Hacettepe Journal of Mathematics and Statistics
Volume 46 (2) (2017), 193 – 198

Centralizers and the maximum size of the pairwise noncommuting elements in finite groups

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Abstract

In this article, we determine the structure of all nonabelian groups G such that G has the minimum number of the element centralizers among nonabelian groups of the same order. As an application of this result, we obtain the sharp lower bound for $\omega(G)$ in terms of the order of G where $\omega(G)$ is the maximum size of a set of the pairwise noncommuting elements of G.

Keywords: finite group, centralizer, CA-group.

2000 AMS Classification: 20D60

Received : 02.02.2016 *Accepted* : 14.07.2016 *Doi* : 10.15672/HJMS.20164519332

1. Introduction and main results

Throughout this paper G will be a finite group and $Z(G)$ will be its center. For a positive integer n, let Z_n and D_{2n} be the cyclic group of order n and the dihedral group of order $2n$, respectively. For a group G, we define $Cent(G) = \{C_G(x) : x \in G\}$ where $C_G(x)$ is the centralizer of the element x in G. It is clear that G is abelian if and only if $|Cent(G)| = 1$. Also it is easy to see that there is no group G with $|Cent(G)| = 2$ or 3. Starting with Belcastro and Sherman [7], many authors have investigated the influence of $|Cent(G)|$ on the group G (see [1], [4], [6], [7], [17-21] and [27-29]). In the present paper, we describe the structures of all groups having minimum number of centralizers among all nonabelian groups of the same order, that is:

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1.1. Theorem. Let G be a nonabelian group of order n. If $|Cent(G)| \leq |Cent(H)|$ for all nonabelian groups H of order n, then one of the following holds:

(1) G is nilpotent, $|Cent(G)| = p+2$ and $\frac{G}{Z(G)} \cong Z_p \times Z_p$ where p is the smallest prime such that p^3 divides n.

(2) G is nonnilpotent, $|Cent(G)| = p^m+2$ and $\frac{G}{Z(G)} \cong (Z_p)^m \rtimes Z_l$ where $l > 0$ and p^m is the smallest prime-power divisor of n such that $p^m - 1$ and n are not relatively prime.

The following corollary are immediate consequence of Theorem 1.1.

1.2. Corollary. Suppose that n is even and G is a nonabelian group of order n. If $|Cent(G)| \leq |Cent(H)|$ for all nonabelian groups H of order n, then $|Cent(G)| = 4$ or $p+2$ where p is the smallest odd prime divisor of n and also $\frac{G}{Z(G)}$ is isomorphic to one of the following groups:

$$Z_2 \times Z_2, Z_p \times Z_p, D_{2p}.$$

1.3. Remark. We notice that both conditions (1) and (2) of Theorem 1.1 may happen for some positive integer n. For example there exist two groups G_1 and G_2 of order 54 such that $|Cent(G_1)| = |Cent(G_2)| = 5$, $\frac{G_1}{Z(G_1)} \cong Z_3 \times Z_3$ and $\frac{G_2}{Z(G_2)} \cong D_6$.

There are interesting relations between centralizers and pairwise noncommuting elements in groups (see Proposition 2.5 and Lemma 2.6 of [1]). Let G be a finite nonabelian group and let X be a subset of pairwise noncommuting elements of G such that $|X| \geq |Y|$ for any other set of pairwise noncommuting elements Y in G. Then the subset X is said to have the maximum size, and this size is denoted by $\omega(G)$. Also $\omega(G)$ is the maximum clique size in the noncommuting graph of a finite group G. The noncommuting graph of a group G is defined as a graph whose $G \setminus Z(G)$ is the set of vertices and two vertices are joined if and only if they do not commute. By a famous result of Neumann [22] answering a question of Erdős, the finiteness of $\omega(G)$ is equivalent to the finiteness of the factor group $\frac{G}{Z(G)}$ which follows that $|Cent(G)|$ is finite. Also,if G has a finite number of centralisers, then it is easy to see that $\omega(G)$ is finite. Various attempts have been made to find $\omega(G)$ for some groups G. Pyber [24] has proved that there exists a constant c such that $|\frac{G}{Z(G)}| \leq c^{\omega(G)}$. Chin [13] has obtained upper and lower bounds of $\omega(G)$ for extra-special groups G of odd order. Isaacs has shown that $\omega(G) = 2m+1$ for any extra-special group G of order 2^{2m+1} (see page 40 of [11]). Brown in [9] and [10] has investigated $\omega(S_n)$ where S_n is the symmetric group on n letters. Also Bertram, Ballester-Bolinches and Cossey gave lower bounds for the maximum size of non-commuting sets for certain solvable groups([5]). Recently authors [17, 20] have determined all groups G with $\omega(G) = 5$ and obtained $\omega(G)$ for certain groups. Known upper bounds for this invariant were recently used to prove an important result in modular represention theory ([13]). In this article we determine the structure of nonabelian groups G of order n such that $\omega(G) \leq \omega(H)$ for all nonabelian groups H of order n.

1.4. Theorem. Let G be a nonabelian group of order n. If $\omega(G) \leq \omega(H)$ for all nonabelian groups H of order n, then one of the following holds:

(1) G is nilpotent, $\omega(G) = p+1$ and $\frac{G}{Z(G)} \cong Z_p \times Z_p$ where p is the smallest prime such that p^3 divides n.

(2) G is nonnilpotent, $\omega(G) = p^m+1$ and $\frac{G}{Z(G)} \cong (Z_p)^m \rtimes Z_l$ where $l > 0$ and p^m is the smallest prime-power divisor of n such that $p^m - 1$ and n are not relatively prime.

Throughout this paper we will use usual notation which can be found in [25] and [15].

2. Proofs of the main results

The following lemmas are useful in the proof of the main theorem.

2.1. Lemma. Let $G, G_1, \cdots, G_n$ be finite groups. Then

1. If $H \leq G$, then $|Cent(H)| \leq |Cent(G)|$;
2. If $G = \prod_{i=1}^{n} G_i$, then $|Cent(G)| = \prod_{i=1}^{n} |Cent(G_i)|$.

Proof. The proof is clear. □

In Lemma 2.7 of [4], it was shown that if p is a prime, then $|Cent(G)| \geq p + 2$ for all nonabelian p-groups G and the equality holds if and only if $\frac{G}{Z(G)} \cong Z_p \times Z_p$. In the following we generalize this result for all nilpotent groups.

2.2. Lemma. Let G be a nilpotent group and p be a prime divisor of $|G|$ such that a Sylow p-subgroup of G is nonabelian. Then $|Cent(G)| \geq p+2$ with equality if and only if $\frac{G}{Z(G)} \cong Z_p \times Z_p$.

Proof. Suppose that P is a Sylow p-subgroup of G. Then we have $|Cent(G)| \geq |Cent(P)| \geq p + 2$ by Lemma 2.1(1) and Lemma 2.7 of [4], as wanted.

Now, assume that $|Cent(G)| = p + 2$. Since G is nilpotent, each Sylow q-subgroup of G is abelian for each prime divisor $q \neq p$ of $|G|$ by Lemma 2.1(2). Consequently $\frac{G}{Z(G)} \cong \frac{P}{Z(P)}$ which is isomorphic to $Z_p \times Z_p$ by Lemma 2.7 of [4]. The converse holds similarly. □

Recall that a minimal nonnilpotent group is a nonnilpotent group whose proper subgroups are all nilpotent. In 1924, O. Schmidt [26] studied such groups. The following result plays an important role in the proof of Theorem 1.1.

2.3. Lemma. Let G be a minimal nonnilpotent group. Then $\frac{G}{Z(G)}$ is Frobenius such that the Frobenius kernel is elementary abelian and the Frobenius complement is of prime order.

Proof. By Theorem 9.1.9 of [25], we have $G = PQ$ where P is a unique Sylow p-subgroup of G and Q is a cyclic Sylow q-subgroup of G for some distinct primes p and q. Also by Exercise 9.1.11 of [25], the Frattini subgroups of P and Q are contained in $Z(G)$. It follows that $\frac{PZ(G)}{Z(G)}$ is elementary and $\frac{QZ(G)}{Z(G)}$ is of order q. Since all Sylow subgroups of $\frac{G}{Z(G)}$ are abelian, Theorem 10.1.7 of [25] gives that $(\frac{G}{Z(G)})' \bigcap Z(\frac{G}{Z(G)}) = \overline{1}$. Since $P = [P, Q]$, we have $\frac{PZ(G)}{Z(G)} \leq \frac{G'Z(G)}{Z(G)}$ and so $Z(\frac{G}{Z(G)})$ is a q-group. On the other hand since G is not nilpotent, $Z(\frac{G}{Z(G)}) = \overline{1}$. Now it is easy to see that $\frac{G}{Z(G)}$ is a Frobenius group. □

2.4. Proposition. Let $\frac{G}{Z(G)} = \frac{K}{Z(G)} \rtimes \frac{H}{Z(G)}$ be a Frobenius group such that H is abelian. If $Z(G) < Z(K)$, then $|Cent(G)| = |Cent(K)| + |\frac{K}{Z(G)}| + 1$ and if $Z(G) = Z(K)$, then $|Cent(G)| = |Cent(K)| + |\frac{K}{Z(G)}|$. Also $\omega(G) = \omega(K) + |\frac{K}{Z(G)}|$.

Proof. See Proposition 3.1 of [18] and its proof. □

Recall that a group G is a CA-group if the centralizer of every noncentral element of G is abelian. R. Schmidt [26] determined all CA-groups (see Theorem A of [14]). Now we are ready to prove the main result.

Proof of Theorem 1.1.

Suppose that G is a nilpotent group. Since G is not abelian, a Sylow q-subgroup of G is not abelian for some prime q. It follows from Lemma 2.2 that $|Cent(G)| \geq q + 2$.

But there exists a nonabelian group $H := Q \times Z_{\frac{n}{q^3}}$ of order n where Q is a nonabelian group of order q^3 and we see that $|Cent(H)| = q + 2$. Since G has the minimum number of the element centralizer, we must have $|Cent(G)| = p + 2$ and p must be the smallest prime such that p^3 divides n. Also $\frac{G}{Z(G)} \cong Z_p \times Z_p$ by Lemma 2.2, as wanted.

Now, assume that G is a nonnilpotent group of order n. Then there exist two prime divisors q and r of n such that q divides $r^k - 1$ for some positive integer k by Corollary 1 of [23]. We claim that if p^m is the smallest prime-power divisor of n such that $gcd(p^m - 1, n) \neq 1$, then $|Cent(G)| \geq p^m + 2$.

Since G is finite and nonnilpotent, G contains a minimal nonnilpotent subgroup M. It follows from Lemma 2.3 that $\frac{M}{Z(M)}$ is Frobenius with the kernel $\frac{K}{Z(M)}$ and the complement $\frac{H}{Z(M)}$. Note that $|\frac{K}{Z(M)}| = p_1^t$ and $|\frac{H}{Z(M)}| = p_2$ for some primes p_1 and p_2 such that $p_2 | p_1^t - 1$. It follows from Proposition 2.4 that $|Cent(M)| \geq |\frac{K}{Z(M)}| + 2 = p_1^t + 2$. Since M is a subgroup of G, we have $|Cent(G)| \geq |Cent(M)| \geq p_1^t + 2$ which is equal or greater than $p^m + 2$ by hypothesis. This proves the claim. Now we want to find the structure of nonnilpotent groups G for which the equality occurs.

Assume that $|Cent(G)| = p^m + 2$. We shall prove that $\frac{G}{Z(G)} \cong (Z_p)^m \rtimes Z_l$ for some positive integer l. By hypothesis and the previous paragraph, there is a subgroup M of G such that $\frac{M}{Z(M)}$ is Frobenius with the kernel $\frac{K}{Z(M)}$ of order p^m and the Frobenius complement $\frac{H}{Z(M)}$ which is cyclic of prime order. Since $|\frac{K}{Z(M)}| + |Cent(K)| \leq |Cent(M)|$ by Proposition 2.4 and $|Cent(M)| \leq |Cent(G)| = p^m + 2$, we have $|Cent(M)| = p^m + 2$ and $|Cent(K)| = 1$. It follows that K is abelian and so M is a CA-group by Theorem A (II) of [14]. Next, we show that G is a CA-group.

Since M is a CA-group, we have $p^m + 1$ is the maximum size of a set of pairwise non-commuting elements of M by Lemma 2.6 of [1] and so the maximum size of a set of pairwise non-commuting elements of G is at least $p^m + 1$. Since $|Cent(G)| = p^m + 2$, the maximum size of a set of pairwise non-commuting elements of G must be $p^m + 1$. Therefore G is CA-group by Lemma 2.6 of [1]. Now we apply Theorem A of [14].

Note, first, that if 8 divides $|G| = n$, then by hypothesis $|Cent(G)| \leq |Cent(D_8 \times Z_{\frac{n}{8}})| = 4$ and so $|Cent(G)| = 4$. Therefore $\frac{G}{Z(G)} \cong Z_2 \times Z_2$ by Fact 3 of [7] and so G is nilpotent, a contradiction. Hence 8 does not divide n and so $|Cent(G)| \geq 5$ by Fact 4 of [7]. Also if 6 divides n, then $|Cent(G)| \leq |Cent(D_6 \times Z_{\frac{n}{6}})| = 5$ which implies $|Cent(G)| = 5$. Therefore $\frac{G}{Z(G)} \cong D_6$ by Fact [7] and so we have the result. Thus we may assume that 6 does not divide n. Now since G is not nilpotent, G satisfies (I), (II) or (III) of Theorem A of [14]. Therefore G has an abelian subgroup A of prime index r or $\frac{G}{Z(G)} = \frac{K}{Z(G)} \rtimes \frac{T}{Z(G)}$ is a Frobenius group with the Frobenius kernel $\frac{K}{Z(G)}$ and the Frobenius complement $\frac{T}{Z(G)}$. In the first case, we have $|G'| = p^m$ by Theorem 2.3 of [6] and so $|\frac{G}{Z(G)}| = p^m r$ by Lemma 4 (page 303) of [8]. Consequently $\frac{G}{Z(G)} = \frac{A}{Z(G)} \rtimes \frac{L}{Z(G)}$ where $|\frac{L}{Z(G)}| = r$ and $|\frac{A}{Z(G)}| = p^m$. By the property of p^m, the number of Sylow r-subgroup of $\frac{G}{Z(G)}$ is p^m and so $\frac{G}{Z(G)}$ is Frobenius. Again by the property of p^m, $\frac{A}{Z(G)}$ is characteristically simple which implies that it is elementary, as wanted.

In the second case, it follows from (II)-(III) of Theorem A of [14] that T is abelian and K is abelian or $K = QZ(G)$ where Q is a normal Sylow q-subgroup of G for some prime q. If K is abelian, then $Z(G) < Z(K)$ and so $|Cent(G)| = |\frac{K}{Z(G)}| + 2$ by Proposition 2.4. Therefore $|\frac{K}{Z(G)}| = p^m$. By the property of p^m, $\frac{K}{Z(G)}$ is elementary. On the other hand $\frac{T}{Z(G)}$ is cyclic by Corollary 6.17 of [15] and so we have the result.

If $K = QZ(G)$, then $|\frac{K}{Z(G)}| = q^a$ and since $\frac{G}{Z(G)}$ is Frobenius, we have $|\frac{T}{Z(G)}|$ divides $q^a - 1$. It follows that $p^m \leq q^a$ by hypothesis. On the other hand $p^m + 2 = |Cent(G)| \geq$

$|\frac{K}{Z(G)}|+2$ by Proposition 2.4 and this implies that $p^m = q^a$. It follows from Proposition 2.4 that $|Cent(K)| = 1$ and K is abelian. The rest of the proof is similar to the previous case.

Proof of Theorem 1.4

The proof is similar to the previous theorem. If G is nilpotent, then $\omega(G) \geq p+1$ where p is the smallest prime such that p^3 divides $|G|$ and the equality holds if and only if $\frac{G}{Z(G)} \cong Z_p \times Z_p$.

Now suppose that G is nonnilpotent. Then G contains a minimal nonnilpotent subgroup M and so $\frac{M}{Z(M)} = \frac{K}{Z(M)} \rtimes \frac{H}{Z(M)}$ is Frobenius such that $|\frac{K}{Z(M)}| = p_1^m$ and $|\frac{H}{Z(M)}| = p_2$ by Lemma 2.3. Since H is abelian and has at least p_1^m conjugates in G, say $H = H_1, H_2, \cdots, H_{p_1^m}$, we see $\{x_1, \cdots, x_{p_1^m}\}$ is a subset of pairwise noncommuting elements of M where $x_i \in H_i \setminus \{1\}$. It follows that $\omega(G) \geq p_1^m + 1 \geq p^m + 1$. The remainder of the proof is similar to Theorem 1.1.

Acknowledgment. The authors would like to thank the referee for his/her careful reading and valuable comments.

References

[1] A. Abdollahi, S. M. Jafarian. Amiri and A. M. Hassanabadi, Groups with specific number of centralizers, Houston J. Math. 33(1) (2007), 43-57.

[2] A. Abdollahi, S. Akbari and H. R. Maimani, Non-commuting graph of a group, J. Algbera 298 (2) (2006), 468-492.

[3] A. Abdollahi, A. Azad, A. Mohamadi Hasanabadi and M. Zarrin, On the clique numbers of non-commuting graphs of certain groups, Algebra Colloq, 17(4) (2010), 611-620.

[4] A. R. Ashrafi, On finite groups with a given number of centralizers, Algebra Colloq. 7(2) (2000), 139-146.

[5] A. Ballester-Bolinches and J. Cossey, On non-commuting sets in finite soluble CC-groups, Publ. Mat. 56 (2012), 467-471

[6] S. J. Baishya, On finite groups with specific number of centralizers, International Electronic Journal of Algebra, 13(2013), 53-62.

[7] S. M. Belcastro and G. J. Sherman, Counting centralizers in finite groups, Math. Mag. 5 (1994), 111-114.

[8] Y. G. Berkovich and E. M. Zhmu'd, Characters of Finite Groups, Part 1, Transl. Math. Monographs 172, Amer. Math. Soc., Providence. RI, 1998.

[9] R. Brown, Minimal covers of S_n by abelian subgroups and maximal subsets of pairwise noncommuting elements, J. Combin. Theory Ser. A **49** (1988), 294-307.

[10] R. Brown, Minimal covers of S_n by abelian subgroups and maximal subsets of pairwise noncommuting elements, II, J. Combin. Theory Ser. A **56** (1991), 285-289.

[11] E. A. Bertram, Some applications of graph theory to finite groups, Discrete Math. 44(1) (1983), 31-43.

[12] A. M. Y. Chin, On noncommuting sets in an extraspecial p-group, J. Group Theory 8(2) (2005), 189–194.

[13] A. Y. M. Chin, On non-commuting sets in an extraspecial p-group, J. Group Theory, 8.2 (2005), 189-194.

[14] S. Dolfi, M. Herzog and E. Jabara, Finite groups whose noncentral commuting elements have centralizers of equal size, Bull. Aust. Math Soc, 82 (2010), 293-304.

[15] I. M. Isaacs, Finite group theory, Grad. Stud. Math, vol. 92, Amer. Math. Soc, Providence, RI, 2008.

[16] The GAP Group, GAP-Groups, Algoritms, and Programming, version 4.4.10, (2007) ,(http://www.gap-system.org).

[17] S. M. Jafarian Amiri and H. Madadi, On the maximum number of the pairwise noncommuting elements in a finite group, J. Algebra Appl, (2016), Vol. 16, No. 1 (2017) 1650197 (9 pages).

[18] S. M. Jafarian Amiri, H. Madadi and H. Rostami, On 9-centralizer groups, J. Algebra Appl, Vol. 14, No. 1 (2015) 1550003 (13 pages).
[19] S. M. Jafarian Amiri and H. Rostami, Groups with a few nonabelian centralizers, Publ. Math. Debrecen, 87 (3-4) (2015), 429-437.
[20] S. M. Jafarian Amiri, H. Madadi and H. Rostami, On F-groups with central factor of order p^4, Math. Slovaca, Accepted.
[21] S. M. Jafarian Amiri, M. Amiri and H. Rostami, Finite groups determined by the number of element centralizers, Comm. Alg., 45(9) (2017), 3792–3797.
[22] B. H. Neumann, A problem of Paul Erdös on groups, J. Austral. Math. Soc. Ser. A **21** (1976), 467-472.
[23] J. Pakianathan, S. Krishnan Shankar, Nilpotent numbers, Amer. Math. Monthly, (2000), 631-634.
[24] L. Pyber, The number of pairwise noncommuting elements and the index of the centre in a finite group, J. Lond. Math. Soc. 35(2) (1987), 287-295.
[25] D. J. S. Robinson, A course in the theory of groups, Springer-Verlag New York, 1996.
[26] R. Schmidt, Zentralisatorverbände endlicher Gruppen, Rend. Sem. Mat. Univ. Padova 44 (1970), 97-131.
[27] M. Zarrin, Criteria for the solubility of finite groups by its centralizers, Arch. Math. 96 (2011), 225-226.
[28] M. Zarrin, On element centralizers in finite groups, Arch. Math. 93(2009), 497-503.
[29] M. Zarrin, On solubility of groups with finitely many centralizers, Bull. Iran. Math. Soc. 39 (2013), 517-521.