla(u_1-u_2)|^p\, \omega\, dx,
\end{aligned}$$

and analogously,

$$\begin{aligned}
&\int_\Omega \Big(|\nabla u_1|^{q-2}\langle \nabla u_1, \nabla(u_1-u_2)\rangle - |\nabla u_2|^{q-2}\langle \nabla u_2, \nabla(u_1-u_2)\rangle \Big)\, \omega\, dx \\
&\ge \tilde\alpha_q \int_\Omega |\nabla(u_1-u_2)|^q\, \omega\, dx \ge 0.
\end{aligned}$$

(iii) By Remark 3.2(i) we have

$$\begin{aligned}
&\left| \int_\Omega (f-\tilde f)\, (u_1-u_2)\, dx + \int_\Omega \langle G-\tilde G, \nabla(u_1-u_2)\rangle\, dx \right| \\
&\le \left(C_\Omega \left\| \frac{f-\tilde f}{\omega} \right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\| \frac{|G-\tilde G|}{\omega} \right\|_{L^{q'}(\Omega,\omega)} \right) \|u_1-u_2\|_X.
\end{aligned}$$

Hence, with $\gamma=\min\{\alpha_p,\tilde{\alpha}_p\}$, we obtain in (3.4)

$$\gamma\,\|u_1-u_2\|_X^p \le \alpha_p\int_\Omega |\Delta(u_1-u_2)|^p\,\omega\,dx+\tilde{\alpha}_p\int_\Omega|\nabla(u_1-u_2)|^p\,\omega\,dx$$
$$\le\left(C_\Omega\left\|\frac{f-\tilde{f}}{\omega}\right\|_{L^{p'}(\Omega,\omega)}+C_{p,q}\left\|\frac{|G-\tilde{G}|}{\omega}\right\|_{L^{q'}(\Omega,\omega)}\right)\|u_1-u_2\|_X.$$

Therefore,

$$\|u_1-u_2\|_X\le\frac{1}{\gamma^{1/(p-1)}}\left(C_\Omega\left\|\frac{f-\tilde{f}}{\omega}\right\|_{L^{p'}(\Omega,\omega)}+C_{p,q}\left\|\frac{|G-\tilde{G}|}{\omega}\right\|_{L^{q'}(\Omega,\omega)}\right)^{1/(p-1)}.$$

□

Corollary 3.5. *Assume $2\le q<p<\infty$. Let the assumptions of Theorem 3.3 be fulfilled, and let $\{f_m\}$ and $\{G_m\}$ be sequences of functions satisfying $\frac{f_m}{\omega}\to\frac{f}{\omega}$ in $L^{p'}(\Omega,\omega)$ and $\left\|\frac{|G_m-G|}{\omega}\right\|_{L^{q'}(\Omega,\omega)}\to 0$ as $m\to\infty$. If $u_m\in X$ is a solution of the problem*

$$(P_m)\begin{cases} Lu_m(x)=f_m(x)-\operatorname{div}(G_m(x)), & \text{in } \Omega,\\ u_m(x)=\Delta u_m(x)=0, & \text{in } \partial\Omega,\end{cases}$$

then $u_m\to u$ in X and u is a solution of problem (P).

Proof. By Corollary 3.4 we have

$$\|u_m-u_r\|_X \le\frac{1}{\gamma^{1/(p-1)}}\left(C_\Omega\left\|\frac{f_m-f_r}{\omega}\right\|_{L^{p'}(\Omega,\omega)}+C_{p,q}\left\|\frac{|G_m-G_r|}{\omega}\right\|_{L^{q'}(\Omega,\omega)}\right)^{1/(p-1)}.$$

Therefore $\{u_m\}$ is a Cauchy sequence in X. Hence, there is $u\in X$ such that $u_m\to u$ in X. We have that u is a solution of problem (P). In fact, since u_m is a solution of (P_m), for all $\varphi\in X$ we have

$$\begin{aligned}
&\int_\Omega|\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx+\int_\Omega|\Delta u|^{q-2}\Delta u\,\Delta\varphi\,\omega\,dx\\
&+\int_\Omega|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle\,\omega\,dx+\int_\Omega|\nabla u|^{q-2}\langle\nabla u,\nabla\varphi\rangle\,\omega\,dx\\
&=\int_\Omega\Big(|\Delta u|^{p-2}\Delta u-|\Delta_m|^{p-2}\Delta u_m\Big)\Delta\varphi\,\omega\,dx\\
&+\int_\Omega\Big(|\Delta u|^{q-2}\Delta u-|\Delta u_m|^{q-2}\Delta u_m\Big)\Delta\varphi\,\omega\,dx\\
&+\int_\Omega\Big(|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle-|\nabla u_m|^{p-2}\langle\nabla u_m,\nabla\varphi\rangle\Big)\omega\,dx\\
&+\int_\Omega\Big(|\nabla u|^{q-2}\langle\nabla u,\nabla\varphi\rangle-|\nabla u_m|^{q-2}\langle\nabla u_m,\nabla\varphi\rangle\Big)\omega\,dx\\
&+\int_\Omega|\Delta u_m|^{p-2}\Delta u_m\,\Delta\varphi\,\omega\,dx+\int_\Omega|\Delta u_m|^{q-2}\Delta u_m\,\Delta\varphi\,\omega\,dx\\
&+\int_\Omega|\nabla u_m|^{p-2}\langle\nabla u_m,\nabla\varphi\rangle\,\omega\,dx+\int_\Omega|\nabla u_m|^{q-2}\langle\nabla u_m,\nabla\varphi\rangle\,\omega\,dx\\
&=I_1+I_2+I_3+I_4+\int_\Omega f_m\varphi\,dx+\int_\Omega\langle G_m,\nabla\varphi\rangle\,dx\\
&=I_1+I_2+I_3+I_4+\int_\Omega f\,\varphi\,dx+\int_\Omega\langle G,\nabla\varphi\rangle\,dx\\
&+\int_\Omega(f_m-f)\,\varphi\,dx+\int_\Omega\langle G_m-G,\nabla\varphi\rangle\,dx, \qquad (3.5)
\end{aligned}$$

where

$$
\begin{aligned}
I_1 &= \int_\Omega \Big(|\Delta u|^{p-2}\Delta u - |\Delta u_m|^{p-2}\Delta u_m\Big)\,\Delta\varphi\,\omega\,dx,\\
I_2 &= \int_\Omega \Big(|\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m\Big)\Delta\varphi\,\omega\,dx,\\
I_3 &= \int_\Omega \Big(|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle - |\nabla u_m|^{p-2}\langle\nabla u_m,\nabla\varphi\rangle\Big)\,\omega\,dx,\\
I_4 &= \int_\Omega \Big(|\nabla u|^{q-2}\langle\nabla u,\nabla\varphi\rangle - |\nabla u_m|^{q-2}\langle\nabla u_m,\nabla\varphi\rangle\Big)\,\omega\,dx.
\end{aligned}
$$

We have that:
(i) By Lemma 2.7(a) there exists $C_p > 0$ such that

$$
\begin{aligned}
|I_1| &\le \int_\Omega \big||\Delta u|^{p-2}\Delta u - |\Delta u_m|^{p-2}\Delta u_m\big|\,|\Delta\varphi|\,\omega\,dx\\
&\le C_p\int_\Omega |\Delta u - \Delta u_m|\,(|\Delta u| + |\Delta u_m|)^{p-2}\,|\Delta\varphi|\,\omega\,dx.
\end{aligned}
$$

Let $r = p/(p-2)$. Since $\dfrac{1}{p} + \dfrac{1}{p} + \dfrac{1}{r} = 1$, by the Generalized Hölder inequality we obtain

$$
\begin{aligned}
&|I_1|\\
&\le C_p\left(\int_\Omega |\Delta u - \Delta u_m|^p\,\omega\,dx\right)^{1/p}\left(\int_\Omega |\Delta\varphi|^p\,\omega\,dx\right)^{1/p}\left(\int_\Omega (|\Delta u| + |\Delta u_m|)^{(p-2)r}\,\omega\,dx\right)^{1/r}\\
&\le C_p\|u-u_m\|_X\,\|\varphi\|_X\||\Delta u| + |\Delta u_m|\|^{(p-2)}_{L^p(\Omega,\omega)}.
\end{aligned}
$$

Now, since $u_m \to u$ in X, then exists a constant $M>0$ such that $\|u_m\|_X \le M$. Hence,

$$
\||\Delta u| + |\Delta u_m|\|_{L^p(\Omega,\omega)} \le \|u\|_X + \|u_m\|_X \le 2M. \tag{3.6}
$$

Therefore,

$$
\begin{aligned}
|I_1| &\le C_p\,(2M)^{p-2}\,\|u-u_m\|_X\,\|\varphi\|_X\\
&= C_1\,\|u-u_m\|_X\,\|\varphi\|_X.
\end{aligned}
$$

Analogously, there exists a constant C_3 such that

$$
|I_3| \le C_3\|u-u_m\|_X\,\|\varphi\|_X.
$$

(ii) By Lemma 2.7(a) there exists a positive constant C_q such that

$$
\begin{aligned}
|I_2| &\le \int_\Omega \big||\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m\big|\,|\Delta\varphi|\,\omega\,dx\\
&\le C_q\int_\Omega |\Delta u - \Delta u_m|\,(|\Delta u| + |\Delta u_m|)^{q-2}\,|\Delta\varphi|\,\omega\,dx.
\end{aligned}
$$

Let $s = q/(q-2)$ (if $2<q<p<\infty$). Since $\dfrac{1}{q} + \dfrac{1}{q} + \dfrac{1}{s} = 1$, by the Generalized Hölder inequality we obtain

$$
\begin{aligned}
&|I_2|\\
&\le C_q\left(\int_\Omega |\Delta u - \Delta u_m|^q\,\omega\,dx\right)^{1/q}\left(\int_\Omega |\Delta\varphi|^q\,\omega\,dx\right)^{1/q}\left(\int_\Omega (|\Delta u| + |\Delta u_m|)^{(q-2)s}\,\omega\,dx\right)^{1/s}\\
&= C_q\,\|\Delta u - \Delta u_m\|_{L^q(\Omega,\omega)}\,\|\Delta\varphi\|_{L^q(\Omega,\omega)}\||\Delta u| + |\Delta u_m|\|^{q-2}_{L^q(\Omega,\omega)}.
\end{aligned}
$$

Now, by Remark 3.2(i) and (3.6) we have

$$\begin{aligned} |I_2| &\leq C_q\, C_{p,q} \|\Delta u - \Delta u_m\|_{L^p(\Omega,\omega)}\, C_{p,q}\|\Delta\varphi\|_{L^p(\Omega,\omega)}\, C_{p,q}^{q-2} \||\Delta u| + |\Delta u_m|\|^{q-2}_{L^p(\Omega,\omega)} \\ &\leq C_q\, C_{p,q}^q \|u - u_m\|_X \|\varphi\|_X\, (2M)^{q-2} \\ &= C_2\, \|u - u_m\|_X\, \|\varphi\|_X. \end{aligned}$$

Analogously, there exists a positive constant C_4 such that

$$|I_4| \leq C_4\, \|u - u_m\|_X\, \|\varphi\|_X.$$

In case $q = 2$, we have $|I_2|, |I_4| \leq C_{p,2}^2\, \|u - u_m\|_X\, \|\varphi\|_X$.
Therefore, we have $I_1, I_2, I_3, I_4 \to 0$ when $m \to \infty$.

(iii) We also have

$$\begin{aligned} &\left| \int_\Omega (f_m - f)\, \varphi\, dx + \int_\Omega \langle G_m - G, \nabla\varphi\rangle\, dx \right| \\ &\left(C_\Omega \left\| \frac{f_m - f}{\omega} \right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\| \frac{|G_m - G|}{\omega} \right\|_{L^{q'}(\Omega,\omega)} \right) \|\varphi\|_X \to 0, \end{aligned}$$

when $m \to \infty$. Therefore, in (3.5), we obtain when $m \to \infty$ that

$$\begin{aligned} &\int_\Omega |\Delta u|^{p-2} \Delta u\, \Delta\varphi\, \omega\, dx + \int_\Omega |\Delta u|^{q-2} \Delta u\, \Delta\varphi\, \omega\, dx \\ &+ \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla\varphi\rangle\, \omega\, dx + \int_\Omega |\nabla u|^{q-2} \langle \nabla u, \nabla\varphi\rangle\, \omega\, dx \\ &= \int_\Omega f\, \varphi\, dx + \int_\Omega \langle G, \nabla\varphi\rangle\, dx, \end{aligned}$$

i.e., u is a solution of problem (P). □

Example 3.6. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, $w(x,y) = (x^2 + y^2)^{-1/2}$ ($\omega \in A_4$, $p = 4$ and $q = 3$), $f(x,y) = \dfrac{\cos(xy)}{(x^2 + y^2)^{1/6}}$ and $G(x,y) = \left(\dfrac{\sin(x+y)}{(x^2 + y^2)^{1/6}}, \dfrac{\sin(xy)}{(x^2 + y^2)^{1/6}} \right)$. By Theorem 3.3 , the problem

$$\begin{cases} \Delta\Big[(x^2 + y^2)^{-1/2}\big(|\Delta u|^2 \Delta u + |\Delta u| \Delta u\big)\Big] \\ - \operatorname{div}\Big[(x^2 + y^2)^{-1/2}\big(|\nabla u|^2 \nabla u + |\nabla u| \nabla u\big)\Big] \\ = f(x) - \operatorname{div}(G(x)), \quad \text{in } \Omega \\ u(x) = \Delta u = 0, \quad \text{in } \partial\Omega \end{cases}$$

has a unique solution $u \in W^{2,4}(\Omega,\omega) \cap W_0^{1,4}(\Omega,\omega)$.

Acknowledgements

The author thanks the referee for his/her useful suggestions and comments which have improved the presentation of the paper.

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