A DQM–Picard iterative approach for solving generalized Emden–Fowler equations

Genelleştirilmiş Emden–Fowler denklemlerinin çözümünde DKM–Picard yinelemeli yaklaşımı

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Abstract

This study focuses on the numerical approximation of singular initial value problems (SIVPs) defined by the generalized Emden–Fowler equation, which appears in many physical and engineering models such as astrophysics and heat conduction. The proposed method is based on the Differential Quadrature Method (DQM), a powerful domain discretization technique that approximates derivatives by weighted sums of function values at discrete points. By employing a relatively small number of collocation points, DQM offers highly accurate approximations even for problems with singularities or steep solution profiles. In this work, the resulting nonlinear system of algebraic equations derived from DQM is solved using Picard iteration, which provides a simple yet effective approach to handle the nonlinear terms. The method is applied to a range of test problems that involve different types of nonlinearities and source terms, including exponential and logarithmic forms. Numerical experiments are conducted to compare the approximate solutions with known exact solutions, and the maximum and minimum absolute errors are calculated to evaluate the performance of the method. The results confirm that the proposed DQM–Picard approach yields highly accurate solutions using significantly fewer discretization points than many traditional numerical techniques. This makes it a promising tool for efficiently solving a broad class of nonlinear singular problems governed by Emden–Fowler type equations. Overall, the study demonstrates the reliability and effectiveness of the method in handling nonlinear singular models with various complexities.

Keywords: Differential quadrature method, Generalized Emden-Fowler equations, Nonlinear singular initial value problems, Numerical methods, Picard iteration.

Öz

Bu çalışma, astrofizik ve ısı iletimi gibi birçok fizik ve mühendislik modelinde ortaya çıkan genelleştirilmiş Emden–Fowler denklemi ile tanımlanan tekil başlangıç değer problemlerinin (TBVP) sayısal yaklaşık çözümüne odaklanmaktadır. Önerilen yöntem, türevleri ayrık noktalardaki fonksiyon değerlerinin ağırlıklı toplamları ile yaklaşık hesaplayan güçlü bir alan ayrıştırma tekniği olan Diferansiyel Kuadratur Metodu'na (DKM) dayanmaktadır. Nispeten az sayıda kollokasyon noktası kullanılarak, DKM tekillikler veya ani değişim gösteren çözüm profillerine sahip problemler için bile yüksek doğrulukta yaklaşık çözümler sunmaktadır. Bu çalışmada, DKM'den elde edilen doğrusal olmayan cebirsel denklem sistemi, doğrusal olmayan terimleri etkili şekilde ele alan basit ancak etkili bir yöntem olan Picard yinelemesi ile çözülmüştür. Yöntem, üstel ve logaritmik gibi farklı kaynak terimler ve doğrusal olmayan yapıların bulunduğu çeşitli test problemlerine uygulanmıştır. Sayısal deneyler, yaklaşık çözümleri bilinen analitik çözümlerle karşılaştırmak ve yöntemin performansını değerlendirmek amacıyla maksimum ve minimum mutlak hata değerleri hesaplanarak gerçekleştirilmiştir. Elde edilen sonuçlar, önerilen DKM–Picard yaklaşımının, birçok geleneksel sayısal teknikten daha az sayıda ayrık noktaya ihtiyaç duyarak yüksek doğrulukta çözümler verdiğini doğrulamaktadır. Bu da yöntemi, Emden–Fowler tipi denklemlerle tanımlanan geniş bir doğrusal olmayan tekil problem sınıfının verimli bir şekilde çözümü için umut verici bir araç haline getirmektedir. Genel olarak, çalışma, yöntemin çeşitli karmaşıklıklara sahip doğrusal olmayan tekil modelleri başarılı ve güvenilir şekilde ele aldığını göstermektedir.

Anahtar kelimeler: *Diferansiyel kuadratur yöntemi, Genelleştirilmiş Emden-Fowler denklemleri, Doğrusal olmayan tekil başlangıç değer problemleri, Nümerik yöntemler, Picard yinelemesi*

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1. Introduction

The Emden–Fowler type equations constitute a broad class of nonlinear ordinary differential equations that arise in diverse scientific and engineering applications. A distinctive feature of many Emden–Fowler equations is the presence of singular behavior at the origin. This singularity often hinders the direct application of standard numerical methods, potentially leading to loss of accuracy or numerical instability. As a result, a wide range of analytical and numerical techniques have been developed, including power series expansions, perturbation methods, finite difference schemes, and spectral methods, to approximate solutions of singular initial value problems (SIVPs) associated with these equations.

In the present work, the singular initial value problem (SIVP) defined by the generalized Emden-Fowler equation

$$y''(x) + \frac{a}{x}y'(x) + f(x)g(y) = h(x) \tag{1}$$

$$y(0) = b; \qquad y'(0) = c \tag{2}$$

is considered. Here the primes denote derivatives with respect to x; $f(x), g(y), h(x)$ are given functions, and b and c are prescribed functions. The nonlinearity in equation (1) depends on the choice of $g(y)$. In particular if $g(y) \equiv 0$ or if $g(y) = y^0$ or $g(y) = y^1$, equation (1) reduces to a linear form, for general (nonlinear) choices of $g(y)$, the equation is nonlinear.

Due to the singularity at $x = 0$ and the strongly nonlinear nature of equation (1), obtaining an exact analytical solution is generally intractable. Existing solution techniques can be broadly classified into two categories: (semi-)analytical methods and purely numerical methods. The first category typically yields solutions in series form. Examples include the Taylor matrix method for $g(y) = y$ (Aslan et al., 2015), the homotopy perturbation method for $h(x) = 0$ (Chowhury & Hashim, 2009), the variational iteration method (Ghorbany & Bacherad, 2017), the Chebyshev operational matrix algorithm for cases where $f(x) = 0$ and/or $h(x) = 0$ (Doha et al., 2013), the Fourier decomposition method for the case $f(x) = 1, h(x) = 0$ (Düz, 2022), the He's variational iteration method (Ravi Kanth & Aruna, 2010).

In contrast, purely numerical methods approximate the solution at the discretized points in the problem domain. Examples include the modified Legendre spectral method (Rismani&Monfared, 2012) and Hermite function collocation method for $f(x) = 1$, $h(x) = 0$ (Parand et al., 2010). More recently, Meral (2025) proposed a differential quadrature method (DQM) combined with Newton linearization to solve the Lane-Emden equation, a special case of the Emden-Fowler equation with $f(x) = 1$, $h(x) = 0$. DQM, being a domain discretization technique, leads to a nonlinear system of algebraic equations, which in Meral (2025) was solved via Newton's method yielding accurate results.

In the present work, the methodology is extended by employing the combined application of DQM with Picard iteration instead of Newton's method, enabling an alternative iterative framework that avoids direct linearization. Furthermore, unlike the Lane–Emden case addressed in Meral (2025), the present study considers the more general Emden–Fowler equation, which includes additional functional flexibility through arbitrary $f(x)$, $g(y)$, and $h(x)$. The proposed scheme is implemented with Gauss–Chebyshev–Lobatto collocation points and demonstrates near machine-precision accuracy even with a modest number of grid points. This spectral discretization effectively captures singular behaviors and steep gradients, while the Picard iterative scheme converges robustly, even from relatively crude initial guesses. Numerical experiments confirm that the method achieves a favorable balance between accuracy and efficiency, with its performance remaining competitive against existing semi-analytical techniques. Furthermore, the flexibility of the framework suggests straightforward extensions to systems of equations and multidimensional problems in future research.

2. The numerical method

This section presents the numerical framework used to solve SIVPs governed by the generalized Emden–Fowler equation. The Differential Quadrature Method (DQM) is employed to discretize the spatial domain by approximating derivatives as weighted sums of function values at selected collocation points. In particular,

Gauss–Chebyshev–Lobatto points are utilized to enhance accuracy near domain boundaries where singularities or steep gradients may occur. The discretization transforms the differential problem into a system of nonlinear algebraic equations. To address the nonlinearity arising from the source term, the Picard iterative scheme is implemented, providing an efficient and straightforward approach for successive approximations until convergence. The combined DQM–Picard method thus offers a robust and accurate computational tool for solving nonlinear singular problems, as detailed in the following subsections.

2.1. Differential quadrature method

The Differential Quadrature Method (DQM), first introduced by Bellman and Casti (1971) and later extended by Bellman et al., (1972), is a numerical technique that approximates derivatives by expressing them as weighted sums of function values at discrete grid points. In the context of the SIVP defined by equations (1) and (2), DQM is utilized to discretize the spatial domain and transform the differential equation into a system of algebraic equations. For the prosent study, the domain $[0,1]$ is discretized using Gauss–Chebyshev–Lobatto (GCL) points (Shu, 2000), which cluster near the boundaries and thus enhance resolution in regions of rapid variation or singularity.

Let N be the number of discretization points. The GCL points (mapped to the interval $[0,1]$) are given by:

$$x_i = \frac{1}{2}\left(1 - cos\left(\frac{(i-1)\pi}{N-1}\right)\right) \quad \text{for } i = 1, 2, \ldots, N. \tag{3}$$

Given these points, the solution $y(x)$ is approximated using Lagrange interpolation polynomials, and its n-th derivative ($n \in \mathbb{Z}^+$) at each node x_i is expressed as:

$$y^{(n)}(x_i) \approx \sum_{j=1}^{N} w_{ij}^{(n)}\, y(x_j) \tag{4}$$

where $w_{ij}^{(n)}$ are the weighting coefficients for the $n-$th derivative, computed from the Lagrange basis functions and the first and second order weighting coefficients are respectively given by

$$w_{ij}^{(1)} = \begin{cases} \dfrac{\prod_{k=1,k \neq i}^{N}(x_i - x_k)}{\prod_{k=1,k \neq j}^{N}(x_j - x_k)(x_i - x_j)} & i \neq j, \\[2em] \displaystyle\sum_{k=1,k \neq i}^{N} w_{ik}^{(1)} & i = j, \end{cases} \tag{5}$$

$$w_{ij}^{(2)} = \begin{cases} 2w_{ij}^{(1)}\left(w_{ii}^{(1)} - \dfrac{1}{x_i - x_j}\right) & i \neq j, \\[2em] -\displaystyle\sum_{k=1,k \neq i}^{N} w_{ik}^{(2)} & i = j. \end{cases} \tag{6}$$

These weighting coefficients are precomputed and stored, allowing for efficient implementation of the discretized form of the differential equation. Moreover, once the first derivative weighting coefficients are calculated, the higher-order derivative coefficients (e.g., second order and above) can be obtained in terms of the lower-order ones, which provides an additional advantage in terms of computational efficiency.

Substituting the DQM approximations into equation (1), the governing differential equation is transformed into a nonlinear algebraic system:

$$\sum_{j=1}^{N} w_{ij}^{(2)}\, y(x_j) + \frac{a}{x_i}\sum_{j=1}^{N} w_{ij}^{(1)}\, y(x_j) + f(x_i)\, g(y(x_i)) = h(x_i), \quad \text{for} \quad i = 1, 2, \ldots, N. \tag{7}$$

Finally, the discretization of the SIVP (1)-(2) using the DQM can be expressed in the compact form

$$A y + \hat{f} = h, \tag{8}$$

where A is an $N \times N$ matrix whose entries are constructed from a combination of the first-derivative weighting coefficients multiplied by $\frac{a}{x_i}$ and the second-derivative weighting coefficients. The vector $\hat{f}$ contains the contributions arising from the nonlinear term $f(x)g(y)$, while h is the discretized form of the source term. It is important to note that the structure of the algebraic system depends on the form of $g(y)$. In particular, when $g(y) \equiv 0$, or when $g(y) = y^0$ (i.e. 1) or $g(y) = y^1$ (i.e. y) the discretized system reduces to a linear system. For other choices of $g(y)$ nonlinear contributions appear and the resulting algebraic system must be solved using an iterative approach, such as the Picard method described in the following subsection.

2.2 Picard iterative scheme

When the discretized system (8) is nonlinear due to the presence of g(y), an iterative approach is required to obtain the solution. In this study, the Picard iteration method is employed, which is based on successive approximations of the nonlinear term.

Starting from an initial guess $y^{(0)}$, the Picard scheme updates the solution according to

$$Ay^{(k+1)} + \hat{f}\left(y^{(k)}\right) = h, \ k = 0,1,2, \cdots \tag{9}$$

Here, $\hat{f}\left(y^{(k)}\right)$ denotes the evaluation of the nonlinear term using the most recent approximation $y^{(k)}$. At each iteration, the system becomes linear in $y^{(k+1)}$, and can be solved efficiently using standard linear algebra techniques. The process is repeated until a convergence criterion

$$\left\|y^{(k+1)} - y^{(k)}\right\| < \varepsilon \tag{10}$$

is satisfied, where ε is a prescribed tolerance.

This iterative scheme ensures a straightforward implementation and typically exhibits good convergence behavior when the initial guess is chosen appropriately.

3. Numerical results and discussion

In this section, the proposed DQM-Picard iterative method is applied to several benchmark SIVPs with known analytical solutions. The performance of the method is assessed by comparing the numerical approximations to the exact solutions. The accuracy of the computed solutions is quantitatively evaluated using the maximum and minimum absolute errors, denoted as

$$\beta_{max} = \max_{x \in (0,1]} \left|y_{exact}(x) - y_{DQMP}(x)\right|, \quad \beta_{min} = \min_{x \in (0,1]} \left|y_{exact}(x) - y_{DQMP}(x)\right|, \tag{11}$$

where $y_{exact}(x)$ and $y_{DQMP}(x)$ represent the exact and numerical(DQM-Picard iterative) solutions, respectively, evaluated at the discretization points. Throughout this section, various types of source terms, including polynomial, logarithmic, and nonlinear exponential terms, are considered to demonstrate the robustness, accuracy, and convergence behavior of the method. Additionally, the numerical results are compared with existing methods such as Haar Wavelet Collocation Method (HWCM) (Alkan, 2017), Optimal Homotopy Asymptotic Method (OHAM) (Iqbal & Javed, 2011), Taylor Wavelet (Saha & Singh, 2024) , and Modified Decomposition methods(Singh & Wazwaz, 2022) to highlight the advantages of the proposed approach.

3.1. Example 1 (polynomial source term)

The first example considers the problem

$$y'' + \frac{2}{x}y' + 20x^2 = 0, \ 0 < x \le 1 \tag{12}$$

with the initial conditions

$$y(0) = 1, y'(0) = 0 \tag{13}$$

The exact solution for this problem is given by $y(x) = 1 - x^4$ (Chowdhury & Hashim, 2009). Table 1 lists the maximum absolute errors obtained for various numbers of discretization points N. It is observed that the errors remain at the level of machine precision for all tested N, indicating that the proposed method yields an essentially exact solution for this problem.

Table 1. Maximum absolute errors for different values of N for Example 1

N	β_{max}
5	$3.33x \ 10^{-16}$
6	$6.66x \ 10^{-16}$
7	$1.28x \ 10^{-15}$
8	$2.33x \ 10^{-15}$
9	$1.44x \ 10^{-15}$
15	$4.91x \ 10^{-15}$
20	$2.77x \ 10^{-15}$

3.2. Example 2 (nonhomogeneous mixed polynomial source)

Consider the following second-order singular initial value problem:

$$y'' + \frac{8}{x}y' + xy = x^5 - x^4 + 44x^2 - 30x \tag{14}$$

$$y(0) = 0, y'(0) = 0 \tag{15}$$

with the exact solution $y(x) = x^4 - x^3$ (Chowdhury & Hashim, 2009).

This problem has previously been studied using methods such as the Haar Wavelet Collocation Method (HWCM) (Alkan, 2017) and the Optimal Homotopy Asymptotic Method (OHAM) (Iqbal & Javed, 2011), both of which provide approximate solutions in the form of truncated series. In these approaches, the approximate solution is expressed symbolically in terms of x, and evaluated at specific points to estimate the numerical solution.

In the HWCM, the problem is discretized by constructing a system of equations based on Haar wavelets, resulting in a global coefficient matrix of size $m \times m$, where m denotes the level of the wavelet. The method solves this system to determine the expansion coefficients, which are then used to reconstruct the approximate solution. In contrast, the OHAM builds a series solution based on the homotopy perturbation approach, where convergence is typically achieved using a limited number of terms.

To enable a fair comparison, the maximum and minimum absolute errors for OHAM for the same problem (Iqbal & Javed, 2011) are calculated using the same Chebyshev-Gauss-Lobatto nodes used in the DQM implementation. These results are presented in Table 2.

Although the minimum absolute errors in OHAM decrease significantly as N increases, the maximum absolute errors remain relatively unchanged across all tested values. DQM, on the other hand, achieves near-machine-precision accuracy for all values of N, indicating a highly accurate and stable solution.

In the case of HWCM, the absolute error values provided in the original study (Alkan, 2017) were computed over the interval (0,1] at uniformly spaced values $x = 0.1, 0.2, ..., 1.0$ i.e., without use of Gauss-type collocation points and results were provided separately for three different values of m. In Table 3, the

maximum and minimum error values among those reported for each m were selected to represent the error range of the method.

Table 2. Comparison of the absolute errors for DQM and OHAM (Iqbal & Javed, 2011)

N	DQM $(\beta_{max}, \beta_{min})$	OHAM (Iqbal & Javed, 2011) $(\beta_{max}, \beta_{min})$
5	$(1.11x\ 10^{-16}, 0)$	$(2.06x\ 10^{-8}, 5.53x\ 10^{-12}\ )$
6	$(1.03x\ 10^{-16}, 1.0 \times 10^{-18}))$	$(1.70x\ 10^{-8}, 4.67x\ 10^{-13}\ )$
7	$(2.43x\ 10^{-16}, 1.0 \times 10^{-18}))$	$(2.08x\ 10^{-8}, 5.47x\ 10^{-14}\ )$
10	$(1.63x\ 10^{-16}, 1.0 \times 10^{-18}))$	$(2.08x\ 10^{-8}, 4.75x\ 10^{-16}\ )$
15	$(1.49x\ 10^{-16}, 1.0 \times 10^{-18}))$	$(2.19x\ 10^{-8}, 2.56x\ 10^{-18}\ )$
20	$(6.28x\ 10^{-16}, 1.0 \times 10^{-18})$	$(2.16x\ 10^{-8}, 6.93x\ 10^{-20}\ )$

Table 3. Maximum and minimum absolute error values for HWCM (Alkan, 2017)

m	HWCM (Alkan, 2017) $(\beta_{max}, \beta_{min})$
16	$(1.09x\ 10^{-4}, 8.22 \times 10^{-5})$
64	$(1.27x\ 10^{-5}, 9.31 \times 10^{-6})$
256	$(2.13x\ 10^{-6}, 1.55 \times 10^{-7})$

The results both in Table 2 and Table 3 indicate that the DQM is capable of achieving highly accurate results—often close to machine precision—with significantly fewer discretization points compared to HWCM and OHAM. In addition, the DQM's algebraic system is smaller and more computationally efficient than the $m \times m$ matrix required in HWCM, or the symbolic manipulations involved in OHAM. Note that in order to obtain the solution for this problem DQM solves a linear algebraic system of equations of size $N - 1$.

3.3. Example 3 (logarithmic source term)

In this example, the initial value problem

$$y'' + \frac{2}{x}y' + 12x \ln x = 0 \tag{16}$$

with initial conditions

$$y(0) = 0, y'(0) = 0 \tag{17}$$

and the exact solution $y(x) = 1 - (lnx - \frac{7}{12})x^3$ (Chowdhury & Hashim, 2009) is solved using the proposed DQM approach.

The convergence behavior of DQM for the problem in Example 3 with logarithmic source term is illustrated in Figure 1. It can be observed that the maximum absolute error decreases rapidly as the number of discretization points N increases, indicating the high–order convergence property of the DQM.

The performance of the method is investigated for different numbers of discretization points N. The maximum absolute errors are computed with respect to the analytical solution.

The results given in Table 4 show that the maximum absolute error decreases rapidly as N increases, reaching values close to machine precision for sufficiently large N reflecting the high accuracy of the method. For $N \approx 40$, the error reaches the order of 10^{-8}, and for $N \approx 100$ it drops to around 10^{-10}. Beyond $N \approx 150$, the error stagnates and shows a slight increase for $N = 200$, which is attributed to the influence of floating–point round–off errors dominating at very fine discretizations. The steep decay of the error for small and moderate N confirms the exponential–type convergence typical of spectral methods.

Table 4. Maximum absolute errors for different values of N for Example 3

N	β_{max}
5	$1.53x \ 10^{-2}$
6	$3.52x \ 10^{-3}$
8	$4.21x \ 10^{-4}$
10	$8.93x \ 10^{-5}$
15	$6.07x \ 10^{-6}$
20	$9.58x \ 10^{-7}$
40	$1.25x \ 10^{-8}$
45	$6.01x \ 10^{-9}$
65	$6.56x \ 10^{-10}$
100	$6.88x \ 10^{-11}$
150	$5.90x \ 10^{-11}$
200	$2.04x \ 10^{-10}$
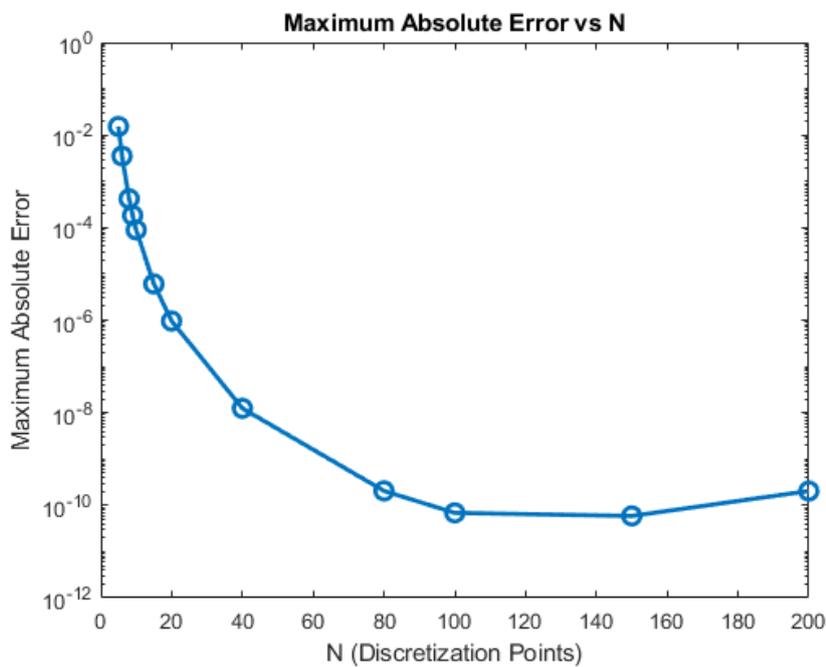
Figure 1. Maximum absolute error versus the number of discretization points N for Example 3

3.4. Example 4 (nonlinear exponential model)

The final test problem addresses a nonlinear initial SIVP involving an exponential nonlinearity given by

$$y'' + \frac{1}{2x}y' = e^{2y} - \frac{1}{2x} \ e^y \tag{18}$$

$$y(0) = \ln\frac{1}{2}, y'(0) = -\frac{1}{2} \tag{19}$$

which admits the exact solution $y(x) = \ln\frac{1}{2+x}$ (Saha & Singh, 2024; Singh & Wazwaz, 2022).

The method is implemented for increasing numbers of discretization points $N \in [5,15]$. The maximum absolute error and the number of Picard iterations required for convergence are recorded for each value of N. Note that a strict tolerance of $\varepsilon = 10^{-10}$ is adopted to ensure high accuracy. The results are illustrated in Figures 2 and 3.

Figure 2 shows the exponential decay of the maximum absolute error with increasing N, confirming the high accuracy of the proposed scheme. For $N \geq 13$, the maximum absolute error falls below the computational precision and is reported as zero.

Figure 3 demonstrates a sharp decrease in the number of iterations required as N increases. While 23 iterations are needed at $N = 5$, only a single iteration is sufficient for $N \geq 13$, highlighting the rapid convergence of the method for larger discretizations.

These observations indicate that the DQM–Picard iterative method is not only highly accurate but also computationally efficient for this nonlinear exponential IVP.
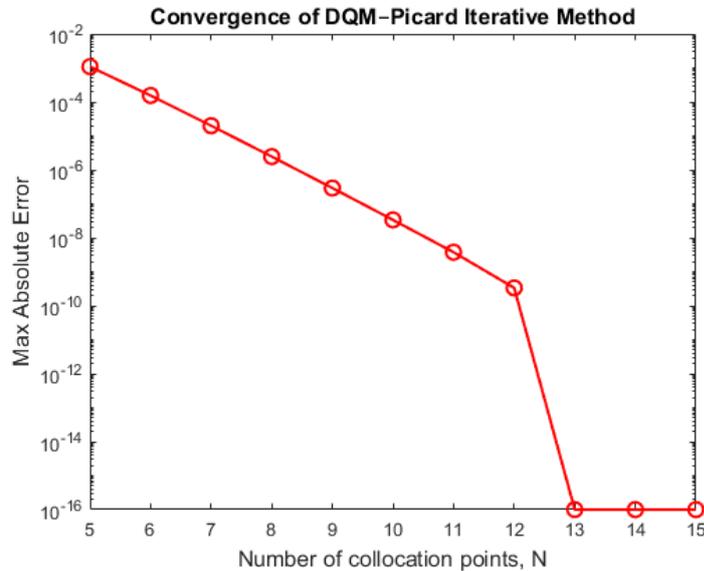

Figure 2. Maximum absolute error versus the number of discretization points N for the proposed DQM-Picard iterative approach for Example 4
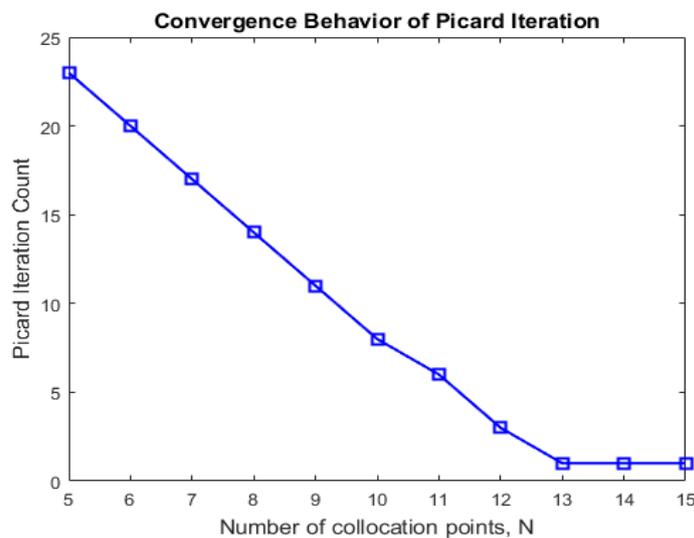

Figure 3. Number of Picard iterations required for convergence versus the number of discretization points N for Example 4

To further assess the robustness of the method, the initial guess is alternatively taken as the constant initial condition vector whose entries are all equal to the initial condition value $\ln\left(\frac{1}{2}\right)$ instead of the exact solution profile. As expected, this choice leads to a moderate increase in the number of iterations and a slight degradation in accuracy. For instance, at $N = 5$, the maximum absolute error rises to 1.09×10^{-3} with 31 iterations required for convergence. However, as N increases, the accuracy rapidly improves, and for $N \geq 12$, the maximum absolute errors decrease to the order of 10^{-10}, approaching machine precision, although convergence consistently requires about 30 iterations. These results confirm that even with a less accurate initial guess, the DQM–Picard iterative method maintains excellent accuracy and reliability.

Table 5 compares the performance of the present method with two recent semi-analytical methods applied to the same problem. The Taylor Wavelet method presented by Saha and Singh, (2024), which provides a series solution with a maximum absolute error of 1.45×10^{-9} for $n = 6$, where n denotes the number of terms in the series. The Modified Decomposition Method introduced by Singh and Wazwaz (2022), which achieves a maximum error of 1.25×10^{-8} with $n = 8$.

Although both methods are semi-analytical and rely on truncated series representations, the proposed DQM–Picard method, being a numerical discretization-based approach, demonstrates superior accuracy. As shown in Table 5 it achieves exact matching with the analytical solution (within machine precision) for $N \geq 13$, while requiring minimal computational effort.

Table 5. Comparison of the maximum absolute errors obtained by different methods for Example 4

Method	Type	Parameter (n or N)	β_{max}
Taylor Wavelet (Saha & Singh, 2024)	Semi-analytical	$n = 6$	1.45×10^{-9}
Modified Decomposition (Singh & Wazwaz, 2022)	Semi-analytical	$n = 8$	1.25×10^{-8}
DQM-Picard (present work, exact initial guess)	Numerical	$N \geq 13$	0 (machine precision)
DQM-Picard (present work, initial guess as the constant initial condition vector)	Numerical	$N = 13$	1.23×10^{-10}

4. Discussion and conclusions

This study presented a numerical approach combining the Differential Quadrature Method (DQM) and Picard iteration for solving nonlinear singular initial value problems modeled by the generalized Emden–Fowler equation, which arise in various physical and engineering contexts. The proposed method demonstrated superior accuracy and computational efficiency compared to existing semi-analytical techniques, achieving near machine precision solutions with relatively low discretization points. The use of Gauss–Chebyshev–Lobatto points proved effective in capturing singular behaviors and steep gradients, while the Picard iterative scheme provided reliable convergence even with less precise initial guesses. Although numerical round-off errors inherent in spectral differentiation matrices limit ultimate precision at very fine discretizations, the overall robustness and applicability of the method were confirmed through diverse test problems involving polynomial, logarithmic, and exponential nonlinearities. Future extensions may include addressing systems of equations and multidimensional problems to expand the method's utility in scientific and engineering applications.

Author contribution

All stages of the study, including the conceptualization, methodology design, implementation, analysis, and manuscript preparation, were carried out solely by the author.

Declaration of ethical code

The materials and methods used in this study do not require ethics committee approval and/or legal-special permission.

Conflicts of interest

The author declares that there is no conflict of interest.

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