Research Article GU J Sci 31(3): 890-896 (2018)

Gazi University

Journal of Science

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Structural Stability For The Bénard Problem With Voight Regularization

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Article Info

Received: 01/03/2018
Accepted: 17/05/2018

Keywords

Bénard Problem
Structural stability
Continuous dependence

Abstract

In this study we consider the Bénard problem involving Voight regularizing terms. We constitute continuous dependence of solutions of the given problem on the coefficients of the Voight regularizing terms.

1. INTRODUCTION

We study continuous dependence of solutions on the coefficients μ and $\varkappa$. Continuous dependence of solutions is a type of structural stability. This type of stability reflect us the effect of small changes in coefficients of equations on the solutions. Some of the results on this subject for linear and nonlinear partial differential equations can be reached (see e.g. [1,2-4,5]).

Our main aim here to study the structural stability for the following system of equations in $\Omega = (0, \mathrm{L}_1) \times (0, \mathrm{L}_2) \times (0,1)$

$$\frac{\partial u}{\partial t} - \nu \Delta u - \mu \Delta u_t + (u.\nabla)u + \nabla p' = e_3 \theta \qquad \text{in } \Omega \times (0,\tau) \tag{1}$$
$$\frac{\partial \theta}{\partial t} - \kappa \Delta \theta - \varkappa \Delta \theta_t + (u.\nabla)\theta - u_3 = 0 \qquad \text{in } \Omega \times (0,\tau) \tag{2}$$
$$\nabla.\mathrm{u} = 0 \qquad \text{in } \Omega \tag{3}$$

where $\tau > 0$, e_3 is the third component of the canonical basis of $\mathbb{R}^3$, u_3 is the third component of u. $u(\boldsymbol{x},t)$ the velocity of the fluid in the box Ω, $p' = p'(\boldsymbol{x},\mathrm{t})$ is modified pressure given by $p' = p - (x_3 + \frac{x_3^2}{2})$ here p is pressure of the fluid, $\theta(\boldsymbol{x},\mathrm{t})$ is the scaled fluctuation which is given by $\theta = T - \left(\frac{T_0}{T_0 - T_1} - x_3\right)$ here $T = T(\boldsymbol{x},\mathrm{t})$ is the temperature of the fluid inside the box Ω, T_0 and T_1 are the temperature of the fluid at the bottom and the top respectively. ν, μ, κ and $\varkappa$ are positive constants and $\boldsymbol{x} = (x_1, x_2, x_3)$. Now we state boundary and initial conditions for (1)-(3) in the following

$$u = 0, \ \theta = 0 \ \text{ at } \ \mathrm{x}_3 = 0, \ \mathrm{x}_3 = 1, \tag{4}$$

$p, u, \theta, \frac{\partial \mathrm{u}}{\partial \mathrm{x_i}}, \frac{\partial \theta}{\partial \mathrm{x_i}}$ $(i = 1,2)$ are periodic in the $\mathrm{x_i}$ directions which means that

$$\varphi(\boldsymbol{x},t) = \varphi(\boldsymbol{x} + L_i e_i, t) \quad i = 1,2 \quad \forall x \in \mathbb{R}^3, \ \forall t > 0 \text{ for a generic function } \varphi, \tag{5}$$

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$$u(\boldsymbol{x},0)=u_0(\boldsymbol{x}),\ \theta(\boldsymbol{x},0)=\theta_0(\boldsymbol{x}). \tag{6}$$

The Bénard problem in the absence of the use of regularization terms has been previously studied by many authors [6-11]. In [7] the existence of global attractor with a finite fractal dimension are proved in 2D and some partial results are given in 3D. In [9] the authors was studied asymptotic behaviour of the weak solutions of this system in 3D. They reported lack of the uniqueness of the Cauchy problem and the continuity of the weak solutions. In [10] we add some Voight regularizing terms to this system and gave the existence-uniqueness and continuity results on the weak solution for the system in 3D. The idea to add these terms to our system comes from Kelvin Voight system (Navier Stokes Voight system). It was given by Oskolkov in [12]. Global regularity for Navier-Stokes Voight system was studied by Kalantarov, Levant and Titi in [13-15].

This outline of the paper is arranged as follows. In section 2 we give some preliminaries and the functional setting of the Bénard problem. In section 3 we prove that solutions of the Bénard problem with some regularizing terms continuously depend on parameters $\mu,$ and $\varkappa$.

2. PRELIMINARIES

In this section some preliminaries and notations usually used them in the mathematical study of hydrodynamics models. Further discussion on this topic, we refer [16,17,18-20].

Let $\mathrm{L^p}(\Omega)$ and $\mathrm{H^k}(\Omega)=W^{k,2}(\Omega)$ be denote the usual Lebesque space and Sobolev space respectively $1\le p\le\infty,\ k\in\mathbb{R}$ and we define the following spaces.

$V:=\{u\in\left(C^\infty(\Omega)\right)^3, u=0 \text{ at } \mathrm{x}_3=1, \mathrm{x}_3=0, u,\ \frac{\partial u}{\partial x_i} \text{are periodic in } x_i \text{ direction i} =1,2$

$\nabla.\mathrm{u}=0 \text{ in } \Omega\,\}$

$\tilde{V}:=\{\theta\in C^\infty(\Omega), \theta=0 \text{ at } \mathrm{x}_3=1, \mathrm{x}_3=0, \theta\,, \frac{\partial\theta}{\partial x_i} \text{are periodic in } x_i \text{ direction i} =1,2\ \}$

$\boldsymbol{H_1}:=$ the closure of V in $\left(L^2(\Omega)\right)^3$,

$\boldsymbol{V_1}:=$ the closure of V in $\left(H^1(\Omega)\right)^3$,

$\boldsymbol{H_2}=$ the closure of $\tilde{V}$ in $L^2(\Omega)$,

$\boldsymbol{V_2}=$ the closure of $\tilde{V}$ in $H^1(\Omega)$.

The inner product on $\boldsymbol{H_1}$ and $\boldsymbol{H_2}$ are given by

$(u,v)=\sum_{i=1}^3\int_\Omega u_iv_i\ dxdydz\quad,(\theta,\vartheta)=\sum_{i=1}^3\int_\Omega \theta_i\vartheta_i\ dxdydz$

respectively, the associated norms are

$\|u\|_{\boldsymbol{H_1}}=(u,u)^{\frac{1}{2}}$ and $\|\theta\|_{\boldsymbol{H_2}}=(\theta,\theta)^{\frac{1}{2}}$.

We also define the inner product on $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ by

$((u,v))=\sum_{i,j=1}^3\int_\Omega \partial_ju_i\partial_jv_i\ dxdydz,\quad \big((\theta,\vartheta)\big)=\sum_{i=1}^3\int_\Omega \partial_j\theta_i\partial_j\vartheta_i\ dxdydz$

respectively, the associated norms are

$\|u\|_{\boldsymbol{V_1}}=((u,u))^{\frac{1}{2}}$, $\|\theta\|_{\boldsymbol{V_2}}=((\theta,\theta))^{\frac{1}{2}}$.

Let $A_i=-\Delta$ be linears operators from $D(A_i)$ into $\boldsymbol{H_i}$ respectively for $i=1,2$ defined by

$(A_iu,v)=\big((u,v)\big)\quad\forall u,v\in D(A_i)$.

A_i are adjoint and positive defined with compact inverse where

$D(A_1)=(H^2(\Omega))^3\cap\boldsymbol{V_1},\ D(A_2)=H^2(\Omega)\cap\boldsymbol{V_2}$.

We define the bilinear form $u,v\in\ \boldsymbol{V_1},\ y\in\boldsymbol{V_2}$

$\boldsymbol{B_1}(u,v)=\boldsymbol{P_1}((u.\nabla)v),\ \boldsymbol{B_2}(u,\theta)=\boldsymbol{P_2}((u.\nabla)\theta),$
here

$\boldsymbol{P_1}:(L^2(\Omega))^3\to\boldsymbol{H_1},\ \boldsymbol{P_2}:L^2(\Omega)\to\boldsymbol{H_2}$
are the projections. It is easy to check that these bilinear operators have the following algebraic properties (see e.g. [19-20]).

$$\langle\boldsymbol{B_1}(u,v),w\rangle_{V_1'}=-\langle\boldsymbol{B_1}(u,w),v\rangle_{V_1'} \tag{7}$$

$$\langle\boldsymbol{B_2}(u,y),z\rangle_{V_2'}=-\langle\boldsymbol{B_2}(u,z),y\rangle_{V_2'} \tag{8}$$

$$\langle\boldsymbol{B_1}(u,v),v\rangle_{V_1'}=0,\ \langle\boldsymbol{B_2}(u,y),y\rangle_{V_2'}=0. \tag{9}$$

Using the bilinear form $\boldsymbol{B_i}$ and the linear operator $\boldsymbol{A_i}$ $(i=1,2)$, we rewrite the system (1)-(3)

$$\frac{du}{dt}+\nu A_1u+\mu A_1u_t+\boldsymbol{B_1}(u,u)=P_1(e_3\theta) \tag{10}$$

$$\frac{d\theta}{dt}+\kappa A_2\theta+\varkappa A_2\theta_t+\boldsymbol{B_2}(u,\theta)=P_2(u_3) \tag{11}$$

$$\mathrm{u}(\boldsymbol{x},0)=\mathrm{u}_0(\boldsymbol{x}),\ \theta(\boldsymbol{x},0)=\theta_0(\boldsymbol{x}). \tag{12}$$

We recall the following 3D interpolation and Sobolev inequalities [21]

$$\|\varphi\|_{L^3}\le c\,\|\varphi\|_{L^2}^{1/2}\|\varphi\|_{H^1}^{1/2} \tag{13}$$

$$\|\varphi\|_{L^6}\le c\|\varphi\|_{H^1}^{1/2} \tag{14}$$

for every $\varphi\in H^1(\Omega)$.
The Poincaré inequality

$$\|\varphi\|^2\ \le\lambda_1^{-1}\|\nabla\varphi\|^2 \tag{15}$$

for all $\varphi\in V$.
We recall the existence uniqueness and continuity results of the weak solutions which are given in the following theorems [10].

Theorem 2.1 [10] *Let* $(u_0,\theta_0)\in V=V_1\times V_2$ *and* $\tau>0$. *The problem (1)-(6) has at least one weak solution* (u,θ) *satisfying:*
$u\in L^2(0,\tau,V_1)\cap L^\infty(0,\tau,V_1)$,
$\theta\in L^2(0,\tau,V_2)\cap L^\infty(0,\tau,V_2)$ *for any* $\tau>0$.

Theorem 2.2 [10] *Let* $u_0\in H^2\cap V_1,\ \theta_0\in H^2\cap V_{2,}\ \tau>0$. *The solution* u *and* θ *of the problem (1)-(6) is in* $C([0,\tau];V_1)\cap\ C([0,\tau];V_2)$*. Furtermore we have*

$$\|u_t\|^2+\mu\|\nabla u_t\|^2+\|\theta_t\|^2+\varkappa\|\nabla\theta_t\|^2\le c. \tag{16}$$

where the constant c is the generic constant depending on the initial datas and the paremeters of (1)-(6).

3. CONTINUOUS DEPENDENCE ON THE COEFFICIENT OF THE VOIGHT REGULARIZATION TERMS

In this section we have studied continuous dependence of the solutions of (1)-(6) on the parameters μ and $\varkappa$. The standard energy methods are employed for the proof. Now we will give the following main theorem.

Theorem 3.1 *Let* $(u_{1,}\theta_1)$, $(u_{2,}\theta_2)$ *be weak solutions of (1)-(6) corresponding to the coefficients of Voight regularization terms* μ_1, $\varkappa_1$ *and* μ_2, $\varkappa_2$ *respectively. Then for any* $t > 0$ *we have*

$\|u\|^2 + \|\theta\|^2 \to 0$ *as* $\mu_1 \to \mu_2$, $\varkappa_1 \to \varkappa_2$

and

$\|\nabla u\|^2 + \|\nabla\theta\|^2 \to 0$ *as* $\mu_1 \to \mu_2$, $\varkappa_1 \to \varkappa_2$

where $u = u_1 - u_2$ $\theta = \theta_1 - \theta_2$.

Proof. To prove continuous dependence on μ and $\varkappa$, let (u_1,θ_1) and (u_2,θ_2) are the solutions of the following boundary initial-value problems for different μ_1, $\varkappa_1$ and $\mu_2, \varkappa_2$.

$$\frac{du_1}{dt} + \nu A_1 u_1 + \mu_1 A_1 u_{1_t} + \boldsymbol{B_1}(u_1, u_1) = P_1(e_3\theta_1)$$
$$\frac{d\theta_1}{dt} + \kappa A_2\theta_1 + \varkappa_1 A_2\theta_{1_t} + \boldsymbol{B_2}(u_1, \theta_1) = P_2\left(u_{1_3}\right)$$
$$u_1(\boldsymbol{x}, 0) = \mathrm{u}_0(\boldsymbol{x}),\ \theta_1(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}),$$

$$\frac{du_2}{dt} + \nu A_1 u_2 + \mu_2 A_1 u_{2_t} + \boldsymbol{B_1}(u_2, u_2) = P_1(e_3\theta_2)$$
$$\frac{d\theta_2}{dt} + \kappa A_2\theta_2 + \varkappa_2 A_2\theta_{2_t} + \boldsymbol{B_2}(u_2, \theta_2) = P_2\left(u_{2_3}\right)$$
$$u_2(\boldsymbol{x}, 0) = \mathrm{u}_0(\boldsymbol{x}),\ \theta_2(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}).$$

Since $\boldsymbol{B_i}$ $(i = 1,2)$ are bilinear, we will write

$$\boldsymbol{B_1}(u_1, u_1) - \boldsymbol{B_1}(u_2, u_2) = \boldsymbol{B_1}(u, u_1) + \boldsymbol{B_1}(u_2, u),$$
$$\boldsymbol{B_2}(u_1, \theta_1) - \boldsymbol{B_2}(u_2, \theta_2) = \frac{\mathbf{1}}{\mathbf{2}}\{\boldsymbol{B_2}(u, \theta_1) + \boldsymbol{B_2}(u, \theta_2) + \boldsymbol{B_2}(u_1, \theta) + \boldsymbol{B_2}(u_2, \theta)\}.$$

Now we determine the difference variables $u=u_1-u_2$, $\theta=\theta_1-\theta_2$, $\mu = \mu_1 - \mu_2$ and $\varkappa = \varkappa_1 - \varkappa_2$ then (u,θ) satisfy

$$\frac{du}{dt} + \nu A_1 u + \mu A_1 u_{1_t} + \mu_2 A_1 u_t + \boldsymbol{B_1}(u, u_1) + \boldsymbol{B_1}(u_2, u) = P_1(e_3\theta)$$
$$\frac{d\theta}{dt} + \kappa A_2\theta + \varkappa A_2\theta_{1_t} + \varkappa_2 A_2\theta_t + \frac{\mathbf{1}}{\mathbf{2}}\{\boldsymbol{B_2}(u, \theta_1) + \boldsymbol{B_2}(u, \theta_2) + \boldsymbol{B_2}(u_1, \theta) + \boldsymbol{B_2}(u_2, \theta)\} = P_2(u_3)$$
$$u(\boldsymbol{x}, 0) = 0,\ \theta(\boldsymbol{x}, 0) = 0.$$

We take the inner product of the last two equations with u and θ respectively and using the (9) we can write

$$\frac{d}{dt}(\|u\|^2 + \mu_2\|\nabla u\|^2) + 2\nu\|\nabla u\|^2 \le 2\left|\left(\mu A_1 u_{1_t}, u\right)\right| + 2|\langle \boldsymbol{B_1}(u, u_1), u\rangle| + 2(P_1(e_3\theta), u)$$

$$\frac{d}{dt}(\|\theta\|^2 + \varkappa_2\|\nabla\theta\|^2) + 2\kappa\|\nabla\theta\|^2 \le 2\left|\left(\varkappa A_2\theta_{1_t}, \theta\right)\right| + |\langle \boldsymbol{B_2}(u, \theta_1), \theta\rangle| + |\langle \boldsymbol{B_2}(u, \theta_2), \theta\rangle| + 2(P_2 u_3, \theta)$$

We can estimate the terms on the right side of the last two inequalities. To do these we use Hölder, Cauchy-Schwarz, Young and Poincaré inequalities. And then we substitute these estimates to the equations and add them together we obtain

$$\frac{d}{dt}(\|u\|^2+\mu_2\|\nabla u\|^2+\|\theta\|^2+\varkappa_2\|\nabla\theta\|^2)+\|u\|^2(\nu\lambda_1-\varepsilon_3-\varepsilon_7)+\|\theta\|^2\left(\kappa\lambda_1-\frac{1}{\varepsilon_3}-\frac{1}{\varepsilon_7}\right)$$
$$+\|\nabla u\|^2\left(\nu-\varepsilon_1-\varepsilon_2-\frac{c^2}{2\sqrt{\lambda_1}}\frac{\|\nabla\theta_1\|^2}{\varepsilon_5}+\frac{c^2}{2\sqrt{\lambda_1}}\frac{\|\nabla\theta_2\|^2}{2\varepsilon_6}+\frac{c^2}{\sqrt{\lambda_1}}\frac{\|\nabla u_1\|^2}{\varepsilon_2}\right)$$
$$+\|\nabla\theta\|^2\left(\kappa-\varepsilon_4-\frac{\varepsilon_5}{2}-\frac{\varepsilon_6}{2}\right)\le\frac{\mu^2}{\varepsilon_1}\left\|\nabla u_{1_t}\right\|^2+\frac{\varkappa^2}{\varepsilon_4}\left\|\nabla\theta_{1_t}\right\|^2$$

where c is the generic constant. Since (u_1,θ_1), (u_2,θ_2) are the weak solutions of the problem, let

$$\underset{0\le t\le\tau}{Sup}(\|\nabla u\|^2)\le d_1,\ \underset{0\le t\le\tau}{Sup}(\|\nabla\theta\|^2)\le d_2,\ \ d=\max\{d_1,d_2\},$$

for positive constants d_1 and d_2.

For $\kappa>1,\ \kappa{\lambda_1}^2\nu\ge 4,\ \nu-1-\frac{4c^2d}{\sqrt{\lambda_1}}\ge 0$, and for appropriate values of ε's we have

$$\frac{d}{dt}(\|u\|^2+\mu_2\|\nabla u\|^2+\|\theta\|^2+\varkappa_2\|\nabla\theta\|^2)+$$
$$a(\|u\|^2+\mu_2\|\nabla u\|^2+\|\theta\|^2+\varkappa_2\|\nabla\theta\|^2)\le 2\mu^2\left\|\nabla u_{1_t}\right\|^2+2\varkappa^2\left\|\nabla\theta_{1_t}\right\|^2 \tag{17}$$

where

$$a=\min\left(\frac{\nu\lambda_1}{2},\frac{\kappa{\lambda_1}^2\nu-4}{\nu\lambda_1},\frac{1}{\mu_2}\left(\nu-1-\frac{4c^2d}{\sqrt{\lambda_1}}\right),\frac{1}{\varkappa_2}(\kappa-1)\right).$$

From the Theorem 2.2 we acquire the continuity result for the weak solution (u_1,θ_1) therefore using (16) we write

$$\underset{0\le t\le\tau}{Sup}\left(\left\|\nabla u_{1_t}\right\|^2\right)\le\frac{c}{\mu_1},\ \underset{0\le t\le\tau}{Sup}\left(\left\|\nabla\theta_{1_t}\right\|^2\right)\le\frac{c}{\varkappa_1}. \tag{18}$$

Using (18) right side of (17) we get

$$\frac{d}{dt}(\|u\|^2+\mu_2\|\nabla u\|^2+\|\theta\|^2+\varkappa_2\|\nabla\theta\|^2)+$$
$$a(\|u\|^2+\mu_2\|\nabla u\|^2+\|\theta\|^2+\varkappa_2\|\nabla\theta\|^2)$$
$$\le\ 2\left(\frac{c}{\mu_1}+\frac{c}{\varkappa_1}\right)[(\mu_1-\mu_2)^2+(\varkappa_1-\varkappa_2)^2].$$

Hence, using the Gronwall's lemma we achieve

$$\|u\|^2+\mu_2\|\nabla u\|^2+\|\theta\|^2+\varkappa_2\|\nabla\theta\|^2\le$$
$$2\left(\frac{c}{\mu_1}+\frac{c}{\varkappa_1}\right)(1-e^{-at})[(\mu_1-\mu_2)^2+(\varkappa_1-\varkappa_2)^2]. \tag{19}$$

From (19) we obtain

$$\|u\|^2 + \|\theta\|^2 \leq 2(\frac{c}{\mu_1} + \frac{c}{\varkappa_1})(1 - e^{-at})[(\mu_1 - \mu_2)^2 + (\varkappa_1 - \varkappa_2)^2]$$

$$\|\nabla u\|^2 + \|\nabla\theta\|^2 \leq 2(\frac{1}{\mu_2} + \frac{1}{\varkappa_2})(\frac{c}{\mu_1} + \frac{c}{\varkappa_1})(1 - e^{-at})[(\mu_1 - \mu_2)^2 + (\varkappa_1 - \varkappa_2)^2].$$

Hence the statement of the theorem holds. Provided that μ_1, μ_2 and $\varkappa_1, \varkappa_2$ are not too small we arrive at the continuous dependence of the solution on the parameters.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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