Süleyman Demirel Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
Cilt 22, Sayı 2, 608-614, 2018

Süleyman Demirel University
Journal of Natural and Applied Sciences
Volume 22, Issue 2, 608-614, 2018

DOI: 10.19113/sdufbed.68167

Canal Surface Around A Spacelike Focal Curve In Lorentz 3-Space

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(Alınış / Received: 11.10.2017, Kabul / Accepted: 20.06.2018, Online Yayınlanma / Published Online: 17.07.2018)

Keywords
Canal surfaces,
Tubular surfaces,
Focal curvatures,
Lorentz space

Abstract: In this paper, we study the tubular surface around a spacelike focal curve in Lorentz 3-Space. First for better understanding of the subject, the definitions and equations of the canal surface around a regular curve in 3-dimensional Euclidean space are given. Section 3, concerned with some important definitions and theorems about focal curves in 3-dimensional Lorentz space. In section 4, we derive equations for canal and tubular surfaces around a spacelike focal curve in 3-dimensional Lorentz. Then we obtain the first and the second fundamental forms on the tubular surfaces in the same space. Gauss and mean curvatures of this surface are obtained. Finally, in this space it is investigated if the parameter curves for the tubular surface are geodesic or asymptotic and related theorems about them are stated and proved.

3-Boyutlu Lorentz Uzayında Bir Uzaybenzeri Focal Eğri Etrafındaki Kanal Yüzeyi

Anahtar Kelimeler
Kanal yüzeyler,
Boru yüzeyler,
Focal eğriler,
Lorentz uzayı

Özet: Bu çalışmada 3-boyutlu Lorentz uzayında bir uzaybenzeri focal eğri etrafındaki tüp yüzeyini incelendik. Öncelikle konunun daha iyi anlaşılması için 3-boyutlu Öklid uzayında bir regüler eğriye göre kanal ve tüp yüzeylerinin tanımları ve denklemleri verildi. 3. bölümde 3-boyutlu Lorentz uzayında focal eğriler ile ilgili önemli tanım ve teoremler verildi. 4. bölümde 3-boyutlu Lorentz uzayında bir uzaybenzeri focal eğri etrafındaki kanal ve tüp yüzeylerin denklemleri çıkarıldı. Daha sonra bu uzayda tüp yüzeyinin birinci ve ikinci temel formları çıkarıldı, Gauss ve ortalama eğrilikleri verildi. Sonunda da bu uzayda tüp yüzeyleri için parametre eğrilerinin geodezik veya asimptot olma durumları incelendi ve onlarla ilgili teoremler verilip ispatlandı.

1. Preliminaries

Let $\mathbb{R}^3 = \{(x_1,x_2,x_3) \mid x_1,x_2,x_3 \in \mathbb{R}\}$ be a 3-dimensional vector space, and let $X = (x_1,x_2,x_3)$ and $Y = (y_1,y_2,y_3)$ be two vectors in $\mathbb{R}^3$. The Lorentz scalar product of X and Y is defined by

$$<X,Y>= x_1y_1 - x_2y_2 + x_3y_3. \quad (1)$$

$\mathbb{E}^3_1 = \left(\mathbb{R}^3, <,>\right)$ is called 3−dimensional Lorentzian space, Minkowski Space or 3−dimensional semi-Euclidean space. Any $X \in \mathbb{R}^3_1$ is named

- spacelike if $<X,X>> 0$ or $X = 0$,
- timelike if $<X,X>< 0$,
- null if $<X,X>= 0$ and $X \neq 0$.

Let X ,$Y \in \mathbb{R}^3_1$ and $s \in I \subset \mathbb{R}$.

- The norm of the vector X in $\mathbb{R}^3_1$ is defined as $\|X\| = |<X,X>|^{\frac{1}{2}}$.
- If $<X,Y>= 0$, then the vectors X and $Y \in \mathbb{R}^3_1$ are said to be orthogonal.
- If $\|X\| = 1$, X is called a unit vector.

Similarly, if the velocity vector $\alpha'(s) = T(s)$ at each point s is locally spacelike, timelike or null (lightlike), then α is spacelike, timelike or null, respectively.
The Lorentzian vector product of X and Y is defined as

$$X \wedge Y = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1). \quad (2)$$

Hyperbolic and Lorentzian spheres of center $M = (m_1,m_2,m_3)$ with radius r in the space $\mathbb{E}^3_1$ can be written as

$$\mathbb{H}^2_0 = \left\{A = (a_1,a_2,a_3) \in \mathbb{E}^3_1 \mid <A-M,A-M>= -r^2\right\}$$

and

$$\mathbb{S}^2_1 = \left\{A = (a_1,a_2,a_3) \in \mathbb{E}^3_1 \mid <A-M,A-M>= r^2\right\},$$

respectively.
If normal vectors at each point of $\mathbb{M}$ are timelike or spacelike vectors, then it is called as spacelike or timelike surface, respecively[6].
Let a curve $\alpha = \alpha(s) : I \longrightarrow \mathbb{E}^3_1$ be given by arclength s. We know that its velocity vector is $T(s) = \alpha'(s) = \dfrac{d\alpha(s)}{ds}$.

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Let as define the unit vector $N = \dfrac{T'(s)}{\|T'(s)\|}$. Finally, define the vector B as $B = N \wedge T$. The family $\{T, N, B\}$ is orthonormal triad. These three vectors are called the tangent, the principal normal and the binormal vectors, respectively. The family $\{T, N, B\}$ is called the Frenet frame.
For a non-lightlike curve α, the rate of change of the Frenet-Serret vector equations may be expressed as

$$\begin{aligned} T' &= \kappa N, \\ N' &= \kappa T + \tau B, \\ B' &= \tau N \end{aligned}$$

the coefficients κ and τ are the first and the second curvatures of the α, respectively [7].
In $\mathbb{E}^3_1$ curvatures of an arbitrary curve X is derived as

$$\kappa = \frac{\|X' \wedge X''\|}{\|X'\|^3}, \qquad \tau = \frac{< X' \wedge X'', X''' >}{\|X' \wedge X''\|^2}. \tag{3}$$

where $\wedge$ is cross product in $\mathbb{E}^3_1$ [3].
If α' and α'' are linearly independent in I, then the curve α is said to be good [8].
From now on, we will assume that the given curves are good curves.
Let

$$f(s) = \frac{1}{2}\left(\|C_\alpha - \alpha\|^2 - r^2\right).$$

If there exist infinitely close joint 4-points between the curve α with its osculating sphere at $s = s_0$ then we have

$$f(s_0) = f^{'}(s_0) = f^{''}(s_0) = f^{'''}(s_0) = 0. \tag{4}$$

The sphere, $\|C_\alpha - \alpha\|^2 = r^2$, with the center C_α obtained in this way is called the osculating Lorentzian sphere.
The plane spanned by the tangent vector and the principle normal vector of a curve is called the osculating plane. A point of a smooth curve in $\mathbb{E}^3_1$ for which the derivative of the curve of order 3 belongs to the osculating plane is called a flattening.
If there exist infinitely close 5-points in the neighbourhood of a point with the osculator sphere at $s = s_0$ of the curve α, it is called a vertex of the curve. Conversely, If there does not exist infinitely close 5-points in the neighbourhood of a point with the osculator sphere at $s = s_0$ of the curve α, it is called a non-vertex of the curve.
From now on, we assume that all points of the given curves are non-vertex.

2. Focal Curves in $\mathbb{E}^3_1$

In this section, we will show that, in $\mathbb{E}^3_1$ it is possible to obtain a Lorentzian tubular surface around a spacelike focal curve.

Definition 2.1. [9] Let $\alpha = \alpha(s) : I \longrightarrow \mathbb{E}^3_1$ be any curve. That the points of C_α are the centres of the osculating spheres of α is called the focal curve of α.

Lemma 2.2. *Let α be a spacelike curve with spacelike binormal in $\mathbb{E}^3_1$ and its Frenet frame be $\{T(s), N(s), B(s)\}$. Then the focal curve C_α of α is*

$$C_\alpha = \alpha + c_1 N + c_2 B \tag{5}$$

and the focal coefficients of C_α are given by

$$c_1 = -\frac{1}{\kappa}, \qquad c_2 = c_1^{'} \frac{1}{\tau} \tag{6}$$

where $\kappa \neq 0$ and $\tau \neq 0$ are the first and the second curvatures of the curve α.

Proof. We can always write the vector $C_\alpha - \alpha$ with respect to the linear independence vectors $\{T(s), N(s), B(s)\}$. Namly

$$C_\alpha - \alpha = c_0 T + c_1 N + c_2 B \tag{7}$$

If we take the Lorentz scalar product with T, N and B both sides of equation (7), then

$$\begin{aligned} &< T, C_\alpha - \alpha >= c_0, \\ &< N, C_\alpha - \alpha >= -c_1, \\ &< B, C_\alpha - \alpha >= c_2. \end{aligned}$$

On the other hand by using equation (4), we may write

$$\begin{aligned} f &= 0 \Rightarrow < C_\alpha - \alpha, C_\alpha - \alpha >= r^2, \\ f' &= 0 \Rightarrow < T, C_\alpha - \alpha >= 0, \\ f'' &= 0 \Rightarrow < N, C_\alpha - \alpha >= \frac{1}{\kappa}, \\ f''' &= 0 \Rightarrow < B, C_\alpha - \alpha >= \left(\frac{1}{\kappa}\right)^{'} \frac{1}{\tau}. \end{aligned}$$

Making use of the equations $c_1 = -\dfrac{1}{\kappa}$ and $c_2 = c_1^{'} \dfrac{1}{\tau}$. Finally, we may write the focal curve as

$$C_\alpha(s) = \alpha(s) - \frac{1}{\kappa(s)} N(s) + \left(-\frac{1}{\kappa(s)}\right)^{'} \frac{1}{\tau(s)} B(s).$$

□

Lemma 2.3. *Let $\alpha = \alpha(s) : I \longrightarrow \mathbb{E}^3_1$ be a spacelike curve with spacelike binormal. If a non-flattening point of α is a vertex, then*

$$c_2' + c_1 \tau = 0.$$

Converse is also true.

Proof. The equation of the Lorentzian spheres with center at C_α is

$$f(s) = \frac{1}{2}\left(\|C_\alpha - \alpha\|^2 - r^2\right).$$

If there exist infinitely close 5-points between α and its osculating sphere at $s = s_0$, then we have

$$f(s_0) = f'(s_0) = f''(s_0) = f'''(s_0) = f^{(4)}(s_0) = 0.$$

Calculating these derivatives we easily obtain the desired result $c_2' + c_1 \tau = 0$. □
The forthcoming theorem, lemmas and corollaries state the relations between α and its focal curve C_α.

Theorem 2.4. *Let $\alpha : I \longrightarrow \mathbb{E}^3_1$ be a spacelike curve with spacelike binormal. Let $\{T, N, B\}$ (resp. $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$) be the Frenet frame to α (resp. C_α). Let κ and τ be first and* *second curvatures of α, respectively. Then we have the connections*

$$\mathbf{t} = \varepsilon_{\mathbf{t}} B, \tag{8}$$

$$\mathbf{n} = \varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}} N, \tag{9}$$

$$\mathbf{b} = -\varepsilon_{\mathbf{n}} T, \tag{10}$$

between $\{T,N,B\}$ and $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ where

$$\varepsilon_{\mathbf{t}} = \frac{c_2' + c_1\tau}{|c_2' + c_1\tau|}, \qquad \varepsilon_{\mathbf{n}} = \frac{\tau}{|\tau|}.$$

Proof. Let σ be the arclength parameter of the focal curve C_α. If we take the derivative of both sides of (5) with respect to the arclength parameter s, we have

$$\frac{dC_\alpha}{ds} = \frac{dC_\alpha}{d\sigma}\frac{d\sigma}{ds} = \left[c_2' + c_1\tau\right] B, \tag{11}$$

and if we take the norm of both sides of (11), we get

$$\frac{ds}{d\sigma} = \frac{1}{|c_2' + c_1\tau|},$$

and if $\varepsilon_{\mathbf{t}} = \dfrac{c_2' + c_1\tau}{|c_2' + c_1\tau|}$, then

$$\mathbf{t} = \varepsilon_{\mathbf{t}} B = \frac{(c_2' + c_1\tau)}{|c_2' + c_1\tau|} B = \frac{dC_\alpha}{d\sigma}. \tag{12}$$

Now, differentiating both sides of (12) with respect to the arclength parameter s, we obtain

$$\mathbf{n} = \varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}} N \tag{13}$$

and

$$\kappa_c = \frac{|\tau|}{|c_2' + c_1\tau|}. \tag{14}$$

On the other hand, we may write

$$\mathbf{b} = \mathbf{t} \wedge \mathbf{n} = (\varepsilon_{\mathbf{t}}\mathbf{B}) \wedge (\varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}}\mathbf{N})$$

and

$$\mathbf{b} = -\varepsilon_{\mathbf{n}} T. \tag{15}$$

Then, taking the derivative of (15) with respect to the arclength parameter s, we obtain

$$\kappa = |\tau_c|\,|c_2' + c_1\tau|. \tag{16}$$

□

Corollary 2.5. *Let $\alpha = \alpha(s) : I \longrightarrow \mathbb{E}_1^3$ be a spacelike curve with spacelike binormal. If the curve α is Lorentzian spherical, then*

$$\begin{aligned} r^2 &= \|C_\alpha - \alpha\|^2, \\ &= \|c_1 N + c_2 B\|^2, \\ &= c_2^2 - c_1^2, \end{aligned}$$

where r is radius of the Lorentzian spherical and differentiating the last equation with respect to the arclenght parameter s we get

$$\left(r^2\right)' = 2c_2\left(c_2' + c_1\tau\right). \tag{17}$$

Converse is also true. According to equation (17), if r is a constant, then

$$c_2 = 0.$$

Because the curve α is a non-vertex curve, $c_2' + c_1\tau \neq 0$.

Corollary 2.6. *If we consider equations (17) and (17), the focal coefficients of c_1, c_2 of the curve α satisfy the following matrix-vector equation*

$$\begin{bmatrix} 1 \\ c_1' \\ c_2' - \dfrac{(r^2)'}{2c_2} \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} 0 \\ c_1 \\ c_2 \end{bmatrix}$$

If the curve α is spherical, $\left(r^2\right)' = 0$

According to this, we can express the following corollary.

Corollary 2.7. *Let κ and τ (resp. κ_c and τ_c) be the first and the second curvatures of α (resp. the first and the second curvatures of the focal curve C_α). If we consider equations (14) and (16), then*

$$\frac{\kappa_c}{|\tau|} = \frac{|\tau_c|}{\kappa} = \frac{1}{|c_2' + c_1\tau|} = \frac{2|c_2|}{|(r^2)'|}.$$

Corollary 2.8. *Because* $\det(\mathbf{t},\mathbf{n},\mathbf{b}) = 1$, *the focal curve C_α is a right-handed curve.*

From now on, we assume that the ranking of $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ will be {space, time, space} or {space, space, time} type.

Lemma 2.9. *Let r be the radius of Lorentzian osculating sphere. If r is constant, then κ is constant and*

$$r = |c_1| = \frac{1}{\kappa},$$

where κ and c_1 are first curvature of the curve α and the first focal coefficient of the focal curve C_α, respectively.

Proof. Since r is constant equation (17) implies either $c_2 = 0$ or $c_2' + c_1\tau = 0$. If $c_2' + c_1\tau = 0$, then the curve is spherical. If $c_2 = 0$, $c_1'\dfrac{1}{\tau} = 0$. This means that $c_1 = -\dfrac{1}{\kappa}$ is constant. □

Lemma 2.10. *If we take the derivative of the Frenet frame $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ of the focal curve C_α with respect to the arclength parameter s, we have*

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & \nu\kappa_c & 0 \\ \nu\kappa_c & 0 & \nu\tau_c \\ 0 & \nu\tau_c & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix},$$

where $\nu = \dfrac{d\sigma}{ds} = |c_2' + c_1\tau|$. If the radius of the osculating sphere r is constant, then

$$\nu = \frac{d\sigma}{ds} = r|\tau|,$$

where s and σ are the arclength parameters of the curve α and the focal curve C_α, respecively.

Now, let us state the equations for canal and tubular surfaces around any good curve in $\mathbb{E}^3$.

3. Canal Surfaces in $\mathbb{E}^3$

Let us recall the definitions and the results of [1, 9]. A canal surface is named as the envelope of a family of 1-parameter spheres. In other words, it is the envelope of a moving sphere with varying radius, defined by the trajectory with center $\alpha(t)$ and a radius function $r(t)$. This moving sphere $S(t)$ touches it at a characteristic circle $K(t)$. If the radius function $r(t) = r$ is a constant, then it is called a tubular or pipe surface. Let $\{T,N,B\}$ be the Frenet vector fields of α, where T, N and B are tangent, principal normal and binormal vectors to α, respectively. Since the canal surface $K(t,\theta)$ is the envelope of a family of one parameter spheres with the center α and radius function r, it is parametrized as

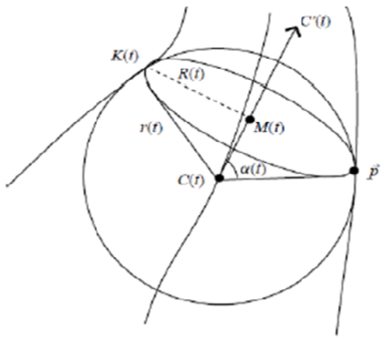

Figure 1. A section of the canal surface (Doğan 2012).

$$
\begin{aligned}
K(t,\theta) &= \alpha(t) - r(t)r'(t)\frac{\alpha'(t)}{\|\alpha'(t)\|} \\
&\pm \cos\theta\, r(t)\frac{\sqrt{\|\alpha'(t)\|^2 - r'(t)^2}}{\|\alpha'(t)\|}N(t) \\
&\pm \sin\theta\, r(t)\frac{\sqrt{\|\alpha'(t)\|^2 - r'(t)^2}}{\|\alpha'(t)\|}B(t).
\end{aligned}
$$

This surface is called the canal surface around the curve α. Clearly, $N(t)$ and $B(t)$ are spanning the plane that contains the characteristic circle. If the spine curve $\alpha(s)$ has an arclenght parametrization $\left(\|\alpha'(s)\| = 1\right)$, then the canal surface is reparametrized as

$$
\begin{aligned}
K(s,\theta) &= \alpha(s) - r(s)r'(s)T(s) \\
&\pm \cos\theta\, r(s)\sqrt{1 - r'(s)^2}N(s) \\
&\pm \sin\theta\, r(s)\sqrt{1 - r'(s)^2}B(s).
\end{aligned}
$$

For the constant radius case $r(s) = r$, the canal surface is called a tubular (pipe) surface and in this case the equation takes the form

$$L(s,\theta) = \alpha(s) + r(\cos\theta N(s) + \sin\theta B(s)),$$

where $0 \leq \theta \leq 2\pi$.

Let a regular curve $\alpha : I \longrightarrow M$ be parametrized so that $\|\alpha'(s)\| = 1$. Then we have

$$
\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix},
$$

where κ and τ are the curvature and the torsion of the curve $\alpha(s)$, respectively.

Now, let us see what happens if we take the focal curve C_α of α instead of the curve α itself in $\mathbb{E}^3_1$.

4. Canal Surfaces in $\mathbb{E}^3_1$

Now, we state and prove an important theorem related to our present study. However, first we need the following definition.

Definition 4.1. A canal surface in $\mathbb{E}^3_1$ is named as the envelope of a family of 1-parameter Lorentzian spheres. In other words, it is the envelope of a moving Lorentzian sphere with varying radius, defined by the trajectory with center $C_\alpha(s)$ and a radius function $r(t)$. This moving sphere $S(t)$ touches it at a Lorentzian characteristic circle $K(t)$. If the radius function $r(t) = r$ is a constant, then it is called as a Lorentzian tubular or pipe surface in $\mathbb{E}^3_1$.

Theorem 4.2. *Let $\alpha = \alpha(s) : I \longrightarrow \mathbb{E}^3_1$ be a spacelike curve with spacelike binormal. Then, the canal surface around its spacelike focal curve $C_\alpha(s)$ can be parametrized as follows*

$$
\begin{aligned}
K(s,t) &= C_\alpha(s) - \frac{r(s)r'(s)}{\nu}B(s) \\
&\mp \lambda(t)\, r(s)\sqrt{1 - \left(\frac{r'(s)}{\nu}\right)^2}\,T(s) \\
&\pm \mu(t)\, r(s)\sqrt{1 - \left(\frac{r'(s)}{\nu}\right)^2}\,N(s).
\end{aligned}
$$

Proof. Let K be any point of the canal surface and C_α be the center of a Lorentzian spheres $\mathbb{S}^2_1(s)$. Then the difference $K(s,t) - C_\alpha(s)$ can be written in terms of the orthogonal vectors $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ as

$$
\begin{aligned}
K(s,t) - C_\alpha(s) &= c(s,t)\mathbf{t}(s) + b(s,t)\mathbf{n}(s) \\
&+ a(s,t)\mathbf{b}(s).
\end{aligned}
$$

By using the connections in (8), the last equation can be rewritten as

$$
\begin{aligned}
K(s,t) - C_\alpha(s) &= -a(s,t)\varepsilon_\mathbf{n}T(s) - b(s,t)\varepsilon_\mathbf{t}\varepsilon_\mathbf{n}N(s) \\
&+ c(s,t)\varepsilon_\mathbf{t}B(s) \qquad (18)
\end{aligned}
$$

where a, b and c have partial derivatives with respect to the variables s and t on I. On the other hand, taking the norm of both sides of equation (18) we obtain

$$\|K(s,t) - C_\alpha(s)\|^2 = r^2(s). \qquad (19)$$

The equation (19) expresses that $K(s,t)$ lies on a Lorentzian sphere $\mathbb{S}^2_1(s)$. Additionally, $K(s,t) - C_\alpha(s)$ is an orthogonal vector to the canal surface which means that

$$
\begin{aligned}
< K(s,t) - C_\alpha(s), K_s > &= 0, \qquad (20) \\
< K(s,t) - C_\alpha(s), K_t > &= 0. \qquad (21)
\end{aligned}
$$

The equations in (20) and (21) indicate that velocity vector of parameter curves K_s and K_t of the canal surface are tangent to $\mathbb{S}_1^2(s)$. By making use of (18) and (19), we immediately obtain the equations

$$\left.\begin{aligned} a^2-b^2+c^2 &= r^2, \\ aa_s-bb_s+cc_s &= rr'. \end{aligned}\right\} \tag{22}$$

Using the partial derivative

$$\begin{aligned} K_s &= (-a_s\varepsilon_{\mathbf{n}}+b\varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}}\kappa)T \\ &\quad +(-a\varepsilon_{\mathbf{n}}\kappa+b_s\varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}}+c\varepsilon_{\mathbf{t}}\tau)N \\ &\quad +(\nu\varepsilon_{\mathbf{t}}+b\varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}}\tau+c_s\varepsilon_{\mathbf{t}})B \end{aligned} \tag{23}$$

of (18) with respect to s, we may rewrite equation (20) as

$$<K(s,t)-C_\alpha(s),K_s>=aa_s-bb_s+cc_s+c\nu=0. \tag{24}$$

Then equation (22) together with (24), lead to the equalities

$$-c\nu=rr'$$

and

$$a^2-b^2=r^2\left[1-\left(\frac{r'}{\nu}\right)^2\right]. \tag{25}$$

from which we obtain

$$\begin{aligned} a &= \mp r\sqrt{1-\left(\frac{r'}{\nu}\right)^2}\cosh t, \\ b &= \mp r\sqrt{1-\left(\frac{r'}{\nu}\right)^2}\sinh t. \end{aligned}$$

If we substitute these values of a and b in (18), we obtain the equation

$$\begin{aligned} K(s,t) &= C_\alpha(s)-\frac{\varepsilon_{\mathbf{t}}r(s)r'(s)}{\nu}B(s) \\ &\quad \mp\varepsilon_{\mathbf{n}}(\cosh t)r(s)\sqrt{1-\left(\frac{r'(s)}{\nu}\right)^2}\,T(s) \\ &\quad \pm\varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}}(\sinh t)r(s)\sqrt{1-\left(\frac{r'(s)}{\nu}\right)^2}\,N(s). \end{aligned} \tag{26}$$

If the radius r is constant, the equation (26) takes the form

$$L(s,t)=C_\alpha(s)+\varepsilon_{\mathbf{n}}\cosh t rT(s)-\varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}}\sinh t rN(s). \tag{27}$$

This means that equation (27) is the Lorentzian tubular surface with parameters s and t. Without loss of generality, in (27) we can take $\varepsilon_{\mathbf{t}}=\varepsilon_{\mathbf{n}}=1$. With this choice, (27) reads as

$$L(s,t)=C_\alpha(s)+r\cosh tT(s)-r\sinh tN(s). \tag{28}$$

□

In the next section, we give the fundamental forms which are crutial for the characterization of the Lorentzian tubular surfaces.

5. Fundamental Forms

Let $\alpha=\alpha(s):I\longrightarrow\mathbb{E}_1^3$ be any unit speed spacelike curve with spacelike binormal. A parametrization $L(s,t)$ of the Lorentzian tubular surface around its spacelike focal curve $C_\alpha(s)$ has given in (28). The partial derivatives of L with respect to the surface parameters s and t can be expressed in terms of Frenet vector fields of α as

$$\begin{aligned} L_s &= -\sinh tT+\cosh tN+r\tau(1-\sinh t)B, \\ L_t &= r\sinh tT-r\cosh tN. \end{aligned}$$

We can also choose a unit normal vector field U as

$$U=\frac{L_s\wedge L_t}{\|L_s\wedge L_t\|}=\cosh tT-\sinh tN,$$

where we know that

$$\|L_s\wedge L_t\|^2=EG-F^2=r^4\tau^2(1-\sinh t)^2. \tag{29}$$

The first fundamental form I of L is defined as

$$I=Edx^2+2Fdxdy+Gdy^2$$

where

$$\begin{aligned} E &= <L_s,L_s>=-1+r^2\tau^2(1-\sinh t)^2, \\ F &= <L_s,L_t>=r, \\ G &= <L_t,L_t>=-r^2. \end{aligned}$$

On the other hand, the second fundamental form II of L is defined as

$$II=edx^2+2fdxdy+gdy^2$$

in which

$$\begin{aligned} e &= <U,L_{ss}>=\kappa+r\tau^2\sinh t(1-\sinh t), \\ f &= <U,L_{st}>=-1, \\ g &= <U,L_{tt}>=r. \end{aligned}$$

Corollary 5.1. *The tubular surface in (28) is a timelike surface.*

Definition 5.2. [1] Let M be any surface and the set $\{E,F,G\}$ be the coefficients of its first fundamental form. M is called a regular surface if $EG-F^2\neq 0$.

Lemma 5.3. *$L(s,t)$ is a regular tube, iff* $\sinh t\neq 1$.

Proof. It can easily be proved by using equation (29) and definition 5.2. □

Theorem 5.4. *The mean and the Gaussian curvatures of a regular surface $L(s,t)$ are*

$$H=\frac{eG-2fF+gE}{2(EG-F^2)}=\frac{1}{2}\left(rK-\frac{1}{r}\right) \tag{30}$$

and

$$K=\frac{eg-f^2}{EG-F^2}=-\frac{\sinh t}{r^2(1-\sinh t)} \tag{31}$$

respectively.

6. Some Special Parameter Curves on The Lorentzian Tubular Surfaces in $\mathbb{E}_1^3$

Theorem 6.1. [5] *Let the curve γ lie on a surface. If γ is an asymptotic curve, then the acceleration vector is orthogonal to the normal vector of the surface.*

Theorem 6.2. *Let $L(s,t)$ be a Lorentzian tubular surface around spacelike focal curve of $\alpha(s)$, then the curves L_s and L_t can not be asymptotic.*

Proof. For the $s-$parameter curves we obtain the first coefficient e of second fundamental form as

$$e = < U, L_{ss} > = \left(\kappa + r\tau^2 \sinh t\right)(1-\sinh t) \neq 0$$

showing that they can not be asymptotic. Similarly, for the $t-$parameter curves we obtain the third coefficient g of second fundamental form as

$$g = < U, L_{tt} > = r \neq 0$$

which implies that they can not be asymptotic. □

Theorem 6.3. [2] *Let the curve γ lie on a surface. If γ is a geodesic curve, then the acceleration vector γ'' and the normal vector U of the surface are linearly dependent. That is, $U \wedge \gamma'' = 0$.*

Theorem 6.4. *Let $L(s,t)$ be a Lorentzian tubular surface around a spacelike focal curve of $\alpha(s)$, then*
(1) The L_s curves can not be geodesic
(2) The L_t curves are geodesic curves.

Proof. For the $s-$parameter curves, we have

$$\begin{aligned} U \wedge L_{ss} &= -\sinh t\left[\tau \cosh t + r\tau'(1-\sinh t)\right] T \\ &\quad + \cosh t\left[\tau \cosh t + r\tau'(1-\sinh t)\right] N \\ &\quad - r\tau^2 (1-\sinh t)\cosh t B. \end{aligned}$$

If the last equation were zero, i.e., $U \wedge L_{ss} = 0.$, we would have

$$\begin{aligned} \sinh t\left[\tau \cosh t + r\tau'(1-\sinh t)\right] &= 0, \\ \cosh t\left[\tau \cosh t + r\tau'(1-\sinh t)\right] &= 0, \\ r\tau^2 (1-\sinh t)(\cosh t) &= 0 \end{aligned} \quad (32)$$

since the vectors $\{T,N,B\}$ are linearly independent. However, since $L(s,t)$ is a regular surface, equation (32) can not be zero. Therefore $U \wedge L_{ss} \neq 0$ which shows that L_s curves can not be geodesics. On the other hand, since

$$U \wedge L_{tt} = U \wedge rU = 0 \quad (33)$$

the $t-$parameter curves L_t are geodesics. Converse is also true and it is trivial. □

Example 6.5. Let γ be a spacelike curve in $\mathbb{E}_1^3$ defined by

$$\begin{aligned} \gamma: \ I &\longmapsto \mathbb{E}_1^3 \\ s &\longmapsto \gamma(s) = \left(\sinh \frac{s}{\sqrt{2}}, \cosh \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right), \end{aligned}$$

where $-4 \leq s \leq 4$. Figure 2 includes the graph of the curve.

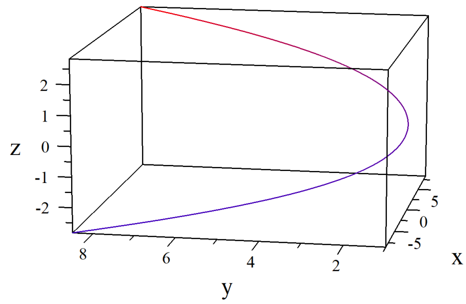

Figure 2. The curve γ of Example 6.5.

Its velocity vector of the curve is

$$\dot{\gamma}(s) = \left(\frac{1}{\sqrt{2}}\cosh \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sinh \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

In this example, we will consider the Lorentz scalar product in (1) and the Lorentzian vectorial product in (2). The Frenet vectors $\{T,N,B\}$ of the curve γ are

$$\begin{aligned} T &= \left(\frac{1}{\sqrt{2}}\cosh \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sinh \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \\ N &= \left(\sinh \frac{s}{\sqrt{2}}, \cosh \frac{s}{\sqrt{2}}, 0\right), \\ B &= \left(-\frac{1}{\sqrt{2}}\cosh \frac{s}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\sinh \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right). \end{aligned}$$

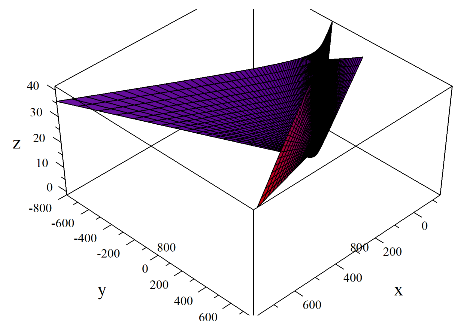

Figure 3. The focal curve C_γ of the curve γ in Example 6.5.

The curvatures of γ are found to be $\kappa = \frac{1}{2}$ and $\tau = -\frac{1}{4}$ by making use ot the equation in (3). Hence, τ/κ is constant. Therefore, the curve γ is the Lorentz circular helix in $\mathbb{E}_1^3$. The focal coefficients of γ can be computed from (6) as $C_1 = -2$ and $C_2 = 0$. For this specific example, by using (5), the focal curve C_γ of γ may be computed as

$$C_\gamma = \gamma - 2N.$$

The last equation and equation (28) with $r = 2$ lead to the components

$$\begin{aligned}
x(s,t) &= -\sinh\frac{s}{\sqrt{2}}+\frac{2}{\sqrt{2}}\cosh t\cosh\frac{s}{\sqrt{2}} \\
&\quad -2\sinh t\sinh\frac{s}{\sqrt{2}}, \\
y(s,t) &= -\cosh\frac{s}{\sqrt{2}}+\frac{2}{\sqrt{2}}\cosh t\sinh\frac{s}{\sqrt{2}} \\
&\quad -2\sinh t\cosh\frac{s}{\sqrt{2}}, \\
z(s,t) &= \frac{s}{\sqrt{2}}+\frac{2}{\sqrt{2}}\cosh t.
\end{aligned}$$

of the tubular surface $L(s,t) = (x(s,t), y(s,t), z(s,t))$. Tubular surface around the focal curve C_γ is shown in Figure 3.

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