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Well-Posedness Analysis for an Inverse Coefficient Nonlinear Wave Equation

Research Article

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Abstract

In this study, a nonlinear hyperbolic equation with a time-dependent unknown inverse coefficient is investigated under periodic boundary conditions. The Fourier method is employed to obtain a solution to the problem, and the existence, uniqueness, and stability of the solution are established.

Keywords: Inverse problem, Nonlinear hyperbolic equation, Periodic boundary conditions.

1. INTRODUCTION

Wave propagation models arise naturally in physics, engineering, and applied mathematics, and classical theory shows that their behavior is governed by hyperbolic partial differential equations. In realistic media, however, waves often exhibit nonlinear, dissipative, or time-dependent effects, making linear models inadequate for describing phenomena such as shock formation, acoustic dissipation, or supersonic motion. These situations lead to nonlinear hyperbolic equations whose analysis is significantly more challenging [1], [2].

Inverse problems for such equations play a central role in many practical applications–including seismic imaging, material characterization, and signal reconstruction–where the physical properties of a medium must be determined from observed wave data. Since these unknown coefficients typically represent essential physical quantities such as density, stiffness, or conductivity, their accurate identification is

crucial. However, inverse problems are frequently ill-posed, requiring careful mathematical treatment to ensure stability, uniqueness, and continuous dependence on the available data [3]– [5].
Boundary value problems with non-local structures, particularly periodic boundary conditions, are especially relevant in processes exhibiting repeated or cyclic behavior [6]. Inverse problems under periodic boundary conditions have been studied in various classical frameworks, including hyperbolic problems [7]– [9].

2. PROBLEM FORMULATION

Consider a nonlinear hyperbolic equation defined on the domain $\Lambda := \{\eta \in (0,\pi); \kappa \in (0,K)\}$:

$$\vee_{\kappa\kappa} - \vee_{\eta\eta} - \wp(\kappa)\vee = f(\eta,\kappa,\vee), \tag{1}$$

with the initial conditions

$$\begin{aligned} &\vee(\eta,0) = \varsigma(\eta), \\ &\vee_{\kappa}(\eta,0) = \mu(\eta), \end{aligned} \tag{2}$$

boundary conditions

$$\begin{aligned} &\vee(0,\kappa) = \vee(\pi,\kappa), \\ &\vee_{\eta}(0,\kappa) = \vee_{\eta}(\pi,\kappa), \end{aligned} \tag{3}$$

and the overdetermination condition

$$\mathrm{P}(\kappa) = \int_0^{\pi} \vee(\eta,\kappa)\,d\eta. \tag{4}$$

Where the functions $\varsigma(\eta), \mu(\eta), \mathrm{P}(\kappa)$ and $f(\eta,\kappa,\vee)$ are given functions on $[0,\pi]$ and $\bar{\Lambda} \times \{-\infty,\infty\}$, respectively; while the function $\vee(\eta,\kappa)$ and the coefficient $\wp(\kappa)$ are unknown.

If the coefficient $\wp(\kappa)$ in equation (1) is known, then the system defined by equations (1)– (3) represents a direct problem. Otherwise, it becomes an inverse problem, which requires an additional condition to ensure solvability. For the inverse problem described by equations (1)– (3), the overdetermination condition (4) is introduced. The primary objective is to determine the unknown coefficient along with the corresponding state function that satisfying the governing equation and all imposed auxiliary conditions.

Definition 1. The task of identifying the pair of unknown functions $(\wp(\kappa), \vee(\eta,\kappa))$ satisfying the system (1)– (4) is referred to as the inverse coefficient problem.

Using the Fourier method to solve equations (1)– (3), we obtain

$$\vee(\eta,\kappa)=\frac{1}{2}\left(\varsigma_0+\mu_0\kappa+\frac{2}{\pi}\int_0^{\kappa}\int_0^{\pi}(\kappa-\tau)\left[\wp(\tau)\vee(\omega,\tau)+f(\omega,\tau,\vee)\right]d\omega d\tau\right)+$$
$$\sum_{l=1}^{\infty}\left(\varsigma_{cl}\cos 2l\kappa+\frac{1}{2l}\mu_{cl}\sin 2l\kappa+\frac{1}{\pi l}\int_0^{\kappa}\int_0^{\pi}[\wp(\tau)\vee(\omega,\tau)+f(\omega,\tau,\vee)]\cos 2l\omega\sin 2l(\kappa-\tau)d\omega d\tau\right)\cos 2l\eta+ \quad (5)$$
$$\sum_{l=1}^{\infty}\left(\varsigma_{sl}\cos 2l\kappa+\frac{1}{2l}\mu_{sl}\sin 2l\kappa+\frac{1}{\pi l}\int_0^{\kappa}\int_0^{\pi}[\wp(\tau)\vee(\omega,\tau)+f(\omega,\tau,\vee)]\sin 2l\omega\sin 2l(\kappa-\tau)d\omega d\tau\right)\sin 2l\eta.$$

Where,

$$\varsigma_0=\frac{2}{\pi}\int_0^{\pi}\varsigma(\eta)d\eta,\ \mu_0=\frac{2}{\pi}\int_0^{\pi}\ \int_0^{\pi}\mu(\eta)d\eta,$$
$$\varsigma_{cl}=\frac{2}{\pi}\int_0^{\pi}\varsigma(\eta)\cos 2l\eta d\eta,\ \mu_{cl}=\frac{2k}{\pi}\int_0^{\pi}\mu(\eta)\cos 2l\eta d\eta,$$
$$\varsigma_{sl}=\frac{2}{\pi}\int_0^{\pi}\varsigma(\eta)\sin 2l\eta d\eta,\ \mu_{sl}=\frac{2k}{\pi}\int_0^{\pi}\mu(\eta)\sin 2l\eta d\eta.$$

Using the overdetermination condition (4) together with the solution (5), the inverse coefficient is obtained as:

$$\wp(\kappa)=\frac{\mathrm{P}''(\kappa)-\int_0^{\pi}f\left(\eta,\kappa,\vee\right)d\eta}{\mathrm{P}(\kappa)}. \quad (6)$$

3. EXISTENCE OF A SOLUTION

Theorem 1. Suppose the following assumptions hold:

A1. $\mathrm{P}(\kappa)\in C^2[0,K],\ \wp(\kappa)\in C[0,K]$.

A2. $\varsigma(\eta)\in C[0,\pi];\mu(\eta)\in C[0,\pi]$.

A3.1 $f(\eta,\kappa,\vee)$ is continuous in each argument on $\Lambda\times(-\infty,\infty)$ and complies with condition:
$$\left|\frac{\partial^{(l)}f(\eta,\kappa,\vee)}{\partial\eta^{(l)}}-\frac{\partial^{(l)}f(\eta,\kappa,\triangledown)}{\partial\eta^{(l)}}\right|\le\delta(\eta,\kappa)\left|\vee-\triangledown\right|,\ l=\overline{0,2},\ \text{where }\delta(\eta,t)\in L_2\left(\Lambda\right),\ \delta(\eta,t)\ge 0$$

A3.2 $f(\eta,\kappa,\vee)\in C[0,\pi],\ \left|f(\eta,\kappa,\vee)\right|\le\Theta,\ t\in[0,K]$.

A3.3 $\int_0^{\pi} f(\eta,\kappa,\vee)d\eta \neq 0, \forall \kappa \in [0,K]$.

Then, the system (1)–(4) has a unique solution.

Proof. We define an iterative procedure for the solution (5) and the inverse coefficient (6) as follows:

$$\vee_0^{(N+1)}(\kappa) = \vee_0^{(0)} + \frac{2}{\pi}\int_0^{\kappa}\int_0^{\pi}(\kappa-\tau)\left[\wp^{(N)}(\tau)\vee^{(N)}(\omega,\tau) + f(\omega,\tau,\vee^{(N)})\right]d\omega d\tau,$$

$$\vee_{cl}^{(N+1)}(\kappa) = \vee_{cl}^{(0)} + \sum_{l=1}^{\infty}\frac{1}{\pi l}\int_0^{\kappa}\int_0^{\pi}\left[\wp^{(N)}(\tau)\vee^{(N)}(\omega,\tau) + f(\omega,\tau,\vee^{(N)})\right]\cos 2l\omega \sin 2l(\kappa-\tau)d\omega d\tau, \quad (7)$$

$$\vee_{sl}^{(N+1)}(\kappa) = \upsilon_{sl}^{(0)} + \sum_{l=1}^{\infty}\frac{1}{\pi l}\int_0^{\kappa}\int_0^{\pi}\left[\wp^{(N)}(\tau)\vee^{(N)}(\omega,\tau) + f(\omega,\tau,\vee^{(N)})\right]\sin 2l\omega \sin 2l(\kappa-\tau)d\omega d\tau,$$

$$\wp^{(\mathrm{N}+1)}(\kappa) = \frac{\mathrm{P}''(\kappa) - \int_0^{\pi} f\left(\eta,\kappa,\vee^{(N)}\right)d\eta}{\mathrm{P}(\kappa)}.$$

Where,

$$\vee_0^{(0)}(\kappa) = \varsigma_0 + \mu_0\kappa,$$
$$\vee_{cl}^{(0)}(\kappa) = \varsigma_{cl}\cos 2l\kappa + \frac{1}{2l}\mu_{cl}\sin 2l\kappa,$$
$$\vee_{sl}^{(0)}(\kappa) = \varsigma_{sl}\cos 2l\kappa + \frac{1}{2l}\mu_{sl}\sin 2l\kappa.$$

If $N = 0$ for the system (7), then

$$\vee_0^{(1)}(\kappa) = \vee_0^{(0)} + \frac{2}{\pi}\int_0^{\kappa}\int_0^{\pi}(\kappa-\tau)\left[\wp^{(0)}(\tau)\vee^{(0)}(\omega,\tau) + f(\omega,\tau,\vee^{(0)})\right]d\omega d\tau,$$
$$\vee_{cl}^{(1)}(\kappa) = \vee_{cl}^{(0)} + \frac{1}{\pi l}\int_0^{\kappa}\int_0^{\pi}\left[\wp^{(0)}(\tau)\vee^{(0)}(\omega,\tau) + f(\omega,\tau,\vee^{(0)})\right]\cos 2l\omega \sin 2l(\kappa-\tau)d\omega d\tau,$$
$$\vee_{sl}^{(1)}(\kappa) = \vee_{sl}^{(0)} + \frac{1}{\pi l}\int_0^{\kappa}\int_0^{\pi}\left[\wp^{(0)}(\tau)\vee^{(0)}(\omega,\tau) + f(\omega,\tau,\vee^{(0)})\right]\sin 2l\omega \sin 2l(\kappa-\tau)d\omega d\tau,$$

$$\wp^{(1)}(\kappa)=\frac{\mathrm{P}''(\kappa)-\int_{0}^{\pi} f\left(\eta,\kappa,\vee^{(0)}\right)d\eta}{\mathrm{P}(\kappa)}.$$

By applying the Cauchy, Bessel, and Hölder inequalities together with the Lipschitz condition, and after performing the necessary rearrangements, we obtain the following estimates:

$$\left\|\vee^{(1)}(\kappa)\right\|\le\frac{\left\|\vee_0^{(0)}(\kappa)\right\|}{2}+\left\|\vee_{cl}^{(0)}(\kappa)\right\|+\left\|\vee_{sl}^{(0)}(\kappa)\right\|+\Upsilon\left(\left\|\wp^{(0)}(\kappa)\right\|+\left\|\delta(\eta,\kappa)\right\|\right)\left\|\vee^{(0)}(\eta,\kappa)\right\|+\Upsilon\Theta,$$

$$\left\|\wp^{(1)}(\kappa)\right\|\le\frac{\left\|\mathrm{P}''(\kappa)\right\|}{\left\|\mathrm{P}(\kappa)\right\|}+\frac{\sqrt{\pi}\left\|\delta(\eta,\kappa)\right\|\left\|\vee^{(0)}(\kappa)\right\|+\sqrt{\pi}\Theta}{\left\|\mathrm{P}(\kappa)\right\|},$$

where

$$\Upsilon:=\sqrt{\pi}\left|K\right|\sqrt{K}\left(\frac{\left|K\right|}{\sqrt{3}}+\frac{2\pi}{\sqrt{6}}\right).$$

Under the assumptions of the theorem, we have $\left\|\vee^{(1)}(\kappa)\right\|<\infty.$

Similarly, for $N+1,$ we obtain

$$\left\|\vee^{(N+1)}(\kappa)\right\|\le\frac{\left\|\vee_0^{(0)}(\kappa)\right\|}{2}+\left\|\vee_{cl}^{(0)}(\kappa)\right\|+\left\|\vee_{sl}^{(0)}(\kappa)\right\|+\Upsilon\left(\left\|\wp^{(N)}(\kappa)\right\|+\left\|\delta(\eta,\kappa)\right\|\right)\left\|\vee^{(N)}(\eta,\kappa)\right\|+\Upsilon\Theta,$$

$$\left\|\wp^{(N+1)}(\kappa)\right\|\le\frac{\left|\mathrm{P}''(\kappa)\right|}{\left\|\mathrm{P}(\kappa)\right\|}+\frac{\sqrt{\pi}}{\left\|\mathrm{P}(\kappa)\right\|}\left(\left\|\delta(\eta,\kappa)\right\|\left\|\vee^{(N)}(\kappa)\right\|+\Theta\right).$$

According to the hypotheses of the theorem, $\left\|\vee^{(N+1)}(\kappa)\right\|<\infty.$

Applying the same approach to the differences of successive approximations yields

$$\left\|\vee^{(1)}(\kappa)-\vee^{(0)}(\kappa)\right\|\le\Upsilon\left(\left\|\wp^{(0)}(\kappa)\right\|+\left\|\delta(\eta,\kappa)\right\|\right)\left\|\vee^{(0)}(\eta,\kappa)\right\|+\Upsilon\Theta,$$

$$\left\|\vee^{(2)}(\kappa)-\vee^{(1)}(\kappa)\right\|\le\Upsilon\left\|\vee^{(2)}(\kappa)\right\|\left\|\wp^{(2)}-\wp^{(1)}\right\|+\Upsilon\left\|\wp^{(1)}(\kappa)\right\|\left\|\vee^{(2)}-\vee^{(1)}\right\|+\Upsilon\left\|\delta(\eta,\kappa)\right\|\left\|\vee^{(1)}-\vee^{(0)}\right\|, \tag{8}$$

$$\left\|\wp^{(2)}(\kappa)-\wp^{(1)}(\kappa)\right\|\leq\frac{\left\|\delta(\eta,\kappa)\right\|\left\|\vee^{(2)}-\vee^{(1)}\right\|}{\left\|\mathrm{P}(\kappa)\right\|}. \tag{9}$$

Substituting (9) into (8), we obtain

$$\left\|\vee^{(2)}(\kappa)-\vee^{(1)}(\kappa)\right\|\leq\frac{1}{Q_1}\Upsilon P\left(\int_0^{\kappa}\int_0^{\pi}\delta^2(\omega,\tau)d\omega d\tau\right)^{\frac{1}{2}}, \tag{10}$$

where

$$P:=\left\|\vee^{(1)}(\kappa)-\vee^{(0)}(\kappa)\right\|,\ \Upsilon=|K|\sqrt{\pi K}\left(\frac{|K|}{\sqrt{3}}+\frac{2\pi}{\sqrt{6}}\right),\ Q_1=\left[1-\left(\Upsilon\frac{\left\|\vee^{(2)}(\kappa)\right\|}{\left\|\mathrm{P}(\kappa)\right\|}\left\|\delta(\eta,\kappa)\right\|\Upsilon\left\|\wp^{(1)}(\kappa)\right\|\right)\right].$$

Similarly, we obtain

$$\left\|\vee^{(3)}(\kappa)-\vee^{(2)}(\kappa)\right\|\leq\Upsilon\left\|\vee^{(3)}(\kappa)\right\|\left\|\wp^{(3)}(\kappa)-\wp^{(2)}(\kappa)\right\|+\Upsilon\left\|\wp^{(2)}(\kappa)\right\|\left\|\vee^{(3)}-\vee^{(2)}\right\|+\Upsilon\left\|\delta(\eta,\kappa)\right\|\left\|\vee^{(2)}-\vee^{(1)}\right\| \tag{11}$$

and

$$\left\|\wp^{(3)}(\kappa)-\wp^{(2)}(\kappa)\right\|\leq\frac{\left\|\delta(\eta,\kappa)\right\|\left\|\vee^{(3)}-\vee^{(2)}\right\|}{\left\|\mathrm{P}(\kappa)\right\|}. \tag{12}$$

Using (12) and (10) in (11), we obtain

$$\left\|\vee^{(3)}(\kappa)-\vee^{(2)}(\kappa)\right\|\leq\frac{\Upsilon^2P}{Q_2Q_1}\left(\int_0^{t}\int_0^{\pi}\delta^2(\omega,\tau)\left(\int_0^{\kappa}\int_0^{\pi}\delta^2(\omega_1,\tau_1)d\omega_1d\tau_1\right)d\omega d\tau\right)^{\frac{1}{2}}.$$

By repeatedly applying the same iterative process until the convergence of $\vee^{(\mathrm{N+1})}$ and $\wp^{(\mathrm{N+1})}$, we obtain

$$\left\|\vee^{(\mathrm{N+1})}-\vee^{(\mathrm{N})}\right\|\leq\frac{\Upsilon^N P}{Q_NQ_{N-1}....Q_1\sqrt{N!}}\left\|\delta(\eta,\kappa)\right\|^N, \tag{13}$$

$$\left\|\wp^{(\mathrm{N+1})}(\kappa)-\wp^{(\mathrm{N})}(\kappa)\right\|\leq\frac{\left\|\delta(\eta,\kappa)\right\|}{\left\|\mathrm{P}(\kappa)\right\|}\left\|\vee^{(\mathrm{N+1})}-\vee^{(\mathrm{N})}\right\|.$$

Thus, $\vee^{(\mathrm{N+1})}\to\vee^{(\mathrm{N})}$ as $\mathrm{N}\to\infty$; therefore, $\wp^{(N+1)}\to\wp^{(N)}$.

Next, let us verify that $\lim_{N\to\infty} \vee^{(N+1)}(\eta,\kappa) = \vee(\eta,\kappa),\ \lim_{N\to\infty} \wp^{(N+1)}(\kappa) = \wp(\kappa).$ Considering the difference between the exact and approximate solutions, we obtain

$$\left\|\vee(\kappa) - \vee^{(\mathrm{N+1})}(\kappa)\right\| \le \Upsilon\left\|\wp(\kappa) - \wp^{(\mathrm{N+1})}(\kappa)\right\|\left\|\vee^{(\mathrm{N+1})}(\eta,\kappa)\right\| + \Upsilon\left\|\delta(\eta,\kappa)\right\|\left\|\vee^{(\mathrm{N+1})} - \vee^{(\mathrm{N})}\right\| + \Upsilon\left(\left\|\wp(\kappa)\right\| + \left\|\delta(\eta,\kappa)\right\|\right)\left\|\vee - \vee^{(\mathrm{N+1})}\right\|. \tag{14}$$

Using $\left\|\wp(\kappa) - \wp^{(\mathrm{N+1})}(\kappa)\right\| \le \dfrac{\sqrt{\pi}\left\|\delta(\eta,\kappa)\right\|}{\left\|\mathrm{P}(\kappa)\right\|}\left\|\vee - \vee^{(\mathrm{N+1})}\right\|$ and (13) in (14), we obtain

$$\left\|\vee(\kappa) - \vee^{(N+1)}(\kappa)\right\| \le S\left(\left\|\wp(\kappa)\right\| + \frac{\sqrt{\pi}\left\|\delta(\eta,\kappa)\right\|\left\|\vee^{(N+1)}(\eta,\kappa)\right\|}{\left\|\mathrm{P}(\kappa)\right\|} + \left\|\delta(\eta,\kappa)\right\|\right)\left(\int_0^{\kappa}\int_0^{\pi}\left[\vee(\omega,\tau) - \vee^{(N+1)}(\omega,\tau)\right]^2 d\omega d\tau\right)^{\frac{1}{2}} + \frac{\Upsilon^{N+1}P}{Q_N \ldots Q_1\sqrt{N+1!}}\left\|\delta(\eta,\kappa)\right\|^{N+1}.$$

Applying Gronwall's inequality leads to

$$\left\|\vee(\kappa) - \vee^{(\mathrm{N+1})}(\kappa)\right\|^2 \le 2\left(\frac{\Upsilon^{\mathrm{N+1}}P}{Q_N \ldots Q_1\sqrt{N+1!}}\right)\left\|\delta(\eta,\kappa)\right\|^{2(\mathrm{N+1})} e^{\left\{2\Upsilon^2\left(\left\|\wp(\kappa)\right\| + \frac{\sqrt{\pi}\left\|\delta(\eta,\kappa)\right\|}{\left\|\mathrm{P}(\kappa)\right\|}\left\|\vee^{(\mathrm{N+1})}(\eta,\kappa)\right\| + \left\|\delta(\eta,\kappa)\right\|\right)^2\right\}}.$$

Hence, $\vee^{(\mathrm{N+1})} \to \vee$ and $\wp^{(\mathrm{N+1})} \to \wp$ as $N \to \infty$.

If two solutions, $\left(\wp(\kappa), \vee(\eta,\kappa)\right)$ and $\left(\lambda(\kappa), \vartheta(\eta,\kappa)\right)$, satisfy (1)–(4), performing the same procedures gives

$$\left\|\wp(\kappa) - \lambda(\kappa)\right\| \le \frac{\sqrt{\pi}\left\|\delta(\eta,\kappa)\right\|}{\left\|\mathrm{P}(\kappa)\right\|}\left\|\vee - \vartheta\right\|, \tag{15}$$

$$\left\|\vee(\kappa) - \vartheta(\kappa)\right\| \le \Upsilon\left\|\wp(\kappa)\right\|\left\|\vee - \vartheta\right\| + \Upsilon\left\|\delta(\eta,\kappa)\right\|\left\|\vee - \vartheta\right\| + \Upsilon\left\|\wp(\kappa) - \lambda(t)\right\|\left\|\vartheta(\eta,\kappa)\right\|. \tag{16}$$

Substituting (15) into (16) yields

$$\left\|\vee(\eta,\kappa) - \vartheta(\eta,\kappa)\right\| \le A\left(\left\|\wp(\kappa)\right\| + \frac{\sqrt{\pi}\left\|\delta(\eta,\kappa)\right\|}{\left\|\mathrm{P}(\kappa)\right\|}\left\|\vartheta(\eta,\kappa)\right\| + \left\|\delta(\eta,\kappa)\right\|\right)\left(\int_0^{\kappa}\int_0^{\pi}\left[\vee(\omega,\tau) - \vartheta(\omega,\tau)\right]^2 d\omega d\tau\right)^{\frac{1}{2}}.$$

Applying Gronwall's inequality, we conclude that $\vee(\eta,t) = \vartheta(\eta,t)$ and $\kappa(t) = \lambda(t)$.

4. ANALYSIS OF SOLUTION STABILITY

Theorem 2. Suppose the conditions of Theorem 1 are satisfied. Then the solution pair $\left(\wp(\kappa), \vee(\eta,\kappa)\right)$ of problem (1)– (4) depends continuously on the input data $\varsigma, \mu, \mathrm{P}$.

Proof. Consider two sets of data

$$\Xi = \left\{\varsigma, \mu, \mathrm{P}\right\},\ \tilde{\Xi} = \left\{\tilde{\varsigma}, \tilde{\mu}, \tilde{\mathrm{P}}\right\},$$

both satisfying the assumptions of Theorem 1. Assume that there exist positive constants $\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3$ such that

$$\left\|\mathrm{P}\right\|_{C^2[0,K]} \le \mathrm{M}_1,\ \left\|\varsigma\right\|_{C[0,K]} \le \mathrm{M}_2,\ \left\|\mu\right\|_{C[0,K]} \le \mathrm{M}_3,$$

and similarly

$$\left\|\tilde{\mathrm{P}}\right\|_{C^2[0,K]} \le \mathrm{M}_1, \left\|\tilde{\varsigma}\right\|_{C[0,K]} \le \mathrm{M}_2, \left\|\tilde{\mu}\right\|_{C[0,K]} \le \mathrm{M}_3.$$

Define the norms

$$\left\|\Xi\right\| = \left\|\varsigma\right\|_{C[0,K]} + \left\|\mu\right\|_{C[0,K]} + \left\|\mathrm{P}\right\|_{C^2[0,K]}, \left\|\tilde{\Xi}\right\| = \left\|\tilde{\varsigma}\right\|_{C[0,K]} + \left\|\tilde{\mu}\right\|_{C[0,K]} + \left\|\tilde{\mathrm{P}}\right\|_{C^2[0,K]}.$$

Let the solutions $\left(\wp, \vee\right)$ and $\left(\tilde{\wp}, \tilde{\vee}\right)$ correspond to data Ξ and $\tilde{\Xi}$, respectively. Following the same methodology as before, we obtain

$$\left\|\wp(\kappa) - \tilde{\wp}(\kappa)\right\| \le \left\|\frac{\mathrm{P}''(\kappa)}{\mathrm{P}(\kappa)} - \frac{\tilde{\mathrm{P}}''(\kappa)}{\tilde{\mathrm{P}}(\kappa)}\right\| + \frac{\sqrt{\pi}\left\|\delta(\eta,\kappa)\right\|}{\mathrm{M}_1}\left\|\vee - \tilde{\vee}\right\|, \tag{17}$$

$$\begin{aligned}\left\|\vee(\eta,\kappa) - \tilde{\vee}(\eta,\kappa)\right\| \le{}& \frac{\left\|\varsigma_0 - \tilde{\varsigma}_0\right\| + \left\|\mu_0 - \tilde{\mu}_0\right\|\left|K\right|}{2} + \sum_{l=1}^{\infty}\left(\left\|\varsigma_{cl} - \overline{\varsigma}_{cl}\right\| + \left\|\varsigma_{sl} - \overline{\varsigma}_{sl}\right\|\right) + \\ & \frac{\pi}{2\sqrt{6}}\sum_{l=1}^{\infty}\left(\left\|\mu_{cl} - \tilde{\mu}_{cl}\right\| + \left\|\mu_{sl} - \tilde{\mu}_{sl}\right\|\right) + \Upsilon\left\|\wp(\kappa) - \tilde{\wp}(\kappa)\right\|\left\|\tilde{\vee}(\eta,\kappa)\right\| + \Upsilon\left(\left\|\wp(\kappa)\right\| + \left\|\delta(\eta,\kappa)\right\|\right)\left\|\vee - \tilde{\vee}\right\|.\end{aligned} \tag{18}$$

Substituting (17) into (18) gives

$$\left\|\vee(\eta,\kappa) - \tilde{\vee}(\eta,\kappa)\right\|^2 \le 2M_4^2\left\|\Xi - \tilde{\Xi}\right\|^2 + 2M_5^2\left\|\delta(\eta,\kappa)\right\|\left\|\vee - \tilde{\vee}\right\|.$$

By applying Gronwall's inequality to the resulting expression, we obtain

$$\left\|\vee(\eta,\kappa)-\tilde{\vee}(\eta,\kappa)\right\|^2 \le 2M_4^2\left\|\Xi-\tilde{\Xi}\right\|^2+2e^{M_5^2\|\delta(\eta,\kappa)\|}.$$

Thus, as $\Xi \to \tilde{\Xi}$, it follows that $\vee \to \tilde{\vee}$. Hence, $\wp \to \tilde{\wp}$.

5.CONCLUSION

A one-dimensional nonlinear hyperbolic equation with an unknown time-dependent inverse coefficient under periodic boundary conditions has been analytically solved. The Fourier method has been employed to construct the solution pair $(\vee, \wp)$.

The conditions for the existence, uniqueness, and continuous dependence of the solution on the given data have been established. Furthermore, the inverse problem has been proved to be well-posed via the Picard successive approximation technique. This study provides a theoretical foundation for inverse-coefficient nonlinear hyperbolic equations. The results obtained are expected to guide future research on more complex multidimensional models, uncertain or variable coefficients, and realistic physical applications.

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