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On Approximate Solution of the Euler-Bernoulli Beam Equation via Galerkin Method

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Abctract
In this paper, numerical solution of a boundary value problem for a fourth-order ordinary differential equation, known as the Euler-Bernoulli beam equation, is presented. The related equation is used extensively in engineering areas such as huge buildings, long bridges across big rivers, planes and cars. The approximate solution of the problem considered is obtained by using the Galerkin method with basis functions that satisfy the boundary conditions given. The accuracy of the proposed method is given through two numerical examples with the help of Maple® program.

Keywords: Euler-Bernoulli beam equation, Galerkin method, Ordinary differential equations

Galerkin Metodu Yardımıyla Euler-Bernoulli Kiriş Denkleminin Yaklaşık Çözümü

Öz
Bu çalışmada Euler-Bernoulli kiriş denklemi olarak bilinen dördüncü mertebeden bir adi diferansiyel denklem için sınır değer probleminin sayısal çözümü sunulmuştur. İlgili denklem büyük binalar, büyük nehirler arasındaki uzun köprüler, uçaklar ve arabalar gibi mühendislik alanlarında yaygın olarak kullanılmaktadır. Ele alınan problemin yaklaşık çözümü, sınır koşullarını sağlayan temel fonksiyonlar ile Galerkin metodu kullanılarak elde edilmektedir. Önerilen yöntemin doğruluğu Maple® programı yardımıyla iki nümerik örnek üzerinden gösterilmektedir.

Anahtar Kelimeler: Euler-Bernoulli kiriş denklemi, Galerkin metodu, Adi diferansiyel denklemler

1. Introduction

The Euler-Bernoulli beam equation that describes steady-state vibrations of a beam on an elastic foundation is the fourth-order differential equation

$$\left(a(x)u''(x)\right)'' + q(x)u(x) = f(x) \quad (1)$$

for $x \in [0, L]$. Here L is the length of the beam, $u(x)$ is the deflection of the beam, $f(x)$ is the transverse distributed load and $q(x)$ is the foundation modulus at the point x (Lesnic, 2006; Thankane and Stys, 2009). Also the function $a(x) = EI$ is the product of the Young's modulus E and moment of inertia I of the beam (Thankane and Stys, 2009; Gunakala et al., 2012). The analysis of beams on elastic foundation is very common in the sciences, especially structural and mechanical engineering. The beams are used as a basis of supporting structures or as the main frame foundation in application areas such as high buildings, bridges between rivers, air vehicles and heavy motor vehicles (Gunakala et al., 2012). Therefore, the beam theory must be used correctly in order to produce such structures successfully and safely. We refer the reader to (Biot, 1937; Han et al., 1999).

We consider the beam equation (1) with the free ends boundary conditions

$$\begin{gathered} u(0) = 0,\ u(L) = 0, \\ u''(0) = 0,\ u''(L) = 0. \end{gathered} \quad (2)$$

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The exact solution $u(x)$ of the problem (1)-(2) can be found by standard methods that are well known in literature of the ordinary differential equations when a, q and f are known simple functions. In general it is not always possible to obtain exact solution of an arbitrary differential equation. It may also be laborious to get the theoretical solution. In these cases, we can use the effective numerical methods to find an approximate solution of the boundary value problems.

The Galerkin method is one of the well-known methods to provide powerful numerical solutions for the ordinary and partial differential equations. Its simplicity makes it perfect for many applications. The approximations to solution of the system of the ordinary differential equations have been obtained by using the Galerkin method in literature (Dubeau et al., 2003; Kostadinova, et al., 2013; Al-Omari et al., 2013). Peradze has been studied the numerical methods to find the approximation solutions of the Kirchhoff-Type Nonlinear Static Beam Equations via Galerkin method in (Peradze, 2009; Peradze, 2016). In (Subaşı et al., 2011; Şener et al., 2013), applications of the Galerkin method to hyperbolic problems have been explained. Smith et al. (1992) have solved the fourth-order Euler-Bernoulli partial differential equations by Galerkin method and tested the results obtained on numerical examples. Younesian et. al. (2012) have obtained the analytical solutions of the beam equation on nonlinear elastic foundations by using the Variational Iteration Method. Thankane and Stys (2009) have presented the mathematical analysis of effective algorithms based on the Finite Difference Method for a beam equation. Gunakala et al. (2012) have given the use of the Galerkin Finite Element Method to solve the beam equation with homogeneous and non-homogeneous boundary conditions by choosing the cubic interpolation functions as the basis functions. Musa (2017) has obtained the solution for bending analysis of a beam on a non-homogeneous foundation by using the Galerkin method.

In this work we present the effective numerical method for solving the steady-state Euler-Bernoulli beam equation with free ends boundary conditions. The proposed method, Galerkin method, uses the basis functions that satisfy the boundary conditions to get the approximate solutions. We get useful approximate weak solutions on some numerical examples by using Maple®.

The paper is organized as follows. In Section 2, we give the definition of the weak solution for the boundary value problems (1)-(2) and explain how Galerkin method is applied this problem. In section 3, we get symbolically the approximation solutions that confirm effectiveness of the method on the numerical examples.

2. Material ve Method

In this section, we will present the Galerkin method similar to carried out in (Ladyzhenskaya, 1985). Assume that $a, q, f \in L^2[0, L]$ and $q(x) \geq 0$ for $x \in [0, L]$. The function $a(x)$ satisfies the condition

$$0 < a_0 \leq a(x) \leq a_1 \qquad (3)$$

where a_0 and a_1 are constants. Then the problem (1)-(2) has a unique solution $u \in \widetilde{H}^2[0, L]$ weak sense where

$$\widetilde{H}^2[0, L] := \{u \in H^2[0, L] : u(0) = u(L) = 0\}$$

(see Lesnic, 2006; Ladyzhenskaya, 1985).

For the weak solution of the problem we mean the function $u \in \widetilde{H}^2[0, L]$ which satisfies the following integral equality:

$$\int_0^L a(x)u''(x)v''(x)dx + \int_0^L q(x)u(x)v(x)\,dx = \int_0^L f(x)v(x)dx \qquad (4)$$

for all $v \in \widetilde{H}^2[0, L]$.

Here the space $L^2[0,L]$ is the complete linear normed space consisting of all measurable (in the sense of Lebesque) functions on $[0,L]$ (Hunter, 2000). $H^2[0,L]$ is the Hilbert space consisting of all the elements $L^2[0,L]$ having generalized derivatives of first and second order from $L^2[0,L]$.

Let's perform an approximation of the solution of the boundary-value problem (1)-(2) by the Galerkin method. Due to this method, we can denote the solution of the problem by u^N and define by formula

$$u^N(x) = \sum_{i=1}^{N} c_i v_i(x) \tag{5}$$

where the functions $v_i(x)$ are the basis functions and the coefficients c_i are constants to be obtained.

Due to boundary conditions (2), the basis functions can be derived from the set

$$\left\{\sqrt{\frac{2}{L}}sin\frac{\pi x}{L}, \sqrt{\frac{2}{L}}sin\frac{2\pi x}{L}, \cdots, \sqrt{\frac{2}{L}}sin\frac{N\pi x}{L}\right\} \tag{6}$$

We write the statement (1) for u^N, multiply $v_l, l = 1,2,\cdots,N$ and integrate over $[0,L]$:

$$\begin{aligned} &\int_0^L [a(x)(u^N)'']'' v_l(x)dx \\ &+ \int_0^L q(x)u^N v_l(x)dx \\ &= \int_0^L f(x)v_l(x)dx, \ (l = 1,2,\cdots,N) \end{aligned} \tag{7}$$

By substituting the approximation (5) into (7) and applying integration by parts, we obtain the system of the linear equations with unknowns $c_i, i = 1,2,\cdots,N$ as

$$\begin{aligned} &\int_0^L [a(x)\sum_{i=1}^{N} c_i v_i''(x)\, v_l''(x)]dx \\ &+ \int_0^L q(x)\sum_{i=1}^{N} c_i v_i(x)\, v_l(x)dx \\ &= \int_0^L f(x)v_l(x)dx \end{aligned} \tag{8}$$

for $l = 1,2,\cdots,N$. The coefficients of the system (8) are constants. The solution of this system gives the coefficients of (5).

3. Findings

In this section, we solve two test problems to show the effectiveness of the method given in the Section 2 with aid of Maple®. For purposes of comparison, we chose examples with known solutions and use the L^2-norm error and absolute error which is defined as $E = |u - u^N|$.

Example 1. For $x \in [0,2]$, we consider the following beam equation

$$\begin{aligned} &(e^x u'')'' + e^x u \\ &= e^x(x^7 - 8x^6 + 66x^5) \\ &+e^x(148x^4 - 584x^3 - 384x^2) \\ &+e^x(1440x - 576) \end{aligned} \tag{9}$$

with corresponding homogeneous boundary conditions

$$\begin{aligned} &u(0) = 0,\ u(2) = 0, \\ &u''(0) = 0,\ u''(2) = 0. \end{aligned} \tag{10}$$

The exact solution of this boundary value problem is the function $u(x) = x^3(x-2)^4$.

Using the basis functions $\{v_l(x)\} = \left\{\sin\frac{l\pi x}{2}\right\}$, $l = 1,2,\cdots,10$ and solving the corresponding system of the linear equations, we have the following approximate solution

$$\begin{aligned} u^{10} = &\ 0.79587824 \sin\frac{\pi x}{2} \\ &+0.20042622 \sin \pi x \\ &-0.21991419 \sin\frac{3\pi x}{2} \\ &-0.07192558 \sin 2\pi x \\ &-0.01845701 \sin\frac{5\pi x}{2} \\ &-0.01135205 \sin 3\pi x \\ &-0.00347032 \sin\frac{7\pi x}{2} \\ &-0.00281533 \sin 4\pi x \\ &-0.00093712 \sin\frac{9\pi x}{2} \\ &-0.00081528 \sin 5\pi x \end{aligned}$$

In Table 1, we present a list of errors rounded off to eight decimal places in space $L^2[0, L]$ for increasing values of N which is the number of used fundamental functions. Similar results can be found for different N values.

Table 1. Some $L^2[0,2]$ errors of Example 1 for some values of N.

N	$\|\boldsymbol{u}(.)-\boldsymbol{u}^N(.)\|^2_{L^2[0,2]}$
10	$0.42309729 \times 10^{-6}$
20	$0.99203057 \times 10^{-9}$
30	$0.29202512 \times 10^{-9}$
40	$0.19999724 \times 10^{-9}$
50	$0.11526815 \times 10^{-9}$
60	$0.10484114 \times 10^{-9}$
70	$0.10247068 \times 10^{-9}$
80	$0.10000000 \times 10^{-9}$

It can be seen from Table 1 that

$$\|u(.)-u^N(.)\|^2_{L^2[0,2]} \to 0$$

when $N \to +\infty$. That is, the error decrease as the number of basis function used increase.

The following Table 2 presents the absolute errors for some x values for Example 1.

Table 2. The absolute errors of Example 1 for some x values when $N = 80$.

x	E
0.1	$0.13329954 \times 10^{-6}$
0.2	$0.74037910 \times 10^{-7}$
0.3	$0.50233288 \times 10^{-7}$
0.4	$0.37021640 \times 10^{-7}$
0.5	$0.28712500 \times 10^{-7}$
0.6	$0.23074682 \times 10^{-7}$
0.7	$0.19014300 \times 10^{-7}$
0.8	$0.15196300 \times 10^{-7}$
0.9	$0.13182000 \times 10^{-7}$

Example 2. Let us consider the functions $a(x), q(x)$ and $f(x)$ in the problem (1), respectively as

$$a(x) = 1,$$

$$q(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{2} \\ x, & \frac{1}{2} < x \le 1 \end{cases},$$

$$f(x) = \begin{cases} x^6 - 3x^5 + 3x^4 - x^3 + 360x^2 - 360x + 72, & 0 \le x \le \frac{1}{2} \\ x^7 - 3x^6 + 3x^5 - x^4 + 360x^2 - 360x + 72, & \frac{1}{2} < x \le 1 \end{cases}$$

and the following boundary conditions

$$u(0) = 0,\ u(1) = 0,$$
$$u''(0) = 0,\ u''(1) = 0.$$

Note that the function $q(x)$ and $f(x)$ are not continuous. The weak solution for this problem is $u(x) = x^6 - 3x^5 + 3x^4 - x^3$. Using Galerkin method for $N = 10$ we get the following the approximate solution

$$
\begin{aligned}
u^{10} &= -0.01243386 \sin \pi x \\
&+0.00343689 \sin 3\pi x \\
&+0.00028895 \sin 5\pi x \\
&+0.00005483 \sin 7\pi x \\
&+0.00001573 \sin 9\pi x \\
&+0.21892454 \times 10^{-10} \sin 2\pi x \\
&-0.41397583 \times 10^{-12} \sin 4\pi x \\
&+0.29585831 \times 10^{-12} \sin 6\pi x \\
&-0.78434871 \times 10^{-12} \sin 8\pi x \\
&+0.14875231 \times 10^{-12} \sin 10\pi x
\end{aligned}
$$

We present the L^2-norm errors for different N values in Table 3 and absolute errors for different x values in Tablo 4 for Example 2.

Table 3. Some $L^2[0,1]$ errors of Example 2 for some values of N.

N	$\|\boldsymbol{u}(.) - \boldsymbol{u}^N(.)\|_{L^2[0,1]}^2$
10	$0.21060126 \times 10^{-10}$
20	$0.46125093 \times 10^{-13}$
30	$0.12242365 \times 10^{-14}$
40	$0.79999699 \times 10^{-16}$
50	$0.11567392 \times 10^{-16}$
60	$0.10758158 \times 10^{-16}$
70	$0.10000000 \times 10^{-16}$
80	$0.75868155 \times 10^{-17}$

It can be seen from Table 3 that

$$\|u(.) - u^N(.)\|_{L^2[0,1]}^2 \to 0$$

when $N \to +\infty$. From Table 3, we can say that the error decrease as the number of basis function used increase.

Table 4. The absolute errors of Example 2 for some x values when $N = 80$.

x	E
0.1	$0.45634082 \times 10^{-9}$
0.2	$0.26124489 \times 10^{-9}$
0.3	$0.20681252 \times 10^{-9}$
0.4	$0.18476063 \times 10^{-9}$
0.5	$0.20290400 \times 10^{-9}$
0.6	$0.19751400 \times 10^{-9}$
0.7	$0.22511145 \times 10^{-9}$
0.8	$0.27719686 \times 10^{-9}$
0.9	$0.46299088 \times 10^{-9}$

4. Result and Discussion

In this study, we introduce the Galerkin method to solve the Euler-Bernoulli boundary value problem. The results obtained indicate that this method can be applied to get accurate numerical solutions of the problem (1)-(2). In other words, Galerkin method is an influential numerical method to solve the problem (1)-(2). When the number of basis functions used increase for Galerkin method, the approximate solution becomes closer to weak solution. That is the accuracy of the method depend on the value of N.

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