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CHEBYSHEV TYPE INEQUALITIES WITH FRACTIONAL DELTA AND NABLA H-SUM OPERATORS

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Abstract. The aim of this study is to establish new discrete inequalities for synchronous functions using fractional order delta and nabla h-sum operators. We give examples to illustrate our results.

1. Introduction

In 1882, P.L. Chebyshev [12] proved the following inequality:

Let f and g be two integrable functions on $[0, 1]$. If both functions are simultaneously increasing or decreasing for the same values of $x \in [0, 1]$, then

$$\int_0^1 f(x)g(x)dx \geq \int_0^1 f(x)dx \int_0^1 g(x)dx. \tag{1}$$

If one function is increasing and the other decreasing for the same values of $x \in [0, 1]$, then

$$\int_0^1 f(x)g(x)dx \leq \int_0^1 f(x)dx \int_0^1 g(x)dx.$$

Since then, generalizations and extensions of such type inequality have appeared in the literature, see [13, 14, 17, 18, 24] and references cited therein.

In 2009, using the fractional order integral, Belarbi and Dahmani [10] proved that:

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Let f and g be two synchronous functions on $[0,\infty)$. Then for all $t>0,\ \alpha>0$, we have

$$J_a^\alpha(fg)(t) \geq \frac{\Gamma(\alpha+1)}{t^\alpha} J_a^\alpha f(t) J_a^\alpha g(t).$$

where J_a^α is $\alpha \geq 0$ order Riemann-Liouville fractional integral operator and defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt.$$

And, the fractional order discrete Chebyshev type inequalities are studied in [3, 11]. Also, there are the fractional analogues of some well-known inequlities in the literature, see [1,2,4,5,15,21]. For more knowledge and applications about discrete and continuous fractional calculus, see [8,19,22].

In this paper, to establish the fractional analogues of Chebyshev inequality, in discrete case, we will use the delta and nabla h-sum operators defined in [9,16,20,23].

2. Preliminaries and basic results

In this section, we give some definitions and results that will be used in the sequel of this paper.

Definition 1 (Synchronous function)**.** *Two functions f and g are called synchronous, respectively asynchronous, on $\mathbb{N}_a$ if for all $\tau, s \in \mathbb{N}_a$, we have $(f(\tau)-f(s))(g(\tau)-g(s)) \geq 0$, respectively $(f(\tau)-f(s))(g(\tau)-g(s)) \leq 0$.*

Firstly, we give the result related to the delta calculus.

Let $h>0$ and $(h\mathbb{N})_a := \{a, a+h, ...\}$, $a \in \mathbb{R}$, and forward jump operator $\sigma(t) = t+h$ for $t \in (h\mathbb{N})_a$.

Definition 2. *Let $\alpha \in \mathbb{R}$, and $h>0$, then the falling $h-$factorial of t is defined by*

$$t_h^{\underline{\alpha}} = h^\alpha \frac{\Gamma(\frac{t}{h}+1)}{\Gamma(\frac{t}{h}+1-\alpha)}.$$

Definition 3 (Delta $h-$sum)**.** *The $\alpha>0$ order fractional delta h-sum of the function $f:(h\mathbb{N})_a \to \mathbb{R}$ is defined by*

$$({}_a\Delta_h^{-\alpha} f)(t) = \frac{h}{\Gamma(\alpha)} \sum_{k=\frac{a}{h}}^{\frac{t}{h}-\alpha} (t-\sigma(kh))_h^{\underline{\alpha-1}} f(kh),$$

where $({}_a\Delta_h^0 \varphi)(t) = \varphi(t)$ and $\sigma(kh) = (k+1)h$.

Definition 4. *Let $\alpha \in (n-1,n]$ and $\mu = n-\alpha$, $n \in \mathbb{N}$. The $\alpha>0$ order fractional delta h-difference of the function $f:(h\mathbb{N})_a \to \mathbb{R}$ is defned by*

$$({}_a\Delta_h^\alpha f)(t) = (\Delta_h^n({}_a\Delta_h^{-\mu} f))(t) = \frac{h}{\Gamma(-\alpha)} \sum_{k=\frac{a}{h}}^{\frac{t}{h}+\alpha} (t-\sigma(kh))_h^{\underline{\mu-1}} f(kh),$$

where $\Delta_h f(t) = \frac{f(t+h)-f(t)}{h}$, *and* $\Delta_h^n f(t) = \Delta_h^{n-1}(\Delta_h f)(t)$.

Let $0 < h \leq 1$ and $(h\mathbb{N})_a := \{a, a+h, ...\}$, $a \in \mathbb{R}$, and backward jump operator $\rho(t) = t - h$ for $t \in (h\mathbb{N})_a$.

Proposition 5. *Let* $a \in \mathbb{R}$, $\alpha > 0$. *Then*

$$_{a+ph}\Delta_h^{-\alpha}(t-a)_h^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)}(t-a)_h^{\underline{\mu+\alpha}}.$$

Proposition 6. *Let* $\alpha \in (n-1, n]$, $n \in \mathbb{N}$ *and* $\nu = (n-\alpha)h$. *Set* $p \in \mathbb{Z}\backslash\{0, 1, ..., n-1\}$ *and* $p - \alpha + 1 \notin \mathbb{Z}$. *Then*

$$_{a+ph}\Delta_h^{\alpha}(t-a)_h^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)}(t-a)_h^{\underline{\mu-\alpha}}.$$

Now, we give the preliminaries about the nabla calculus.

Let $0 < h \leq 1$ and backward jump operator $\rho(t) = t - h$ for $t \in (h\mathbb{N})_a$.

Definition 7. *Let* $\alpha \in \mathbb{R}$ *and* $0 < h \leq 1$, *then the rising* $h-$*factorial of* t *is defined by*

$$t_h^{\overline{\alpha}} = h^{\alpha}\frac{\Gamma(\frac{t}{h}+\alpha)}{\Gamma(\frac{t}{h})}.$$

Definition 8 (Nabla $h-$sum)**.** *For a function* $f : (h\mathbb{N})_a \to \mathbb{R}$, *the fractional nabla* $h-$*sum of order* $\alpha > 0$ *is defined by*

$$\begin{aligned} \left({}_a\nabla_h^{-\alpha} f\right)(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-\rho_h(s))_h^{\overline{\alpha-1}} f(s)\nabla_h s \\ &= \frac{h}{\Gamma(\alpha)}\sum_{k=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(kh))_h^{\overline{\alpha-1}} f(kh),\ t \in (h\mathbb{N})_a, \end{aligned}$$

where $\nabla_h = \frac{f(t)-f(t-h)}{h}$ *and* $\rho(kh) = (k-1)h$.

Definition 9. *The fractional nabla* $h-$*difference order* $0 < h \leq 1$ *(starting from* a*) is defined by*

$$\begin{aligned} \left({}_a\nabla_h^{\alpha} f\right)(t) &= \left(\nabla_{h\,a}\nabla_h^{-(1-\alpha)} f\right)(t) \\ &= \frac{1}{\Gamma(1-\alpha)}\nabla_h \sum_{k=a/h+1}^{t/h} (t-\rho(kh))_h^{\overline{-\alpha}} f(kh)h,\ t \in (h\mathbb{N})_{a+h}. \end{aligned}$$

Proposition 10. *Let* $\alpha > 0$, $\mu > -1$, $h > 0$, *and* $t \in (h\mathbb{N})_a$. *Then*

$$_a\nabla_h^{-\alpha}(t-a)_h^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)}(t-a)_h^{\overline{\mu+\alpha}}.$$

Remark 11. *Taking $h = 1$ in Definitions 3 and 8, we obtain*

$$\left({}_a\Delta_{h=1}^{-\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=a}^{t-\alpha} (t-\sigma(k))^{\underline{\alpha-1}} f(k), \tag{2}$$

and

$$\left({}_a\nabla_{h=1}^{\alpha} f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \nabla \sum_{k=a+1}^{t} (t-\rho(k))_h^{\overline{-\alpha}} f(k). \tag{3}$$

(2) *and* (3) *are fractional order delta and nabla sum operators defined by Atici and Eloe [6, 7].*

3. Delta Chebyshev's inequality

In this chapter, we give fractional order discrete analogues of (1), using the delta $h-$sum operator.

Theorem 12. *Let $v > 0$ and f and g are two synchronous functions on $(h\mathbb{N})_a$. Then, we have*

$$\left({}_a\Delta_h^{-v} fg\right)(t) \geq \frac{\Gamma(1+v)}{(t-a)_h^{\underline{v}}} \left({}_a\Delta_h^{-v} f\right)(t) \left({}_a\Delta_h^{-v} g\right)(t), \tag{4}$$

for all $t \in (h\mathbb{N})_a$.

Proof. Since the functions f and g are synchronous on $(h\mathbb{N})_a$, we can write

$$\left(f(\tau) - f(s)\right)\left(g(\tau) - g(s)\right) \geq 0, \tag{5}$$

for all $\tau, s \in (h\mathbb{N})_a$. From (5), we have

$$f(\tau)g(\tau) + f(s)g(s) \geq f(\tau)g(s) + f(s)g(\tau). \tag{6}$$

Taking v order delta $h-$sum of (6) respect to variable τ, gives us

$$\begin{aligned} &\left({}_a\Delta_h^{-v} fg\right)(t) + f(s)g(s)\left[{}_a\Delta_h^{-v}(1)\right] \\ &\geq g(s)\left({}_a\Delta_h^{-v} f\right)(t) + f(s)\left({}_a\Delta_h^{-v} g\right)(t) \end{aligned} \tag{7}$$

And again, taking v order delta $h-$sum of (7) respect to variable s, we get

$$\begin{aligned} &\left({}_a\Delta_h^{-v} fg\right)(t)\left[{}_a\Delta_h^{-v}(1)\right] + \left({}_a\Delta_h^{-v} fg\right)(t)\left[{}_a\Delta_h^{-v}(1)\right] \\ &\geq \left({}_a\Delta_h^{-v} g\right)(t)\left({}_a\Delta_h^{-v} f\right)(t) + \left({}_a\Delta_h^{-v} f\right)(t)\left({}_a\Delta_h^{-v} g\right)(t), \end{aligned}$$

and so

$$\left[{}_a\Delta_h^{-v}(1)\right]\left({}_a\Delta_h^{-v} fg\right)(t) \geq \left({}_a\Delta_h^{-v} g\right)(t)\left({}_a\Delta_h^{-v} f\right)(t).$$

As the last step, we calculate the ${}_a\Delta_h^{-v}(1)$. From Proposition 5, for $p = 0$, we have

$$\begin{aligned} {}_a\Delta_h^{-v}(t-a)_h^{\underline{0}} &= {}_a\Delta_h^{-v}(1) \\ &= \frac{1}{\Gamma(1+v)}(t-a)_h^{\underline{v}}. \end{aligned}$$

Finally, using this result, we have

$$\left({}_a\Delta_h^{-v}fg\right)(t) \geq \frac{\Gamma(1+v)}{(t-a)_h^{\frac{v}{h}}}\left({}_a\Delta_h^{-v}g\right)(t)\left({}_a\Delta_h^{-v}f\right)(t),$$

and this is the desired inequality. □

Theorem 13. *Let $v, \mu > 0$ and f and g are two synchronous functions on $(h\mathbb{N})_a$. Then, we have*

$$\begin{aligned}&\frac{(t-a)_h^{\frac{\mu}{h}}}{\Gamma(1+\mu)}\left({}_a\Delta_h^{-v}fg\right)(t)+\frac{(t-a)_h^{\frac{v}{h}}}{\Gamma(1+v)}\left({}_a\Delta_h^{-\mu}fg\right)(t)\\&\geq\left({}_a\Delta_h^{-\mu}g\right)(t)\left({}_a\Delta_h^{-v}f\right)(t)+\left({}_a\Delta_h^{-\mu}f\right)(t)\left({}_a\Delta_h^{-v}g\right)(t),\end{aligned}\tag{8}$$

for all $t \in (h\mathbb{N})_a$.

Proof. Proceeding as in the proof of Theorem 12, we obtain

$$\begin{aligned}&\left({}_a\Delta_h^{-v}fg\right)(t)+f(s)g(s)\left[{}_a\Delta_h^{-v}(1)\right]\\&\geq g(s)\left({}_a\Delta_h^{-v}f\right)(t)+f(s)\left({}_a\Delta_h^{-v}g\right)(t).\end{aligned}\tag{9}$$

By taking μ order delta $h-$sum of (9) respect to variable s, we have

$$\begin{aligned}&\left({}_a\Delta_h^{-v}fg\right)(t)\left[{}_a\Delta_h^{-\mu}(1)\right]+\left({}_a\Delta_h^{-\mu}fg\right)(t)\left[{}_a\Delta_h^{-v}(1)\right]\\&\geq\left({}_a\Delta_h^{-\mu}g\right)(t)\left({}_a\Delta_h^{-v}f\right)(t)+\left({}_a\Delta_h^{-\mu}f\right)(t)\left({}_a\Delta_h^{-v}g\right)(t).\end{aligned}\tag{10}$$

And using Proposition 5, from (10) we get

$$\begin{aligned}&\frac{(t-a)_h^{\frac{\mu}{h}}}{\Gamma(1+\mu)}\left({}_a\Delta_h^{-v}fg\right)(t)+\frac{(t-a)_h^{\frac{v}{h}}}{\Gamma(1+v)}\left({}_a\Delta_h^{-\mu}fg\right)(t)\\&\geq\left({}_a\Delta_h^{-\mu}g\right)(t)\left({}_a\Delta_h^{-v}f\right)(t)+\left({}_a\Delta_h^{-\mu}f\right)(t)\left({}_a\Delta_h^{-v}g\right)(t),\end{aligned}$$

so this completes the proof. □

Remark 14. *If we take $v = \mu$ in (8), then we obtain (4).*

Example 15. *Take $f(t) = (t-a)_h^{\frac{\alpha}{h}}$ and $g(t) = (t-a)_h^{\frac{\beta}{h}}$, $t \in (h\mathbb{N})_a^b = \{a, a+h, ..., b\}$. Since $f(t)$ and $g(t)$ are increasing for $t \in (h\mathbb{N})_a^b$, one can conclude that these functions are synchronous. Hence, using Theorem 13, we obtain*

$$\begin{aligned}&\frac{(t-a)_h^{\frac{\mu}{h}}}{\Gamma(1+\mu)}\left({}_a\Delta_h^{-\nu}fg\right)(t)+\frac{(t-a)_h^{\frac{\nu}{h}}}{\Gamma(1+\nu)}\left({}_a\Delta_h^{-\mu}fg\right)(t)\\\geq\;&\left({}_a\Delta_h^{-\mu}g\right)(t)\left({}_a\Delta_h^{-\nu}f\right)(t)+\left({}_a\Delta_h^{-\mu}f\right)(t)\left({}_a\Delta_h^{-\nu}g\right)(t)\\=\;&\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\mu)}(t-a)_h^{\frac{\beta+\mu}{h}}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\nu)}(t-a)_h^{\frac{\alpha+\nu}{h}}\\&+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\mu)}(t-a)_h^{\frac{\alpha+\mu}{h}}\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\nu)}(t-a)_h^{\frac{\beta+\nu}{h}}\end{aligned}$$

Taking $\nu = \mu$, we get the inequality

$$\frac{(t-a)_h^{\underline{\nu}}}{\Gamma(1+\nu)}\left({}_a\Delta_h^{-\nu}fg\right)(t) \geq \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\nu)}(t-a)_h^{\underline{\beta+\nu}}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\nu)}(t-a)_h^{\underline{\alpha+\nu}}.$$

Finally, we give a generalization of Theorem 12.

Theorem 16. *Let $v > 0$ and $f_k, 1 \leq k \leq n,\ n \in \mathbb{N}$, are functions such that $\prod_{k=1}^{l-1} f_k$ and f_l are synchronous for $l \in \{2, ..., n\}$, and $f_k \geq 0$ for $3 \leq k \leq n$. Then, we have*

$$\left({}_a\Delta_h^{-v}\prod_{k=1}^{n} f_k\right)(t) \geq \left(\frac{\Gamma(1+v)}{(t-a)_h^{\underline{v}}}\right)^{n-1}\prod_{k=1}^{n}\left({}_a\Delta_h^{-v}f_k\right)(t), \tag{11}$$

for all $t \in (h\mathbb{N})_a$.

Proof. The proof can be obtained by applying the (4) consecutively. □

Remark 17. *If we take $f_1 = f$ and $f_2 = g$ in (11) for $n = 2$, then we obtain (4).*

4. Nabla Chebysev's inequality

In this chapter, we give the nabla analogues of Theorems 12, 13 and 16.

Theorem 18. *Let $v > 0$ and f and g are two synchronous functions on $(h\mathbb{N})_a$. Then, we have*

$$\left({}_a\nabla_h^{-v}fg\right)(t) \geq \frac{\Gamma(1+v)}{(t-a)_h^{\overline{v}}}\left({}_a\nabla_h^{-v}f\right)(t)\left({}_a\nabla_h^{-v}g\right)(t), \tag{12}$$

for all $t \in (h\mathbb{N})_a$.

Proof. Taking v order nabla $h-$sum of (6) respect to variable τ, gives us

$$\begin{aligned}&\left({}_a^{-v}\nabla_h fg\right)(t) + f(s)g(s)\left[{}_a\nabla_h^{-v}(1)\right]\\ &\geq g(s)\left({}_a\nabla_h^{-v}f\right)(t) + f(s)\left({}_a\nabla_h^{-v}g\right)(t)\end{aligned} \tag{13}$$

And, taking v order nabla $h-$sum of (13) respect to variable s, we get

$$\begin{aligned}&\left({}_a\nabla_h^{-v}fg\right)(t)\left[{}_a\nabla_h^{-v}(1)\right] + \left({}_a\nabla_h^{-v}fg\right)(t)\left[{}_a\nabla_h^{-v}(1)\right]\\ &\geq \left({}_a\nabla_h^{-v}g\right)(t)\left({}_a\nabla_h^{-v}f\right)(t) + \left({}_a\nabla_h^{-v}f\right)(t)\left({}_a\nabla_h^{-v}g\right)(t)\ .\end{aligned}$$

Using the Proposition 10, we get (12). Therefore proof is completed. □

Example 19. *Take $f(t) = t_h^{\overline{\alpha}}$ and $g(t) = t_h^{\overline{\beta}}$, $t \in (h\mathbb{N})_0^b = \{0, h, 2h, ..., b\}$. From [23], we know that $f(t)$ and $g(t)$ are increasing for $t \in (h\mathbb{N})_0^b$, so $f(t)$ and $g(t)$ are synchronous functions. Therefore, we can use Theorem 18. Then, we have*

$$\left({}_0\nabla_h^{-v}fg\right)(t) \geq \frac{\Gamma(1+v)}{t_h^{\overline{v}}}\left({}_0\nabla_h^{-v}f\right)(t)\left({}_0\nabla_h^{-v}g\right)(t),$$

and using Proposition 10

$$ {}_0\nabla_h^{-v}\left(t_h^{\overline{\alpha}}.t_h^{\overline{\beta}}\right) \geq \frac{\Gamma(1+v)\Gamma\left(\frac{t}{h}\right)}{\Gamma\left(\frac{t}{h}+\nu\right)}\left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\nu)}t_h^{\overline{\alpha+\nu}}\right)\left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\nu)}t_h^{\overline{\beta+\nu}}\right). $$

Theorem 20. *Let $v,\mu>0$ and f and g are two synchronous functions on $(h\mathbb{N})_a$. Then, we have*

$$ \begin{aligned} &\frac{(t-a)_h^{\overline{\mu}}}{\Gamma(1+\mu)}\left({}_a\nabla_h^{-v}fg\right)(t)+\frac{(t-a)_h^{\overline{v}}}{\Gamma(1+v)}\left({}_a\nabla_h^{-\mu}fg\right)(t) \\ &\geq \left({}_a\nabla_h^{-\mu}g\right)(t)\left({}_a\nabla_h^{-v}f\right)(t)+\left({}_a\nabla_h^{-\mu}f\right)(t)\left({}_a\nabla_h^{-v}g\right)(t), \end{aligned} \tag{14} $$

for all $t\in(h\mathbb{N})_a$.

Proof. Taking μ order nabla $h-$sum of (13) respect to variable s, we get

$$ \begin{aligned} &\left({}_a\nabla_h^{-v}fg\right)(t)\left[{}_a\nabla_h^{-\mu}(1)\right]+\left({}_a\nabla_h^{-\mu}fg\right)(t)\left[{}_a\nabla_h^{-v}(1)\right] \\ &\geq \left({}_a\nabla_h^{-\mu}g\right)(t)\left({}_a\nabla_h^{-v}f\right)(t)+\left({}_a\nabla_h^{-\mu}f\right)(t)\left({}_a\nabla_h^{-v}g\right)(t). \end{aligned} $$

From Proposition 10, we get (14), so proof is completed. □

Remark 21. *If we take $v=\mu$ in (14), then we obtain (13).*

Finally, we give a generalization of Theorem 18 without proof.

Theorem 22. *Let $v>0$ and $f_k, 1\leq k\leq n,\ n\in\mathbb{N}$, are functions such that $\prod_{k=1}^{l-1} f_k$ and f_l are synchronous for $l\in\{2,...,n\}$, and $f_k\geq 0$ for $3\leq k\leq n$. Then, we have*

$$ \left({}_a\nabla_h^{-v}\prod_{k=1}^{n} f_k\right)(t) \geq \left(\frac{\Gamma(1+v)}{(t-a)_h^{\overline{v}}}\right)^{n-1}\prod_{k=1}^{n}\left({}_a\nabla_h^{-v}f_k\right)(t), \tag{15} $$

for all $t\in(h\mathbb{N})_a$.

Remark 23. *If we take $f_1=f$ and $f_2=g$ in (15), then we obtain (12).*

5. Conclusions

In this study, we obtained Chebyshev type inequalities using fractional order delta h-sum and nabla h-sum operators. Our results are more general than results those published before. To see that,

(i) Taking $h=1$ in Theorems 12, 13 and 16, we obtain the inequalities given by Bohner and Ferreira [11],

(ii) Taking $h=1$ in Theorems 18, 20 and 22, we get the inequalities introduced in [3].

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References

[1] Anastassiou, G. A., Nabla fractional calculus on time scales and inequalities, *J. Concr. Appl. Math.*, 11(1) (2013), 96–111.

[2] Andrić, M., Pečarić, J., Perić, I., A multiple Opial type inequality for the Riemann-Liouville fractional derivatives, *J. Math. Inequal.,* 7(1) (2013), 139–150. https://doi.org/10.7153/jmi-07-13

[3] Aslıyüce, S., Güvenilir, A. F., Chebyshev type inequality on nabla discrete fractional calculus, *Fract. Differ. Calc.,* 6(2) (2016), 275–280. https://doi.org/10.7153/fdc-06-18

[4] Aslıyüce, S., Güvenilir, A. F., Fractional Jensen's Inequality, *Palest. J. Math.,* 7(2) (2018), 554–558.

[5] Aslıyüce, S., Wirtinger type inequalities via fractional integral operators, *Stud. Univ. Babeş-Bolyai Math.,* 64(1) (2019) 1, 35–42. https://doi.org/10.24193/subbmath.2019.1.04

[6] Atici, F, M., Eloe, P. W., A transform method in discrete fractional calculus, *Int. J. Difference Equ.,* 2(2007), 165–176.

[7] Atici, F, M., Eloe, P. W., Discrete fractional calculus with the nabla operator, *Electron J. Qual. Theory Differ. Equ.,* 3(2009), 12pp.

[8] Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J. J., Fractional Calculus. Models and Numerical Methods. World Scientific Publishing Co. Pte. Ltd., Hackensack, 2017.

[9] Bastos, N. R. O., Ferreira, R. A. C, Torres, D. F. M., Discrete-time fractional variational problems, *Signal Processing*, 91(2011), 513-524. https://doi.org/10.1016/j.sigpro.2010.05.001

[10] Belarbi, S., Dahmani, Z., On some new fractional integral inequalities, *JIPAM. J. Inequal. Pure Appl. Math.,* 10(3) (2009), Article 86, 5 pp.

[11] Bohner, M., Ferreira, R. A. C., Some discrete fractional inequalities of Chebyshev type, *Afr. Diaspora J. Math.,* 11(2) (2011), 132–137.

[12] Chebyshev, P.L., Sur les expressions approximatives des integrales definies par les autres prises entre les memes limites, *Proc. Math. Soc. Charkov,* 2,(1882), 93-98.

[13] Dragomir, S. S., Crstici, B., A mapping associated to Chebyshev's inequality for integrals, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.,* 10(1999), 63–67.

[14] Dragomir, S. S., Operator Inequalities of the Jensen, Čebyšev and Grüss Type, Springer Briefs in Mathematics. Springer, New York, 2012.

[15] Ferreira, R. A. C., A discrete fractional Gronwall inequality, *Proc. Amer. Math. Soc.,* 140(5) (2012), 1605–1612. https://doi.org/10.1090/S0002-9939-2012-11533-3

[16] Ferreira, R. A. C., Torres, D. F. M., Fractional h-difference equations arising from the calculus of variations, *Appl. Anal. Discrete Math.,* 5(1) (2011), 110–121. https://doi.org/10,2298/AADM110131002F

[17] Flores-Franulič, A., Román-Flores, H., A Chebyshev type inequality for fuzzy integrals, *Appl. Math. Comput.,* 190(2) (2007), 1178–1184. https://doi.org/10.1016/j.amc.2007.02.143

[18] Gonska, H., Raşa, I., Rusu, M., Chebyshev-Grüss-type inequalities via discrete oscillations, *Bul. Acad. Ştiinţe Repub. Mold. Mat.,* 74(1) (2014), 63–89.

[19] Goodrich, C., Peterson, A. C., Discrete Fractional Calculus, Springer, Cham, 2015.

[20] Mozyrska, D., Girejko, E., Overview of Fractional h-Difference Operators., Advances in Harmonic Analysis and Operator Theory, 253–268, Oper. Theory Adv. Appl., 229, Birkhäuser/Springer Basel AG, Basel, 2013.

[21] Persson, L., Oinarov, R., Shaimardan, S., Hardy-type inequalities in fractional h-discrete calculus, *J. Inequal. Appl.,* 2018(73) (2018), 14 pp. https://doi.org/10.1186/s13660-018-1662-6

[22] Podlubny, I. Fractional Ddifferential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Academic Press, San Diego, CA, 1999.

[23] Suwan, I., Owies, S., Abdeljawad, T., Monotonicity results for h-discrete fractional operators and application, *Adv Differ Equ.,* (2018) 2018: 207. https://doi.org/10.1186/s13662-018-1660-5

[24] Wen, J. J., Pečarić, J., Han, T. Y., Weak monotonicity and Chebyshev type inequality, *Math. Inequal. Appl.,* 18(1) (2015), 217–231. https://doi.org/10.7153/mia-18-16