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STRESS-STRENGTH RELIABILITY for $P(X_{r:n_1} < Y_{k:n_2})$ in the EXPONENTIAL CASE

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Abstract : This paper deals with the estimation problem of the multicomponent stress-strength reliability parameter when stress, strength variates are given by two independent one-parameter exponential distributions with different parameters. It is assumed that $Y_1, \ldots, Y_{n_2}$ are the random strengths of n_2 components subjected to random stresses $X_1, \ldots, X_{n_1}$. Our study is concentrated on the probability$P(X_{r:n_1} < Y_{k:n_2})$ and the problem of frequentist and Bayesian estimation of $P(X_{r:n_1} < Y_{k:n_2})$ based on X- and Y-samples are discussed. Some special cases are considered and the small sample comparison of the reliability estimates is made through Monte Carlo simulation.

Key words: Stress-strength reliability, Squared error loss function, Uniformly minimum variance unbiased estimator, Maximum likelihood estimator, Parallel and series systems.
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1. Introduction

In reliability context, the probability that the random variable X (stress) is exceeded by its strength which is a realization of a random variable Y is called stress-strength reliability and is equal to $R := Pr(X < Y)$. Parametric and non-parametric inferences on $R = P(X < Y)$ have been discussed in the literature extensively. The estimator of $P(X < Y)$ when X and Y follow independent exponential random variables are discussed by several authors, for example see the works by, Enis and Geisser (1971), Tong (1974), Kelley et al. (1976), Shah and Sathe (1981) and Chao (1982). Reiser and Guttman (1987) are compared point estimations of R in the normal case. Empirical Bayes estimation of $P(X < Y)$ is discussed in Ahmad and Fakhry (1997), when X and Y are Burr Type-X random variables. We refer the readers to Kotz et al. (2003) and references therein for an extensive review of the topic up to 2003. This book collects and digests theoretical and practical results on the theory and applications of the stress-strength relationships in industrial and economic systems. Kunda and Gupta (2005) considered the estimation of $R = P(X < Y)$, when X and Y are independent and have generalized exponential distribution. Saraçoğlu and Kaya (2007) considered frequentist and Bayesian estimation problem of reliability $R = P(X < Y)$ in the Gompertz case. Eryilmaz (2008a) obtained minimum variance unbiased (MVU) estimator of the reliability of consecutive k-out-of-n:G system, when the stress and strength distributions are exponential with unknown scale parameters. Eryilmaz (2010) studied stress-strength reliability for a general coherent system and illustrated the estimation procedure for exponential stress-strength distributions.

Multicomponent stress-strength reliability also has been studied by several authors, see for examples, Bhattacharyya and Johnson (1974), Pandey et al. (1992) and Eryilmaz (2008b). Let us denote the rth and the kth order statistics from X-sample with sample size n_1 and Y-sample with sample size n_2, by $X_{r:n_1}$ and $Y_{k:n_2}$, respectively. In this paper, we assume $Y_1, \ldots, Y_{n_2}$ are the random strengths of n_2 component subjected to random stresses $X_1, \ldots, X_{n_1}$. We obtain the reliability

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of stress-strength models based on rth order stress component, $X_{r:n_1}$, and kth order component strength, $Y_{k:n_2}$, i.e, $P(X_{r:n_1} < Y_{k:n_2})$ which contains all arrangements of components. For example taking $r = n_1$ and $k = 1$ leads to the reliability of series stress-strength system. And, $r = n_1 = 1$ and $k = n_2 - s + 1$ leads the reliability of a system with n_2 components where the system functions when at least s $(1 \le s \le n_2)$ components survive a common random stress X. So, the probability $P(X_{r:n_1} < Y_{k:n_2})$ generalizes various stress-strength reliability models for particular selection of r and k.

The rest of this paper is structured as follows: First, we consider special cases of $R_{r,k} = P(X_{r:n_1} < Y_{k:n_2})$ and determine the reliability of the system for this cases. Then, maximum likelihood estimator (MLE), uniformly minimum variance unbiased estimator (UMVUE) and Bayes estimator of $R_{r,k}$ are obtained, these are presented in Section 3. In Section 4, a simulation study is performed to compare the estimators of $R_{n_1,1}$. Section 5 contains a brief summary.

2. Model description

Let X and Y be two random variables with exponential distribution with means $1/\alpha$ and $1/\beta$, respectively. Then, it is known that the pdf of X and Y are given by

$$f_X(x) = \alpha e^{-\alpha x},\ x > 0,\ \alpha > 0, \tag{2.1}$$

and

$$f_Y(y) = \beta e^{-\beta y},\ y > 0,\ \beta > 0, \tag{2.2}$$

respectively. Suppose $X_1, \dots, X_{n_1}$ and $Y_1, \dots, Y_{n_2}$ are two independent samples from X and Y, respectively. The stresses and the strengths, are assumed to be independent. Under these assumptions, we find

$$R_{r,k} = P(X_{r:n_1} < Y_{k:n_2}) = \int_0^\infty F_{X_{r:n_1}}(y) f_{Y_{k:n_2}}(y)\ dy, \tag{2.3}$$

where $F_{X_{r:n_1}}(y)$ and $f_{Y_{k:n_2}}(y)$ stand for the rth cdf and the kth pdf of $X_{r:n_1}$ and $Y_{k:n_2}$, respectively. We recall that for a random sample $X_1, \dots, X_m$, the pdf and cdf of the ith order statistic are given by

$$f_{X_{i:m}}(x) = i\binom{m}{i} F^{i-1}(x)[1-F(x)]^{m-i} f(x), \tag{2.4}$$

and

$$F_{X_{i:m}}(x) = \sum_{j=i}^{m} \binom{m}{j} F^j(x)[1-F(x)]^{m-j}, \tag{2.5}$$

respectively, see David and Nagaraja (2003) for more details. By substituting (2.1), (2.2), (2.4) and (2.5) into (2.3), and doing some calculations, we obtain

$$\begin{aligned} R_{r,k} &= k\binom{n_2}{k} \sum_{j=r}^{n_1} \binom{n_1}{j} \int_0^\infty \beta(1-e^{-y\alpha})^j (1-e^{-y\beta})^{k-1} e^{[-y(\alpha(n_1-j)+\beta(n_2-k+1))]}\, dy \\ &= k\binom{n_2}{k} \sum_{j=r}^{n_1} \sum_{i=0}^{j} \sum_{l=0}^{k-1} \binom{n_1}{j}\binom{j}{i}\binom{k-1}{l} \frac{(-1)^{l+i}\beta}{\alpha(i+n_1-j)+(l+n_2-k+1)\beta}. \end{aligned} \tag{2.6}$$

In what follows, some special cases of (2.6) are considered.

2.1. Special cases

For some special cases of $R_{r,k}$, we obtained a simple expression for the reliability of the system with different arrangement of the components.

(i) For $r=n_1$ and $k=1$, minimum strength component is subjected to maximum stress component. In this case, the probability $R_{n_1,1}$ is the reliability of a series system with n_2 component

$$R_{n_1,1}=n_2\sum_{i=0}^{n_1}\binom{n_1}{i}(-1)^i\frac{\beta}{i\alpha+n_2\beta}. \tag{2.7}$$

(ii) When $r=n_1$ and $k=n_2$, maximum strength component is subjected to maximum stress component. Then, R_{n_1,n_2} is reliability of a parallel system with n_2 component

$$R_{n_1,n_2}=n_2\sum_{i=0}^{n_1}\sum_{l=0}^{n_2-1}\binom{n_1}{i}\binom{n_2-1}{l}\frac{(-1)^{i+l}\beta}{i\alpha+(l+1)\beta}. \tag{2.8}$$

(iii) When $r=1$ and $k=1$, minimum strength component is subjected to minimum stress component. Then

$$R_{1,1}=n_2\sum_{j=1}^{n_1}\sum_{i=0}^{j}\binom{n_1}{j}\binom{j}{i}\frac{(-1)^i\beta}{(i+n_1-j)\alpha+n_2\beta}. \tag{2.9}$$

(iv) For $r=n_1$ and $k=k$, the kth strength order component is subjected to maximum stress component. In fact in this case, $R_{n_1,k}$ is reliability of the k-out-of-n_2 system

$$R_{n_1,k}=k\binom{n_2}{k}\sum_{i=0}^{n_1}\sum_{l=0}^{k-1}\binom{n_1}{i}\binom{k-1}{l}\frac{(-1)^{l+i}\beta}{i\alpha+(l+n_2-k+1)\beta}. \tag{2.10}$$

3. Estimation of reliability

When the parameters α and β are known, then the exact value of $R_{r,k}$ is simply calculated, otherwise we have to obtain an estimate of the reliability. In this section, we provide three common estimators namely the UMVUE, MLE and Bayes estimator for reliability of $R_{r,k}$.

3.1. MLE

Let $X_1,\ldots,X_{n_1}$ be a random sample of size n_1 from (2.1) and $Y_1,\ldots,Y_{n_2}$ be a random sample of size n_2 from (2.2). Then, the log likelihood function of the observed samples is readily given by

$$\log L(\alpha,\beta)=n_1\log\alpha+n_2\log\beta-\alpha\sum_{i=1}^{n_1}x_i-\beta\sum_{j=1}^{n_2}y_j.$$

Then the MLE of α and β denoted by $\tilde{\alpha}$ and $\tilde{\beta}$, receptively, immediately obtained as

$$\tilde{\alpha}=\frac{n_1}{\sum_{i=1}^{n_1}X_i}, \tag{3.1}$$

and

$$\tilde{\beta}=\frac{n_2}{\sum_{i=1}^{n_2}Y_i}. \tag{3.2}$$

Due to the invariance property of the maximum likelihood estimator the MLE of $R_{r,k}$, denoted by $\tilde{R}_{r,k}$, can be easily obtained by replacing α and β by their MLE's as in (3.1) and (3.2), respectively. Hence the MLE of $R_{r,k}$ is given by

$$\tilde{R}^M_{r,k}=k\binom{n_2}{k}\sum_{j=r}^{n_1}\sum_{i=0}^{j}\sum_{l=0}^{k-1}\binom{n_1}{j}\binom{j}{i}\binom{k-1}{l}\frac{(-1)^{l+i}\bar{X}}{(i+n_1-j)\bar{Y}+(l+n_2-k+1)\bar{X}}. \tag{3.3}$$

To obtain $E(\tilde{R}^M_{r,k})$, by noting that $W = \dfrac{\beta\bar{Y}}{\alpha\bar{X}}$ has F-distribution with $2n_2$ and $2n_1$ degree of freedom, it is enough to find

$$E\left(\frac{\bar{X}}{(i+n_1-j)\bar{Y}+(l+n_2-k+1)\bar{X}}\right).$$

For example in the case of series system, we obtain

$$\begin{aligned}
E\left(\frac{\bar{X}}{i\bar{Y}+n_2\bar{X}}\right) &= E\left[\frac{\alpha i}{\beta}W+n_2\right]^{-1} \\
&= \int_0^\infty \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)}(\frac{n_2}{n_1})^{n_2}\frac{w^{n_2-1}}{(1+\frac{n_2}{n_1}w)^{n_1+n_2}}\frac{1}{n_2+\frac{i\alpha}{\beta}w}dw, \\
&= \frac{1}{n_2}(\frac{i\alpha n_1}{\beta n_2^2})^{n_1}\int_0^1 \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)}u^{n_1}(1-u)^{n_2-1}(1-u(1-\frac{i\alpha n_1}{\beta n_2}))^{-(n_1+n_2)}du, \\
&= \begin{cases} (\frac{\alpha n_1 i}{\beta n_2^2})^{n_1}\frac{n_1}{(n_1+n_2)n_2}F(n_1+n_2,n_1+1,n_+n_2+1;C), & |C|<1, \\ \\ (\frac{\beta n_2^2}{\alpha n_1 i})^{n_2}\frac{n_1}{(n_1+n_2)n_2}F(n_1+n_2,n_2,n_+n_2+1;\frac{C}{C-1}), & C<-1, \end{cases}
\end{aligned} \tag{3.4}$$

where $C = 1-\frac{i\alpha n_1}{\beta n_2}$ and $F(a,b,c;z)$ is hypergeometric series defined by (see e.g. Abramowtiz and Stegun, 1992, page 556)

$$F(a,b,c;z) = \sum_{j=1}^{\infty}\frac{a(a+1)\dots(a+j-1)b(b+1)\dots(b+j-1)}{c(c+1)\dots(c+j-1)}\frac{z^j}{j!}. \tag{3.5}$$

The series in (3.5) is convergent for $|z|<1, c\neq 0,-1,-2,\dots,$ and reduces to a finite sum if a or b is zero or a negative integer. Since $|\frac{C}{1-C}|<1$ for $C<-1$, the hypergeometric series in (3.4) are always convergent. We used the integral form of hypergeometric series in (3.4) which is given by

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-b)}\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-ux)^{-a}du.$$

Therefore, the excepted value of $\tilde{R}_{n_1,1}$ is

$$\begin{aligned}
E(\tilde{R}^M_{n_1,1}) &= n_2\sum_{i=0}^{n_1}\binom{n_1}{i}(-1)^i \\
&\times \begin{cases} (\frac{\alpha n_1 i}{\beta n_2^2})^{n_1}\frac{n_1}{(n_1+n_2)n_2}F(n_1+n_2,n_1+1,n_+n_2+1;C), & |C|<1, \\ \\ (\frac{\beta n_2^2}{\alpha n_1 i})^{n_2}\frac{n_1}{(n_1+n_2)n_2}F(n_1+n_2,n_2,n_+n_2+1;\frac{C}{C-1}). & C<-1, \end{cases}
\end{aligned}$$

We have not an analytical expression for MSE of $\tilde{R}_{r,k}$, this can be done by using numerical computations.

3.2. UMVUE

Let us denote the UMVUE of $R_{r,k}$ by $\hat{R}^U_{r,k}$. To obtain the UMVUE of $R_{r,k}$, from (2.6) and using the linear property of UMVUE, it is enough to find UMVUE for

$$\phi(\alpha,\beta) = \frac{\beta}{\alpha(i+n_1-j)+(l+n_2-k+1)\beta}. \tag{3.6}$$

To do this, let $X_1,\ldots,X_{n_1}$ and $Y_1,\ldots,Y_{n_2}$ be two independent random samples from (2.1) and (2.2), respectively. For obtaining UMVU estimator, we use Rao-Blackwell method (see, Rao, 1973, for more details). An unbiased estimator of $\phi(\alpha,\beta)$ is given by

$$h(X_1,Y_1,Y_2)=\begin{cases}1, & X_1>(i+n_1-j)Y_1 \text{ and } Y_2>(l+n_2-k)Y_1,\\ 0, & \text{otherwise.}\end{cases}$$

Since $T=(T_1,T_2)=\left(\sum_{j=1}^{n_1}X_j,\sum_{j=1}^{n_2}Y_j\right)$ is a complete sufficient statistic for (α,β), then from Rao-Blackwell Theorm, the UMVUE of $\phi(\alpha,\beta)$ is given by

$$\begin{aligned}\hat{\phi}(\alpha,\beta) &= E[h(X_1,Y_1,Y_2)|T]\\ &= P(X_1>(i+n_1-j)Y_1,Y_2>(l+n_2-k)Y_1\mid T).\end{aligned} \tag{3.7}$$

Letting $S_1=X_1/T_1,\ S_2=Y_1/T_2,\ S_3=Y_2/T_2$ and $V=T_2/T_1$, then, the equation (3.7) can be expressed as

$$\hat{\phi}(\alpha,\beta)=P\{S_1>(i+n_1-j)VS_2,S_3>(l+n_2-k)S_2\mid T\}. \tag{3.8}$$

Obvious that S_1, S_2 and S_3 are ancillary statistics, therefore by using Basu's Theorem (see, Basu, 1955) (S_1,S_2,S_3) is independent of T. Consequently, we have

$$f_{S_1,S_2,S_3}(s_1,s_2,s_3\mid T)=(n_1-1)(n_2-1)(n_2-1)(1-s_1)^{n_1-2}(1-s_2)^{n_2-2}(1-s_3)^{n_2-2},$$
$$0<s_i<1, i=1,2,3. \tag{3.9}$$

Using (3.9), for the case $(i+n_1-j)V\le 1$ if $l+n_2-k\le 1$, we find

$$\begin{aligned}\hat{\phi}(\alpha,\beta) &= P\{S_1>(i+n_1-j)VS_2,S_3>(l+n_2-k)S_2\mid T\}\\ &= \int_0^1(1-(i+n_1-j)Vs_2)^{n_1-1}(1-(l+n_2-k)s_2)^{n_2-1}(n_2-1)(1-s_2)^{n_2-2}\,ds_2.\end{aligned}$$

When $l+n_2-k>1$, we have

$$\hat{\phi}(\alpha,\beta)=\int_0^{\frac{1}{l+n_2-k}}(1-(i+n_1-j)Vs_2)^{n_1-1}(1-(l+n_2-k))s_2)^{n_2-1}(n_2-1)(1-s_2)^{n_2-2}\,ds_2.$$

Similarly for the case $(i+n_1-j)V>1$, we have

$$\hat{\phi}(\alpha,\beta)=\int_0^{\frac{1}{(i+n_1-j)V}}(1-(i+n_1-j)Vs_2)^{n_1-1}(1-(l+n_2-k)s_2)^{n_2-1}(n_2-1)(1-s_2)^{n_2-2}\,ds_2,$$

when $l+n_2-k>1$, we find

$$\hat{\phi}(\alpha,\beta)=\int_0^{E}(1-(i+n_1-j)Vs_2)^{n_1-1}(1-(l+n_2-k)s_2)^{n_2-1}(n_2-1)(1-s_2)^{n_2-2}\,ds_2,$$

where $E=\min\{\frac{1}{(i+n_1-j)V},\frac{1}{l+n_2-k}\}$. Summing up, the UMVUE of $\phi(\alpha,\beta)$ is given by

$$\hat{\phi}(\alpha,\beta)=\begin{cases}Q(V,1), & (i+n_1-j)V\le 1,\ l+n_2-k\le 1\\ Q(V,\frac{1}{l+n_2-k}), & (i+n_1-j)V\le 1,\ l+n_2-k>1\\ Q(V,\frac{1}{(i+n_1-j)V}), & (i+n_1-j)V>1,\ l+n_2-k\le 1\\ Q(V,\min\{\frac{1}{(i+n_1-j)V},\frac{1}{l+n_2-k}\}), & (i+n_1-j)V>1,\ l+n_2-k>1,\end{cases}$$

where $T_1 = \sum_{j=1}^{n_1} X_j$, $T_2 = \sum_{j=1}^{n_2} Y_j$, $V = T_2/T_1$ and

$$Q(V,p) = \int_0^p (1-(i+n_1-j)Vs_2)^{n_1-1}(1-(l+n_2-k)s_2)^{n_2-1}(n_2-1)(1-s_2)^{n_2-2}\, ds_2.$$

Since $R_{r,k}$ is a linear function of $\phi(\alpha,\beta)$, then the UMVUEs of $R_{r,k}$ is readily obtained; see, e.g., Rao (1973, p. 318).

$$\hat{R}^U_{r,k} = k\binom{n_2}{k}\sum_{j=r}^{n_1}\sum_{i=0}^{j}\sum_{l=0}^{k-1}\binom{n_1}{j}\binom{j}{i}\binom{k-1}{l}(-1)^{l+i}\hat{\phi}(\alpha,\beta). \tag{3.10}$$

The UMVU estimator obtained in (3.10) leads to computational complexities, however the variance of $\hat{R}_{r,k}$ can be easily obtained numerically.

3.3. Bayes estimator of $R_{r,k}$

In this section, we consider the Bayes estimator of $R_{r,k}$ with respect to the squared error loss (SEL) function. Let $X_1,\dots,X_{n_1}$ and $Y_1,\dots,Y_{n_2}$ be two independent random samples taken from one-parameter exponential distribution with parameters α and β as in (2.1) and (2.2), respectively. We consider conjugate priors for α and β, i.e.

$$\pi(\alpha) = \frac{\gamma^\mu}{\tau(\mu)}\alpha^{\mu-1}\exp\{-\gamma\alpha\}, \alpha>0, \gamma>0, \mu>0, \tag{3.11}$$

and

$$\pi(\beta) = \frac{\lambda^\upsilon}{\tau(\upsilon)}\beta^{\upsilon-1}\exp\{-\lambda\beta\}, \beta>0, \upsilon>0, \lambda>0, \tag{3.12}$$

respectively. From (2.1), (2.2), (3.11) and (3.12) the posterior distribution of α and β is as follows

$$\begin{aligned}\pi(\alpha,\beta|\underline{x},\underline{y}) &= \frac{f(\underline{x},\underline{y}\mid\alpha,\beta)\pi(\alpha,\beta)}{\int_\alpha\int_\beta f(\underline{x},\underline{y}\mid\alpha,\beta)\pi(\alpha,\beta)\, d\alpha\, d\beta}\\ &= \frac{(\gamma+n_1\bar{x})^{n_1+\mu}(n_2\lambda+\bar{y})^{n_2+\upsilon}}{\tau(n_1+\mu)\tau(n_2+\upsilon)}\alpha^{n_1+\mu-1}\beta^{n_2+\upsilon-1}\\ &\quad\times\exp\{-\alpha(\gamma+n_1\bar{x})-\beta(\lambda+n_2\bar{y})\},\end{aligned} \tag{3.13}$$

where $\underline{x}=(x_1,...,x_{n_1})$ and $\underline{y}=(y_1,...,y_{n_2})$. Note that, from (2.6), $R_{r,k}$ can be expressed as

$$\begin{aligned}R_{r,k} = k\binom{n_2}{k}\sum_{l=0}^{k-1}(-1)^l\frac{1}{l+n_2-k+1} + k\binom{n_2}{k}\sum_{j=r}^{n_1-1}\sum_{i=1}^{j}\sum_{l=0}^{k-1}\binom{n_1}{j}\binom{j}{i}\binom{k-1}{l}(-1)^{l+i}\\ \times[\frac{\alpha(i+n_1-j)}{\beta}+l+n_2-k+1]^{-1}.\end{aligned} \tag{3.14}$$

It is known that the Bayes estimator under SEL function is the mean of the posterior distribution. Hence, using (3.13) and (3.14), denoting the Bayes estimator of $R_{r,k}$ by $\hat{R}^B_{r,k}$, we have

$$\begin{aligned}\hat{R}^B_{r,k} &= k\binom{n_2}{k}\sum_{l=0}^{k-1}(-1)^l\frac{1}{B}\\ &+k\binom{n_2}{k}\sum_{j=r}^{n_1-1}\sum_{i=1}^{j}\sum_{l=0}^{k-1}\binom{n_1}{j}\binom{j}{i}\binom{k-1}{l}(-1)^{l+i}\frac{(\gamma+n_1\bar{x})^{n_1+\mu}(\lambda+n_2\bar{y})^{n_2+\upsilon}}{\tau(n_1+\mu)\tau(n_2+\upsilon)}\\ &\times\int_0^\infty\int_0^\infty\frac{\alpha^{n_1+\mu-1}\beta^{n_2+\upsilon-1}}{\frac{\alpha}{\beta}A+B}\exp\{-\alpha(\gamma+n_1\bar{x})-\beta(\lambda+n_2\bar{y})\}\, d\alpha\, d\beta,\end{aligned} \tag{3.15}$$

where $A = i + n_1 - j$ and $B = l + n_2 - k + 1$.
Consider a one-to-one transformation $U = \frac{B\beta}{A\alpha + B\beta}$ and $W = A\alpha + B\beta$ with the inverse $\alpha = \frac{W(1-U)}{A}$ and $\beta = \frac{UW}{B}$. The Jacobian $\mid J(U,W) \mid$ here is $det\begin{pmatrix} \frac{\partial\alpha}{\partial U} & \frac{\partial\alpha}{\partial W} \\ \frac{\partial\beta}{\partial U} & \frac{\partial\beta}{\partial W}\end{pmatrix} = det\begin{pmatrix} \frac{-W}{A} & \frac{1-U}{A} \\ \frac{W}{B} & \frac{U}{B}\end{pmatrix}$. Hence, the term double integral in equation (3.15) can be reexpressed as

$$\frac{\Gamma(n_1+n_2+\mu+\upsilon)}{B^{n_2+\upsilon+1}A^{n_1+\mu}}\int_0^1 u^{n_2+\upsilon}(1-u)^{n_1+\mu-1}\left((1-u)\frac{\gamma+n_1\bar{x}}{A}+u\frac{\lambda+n_2\bar{y}}{B}\right)^{-(n_1+n_2+\mu+\upsilon)}du. \quad (3.16)$$

By substituting (3.16) in (3.15) and take $D = 1 - \frac{A(\lambda+n_2\bar{y})}{B(\gamma+n_1\bar{x})}$, finally the Bayes estimator of $R_{r,k}$ is given by

$$\hat{R}^B_{r,k} = k\binom{n_2}{k}\sum_{l=0}^{k-1}(-1)^l\frac{1}{l+n_2-k+1} + \frac{1}{B}k\binom{n_2}{k}\sum_{j=r}^{n_1-1}\sum_{i=1}^{j}\sum_{l=0}^{k-1}\binom{n_1}{j}\binom{j}{i}\binom{k-1}{l}(-1)^{l+i}$$
$$\times\begin{cases} G^{n_2+\upsilon}\frac{n_2+\upsilon}{n_1+n_2+\mu+\upsilon}F(n_1+n_2+\mu+\upsilon, n_2+\upsilon+1, n_1+n_2+\mu+\upsilon+1; D), \ |D|<1, \\ \\ G^{-n_1-\mu}\frac{n_2+\upsilon}{n_1+n_2+\mu+\upsilon}F(n_1+n_2+\mu+\upsilon, n_1+\mu, n_1+n_2+\mu+\upsilon+1; \frac{D}{D-1}). \ D<-1,\end{cases} \quad (3.17)$$

where $G = \frac{A(\lambda+n_2\bar{y})}{B(\gamma+n_1\bar{x})}$.

We have not a closed form for the risk of $\hat{R}^B_{r,k}$, hence numerical computations are needed. As we recognize, in this case the results lead to some complexities and it is hard to compute the risk of $\hat{R}^B_{r,k}$ in (3.17). Here, we employ Lindley's approximation method to obtain Bayes estimates of $R_{r,k}$. Notice that there exists other methods and approximations such as Markov Chain Monte Carlo (MCMC) method and Tierney-Kadane approximation; see DasGupta (2008) for more details.

3.3.1. Lindley's approximation

We consider Lindley's approximation (see, Lindley, 1980) form expanding about the posterior mode. For the two parameter case $\lambda = (\lambda_1, \lambda_2)$, Lindley's approximation leads to

$$\hat{U}_B = E[U(\lambda)\mid x] = U(\lambda) + \frac{1}{2}[B + Q_{30}B_{12} + Q_{21}C_{12} + Q_{12}C_{21} + Q_{03}B_{21}], \quad (3.18)$$

where $B = \sum_{i=1}^2\sum_{j=1}^2 U_{ij}\tau_{ij}$, $Q_{\eta\xi} = \partial^{n+\xi}Q/\partial^n\lambda_1\partial^\xi\lambda_2$, $\eta,\xi = 0,1,2,3$, $\eta+\xi = 3$, for $i,j = 1,2$, $U_i = \partial U/\partial\lambda_i$, $U_{ij} = \partial^2U/\partial\lambda_i\partial\lambda_j$ and for $i \neq j$, $B_{ij} = (U_i\tau_{ii} + U_j\tau_{ij})\tau_{ii}$, $C_{ij} = 3U_i\tau_{ii}\tau_{ij} + U_j(\tau_{ii}\tau_{jj} + 2\tau_{ij}^2)$ τ_{ij} is the (i,j)th element in the inverse of matrix $Q^* = (-Q^*_{ij})$, $i,j = 1,2$ such that $Q^* = \partial^2Q/\partial\lambda_1\partial\lambda_2$. Expansion (3.17) is to be evaluated at $(\hat{\lambda_1}, \hat{\lambda_2})$, the mode of the posterior density.
In our case, $(\lambda_1,\lambda_2) = (\alpha,\beta)$ and Q is given by

$$Q = \log q \propto (n_1+\mu-1)\log\alpha - \alpha(\gamma + \sum_{j=1}^{n_1}x_j) + (n_2+\upsilon-1)\log\beta - \beta(\lambda + \sum_{j=1}^{n_2}y_j), \quad (3.19)$$

also $U(\alpha,\beta) = R_{r,k}$. The joint posterior mode, denoted by $(\hat{\alpha}_{Mod}, \hat{\beta}_{Mod})$, is obtained from (3.19), we have

$$\hat{\alpha}_{Mod} = \frac{n_1+\mu-1}{\gamma+\sum_{j=1}^{n_1}x_j} \quad \text{and} \quad \hat{\beta}_{Mod} = \frac{n_2+\upsilon-1}{\lambda+\sum_{j=1}^{n_2}y_j}.$$

First, the τ_{ij} elements of the inverse of the matrix $Q^* = (-Q^*_{ij})$ $i,j = 1,2$ are given by $\tau_{11} = \frac{\alpha^2}{n_1+\mu-1}$, $\tau_{22} = \frac{\beta^2}{n_2+\upsilon-1}$ and $\tau_{12} = \tau_{21} = 0$. Furthermore, $Q_{12} = Q_{21} = 0$, $Q_{30} = \frac{2(n_1+\mu-1)}{\alpha^3}$ and $Q_{03} = \frac{2(n_2+\upsilon-1)}{\beta^3}$.

Substituting the above values in (3.18) and take $A = i + n_1 - j$ and $B = l + n_2 - k + 1$ yields the Bayes estimate of the function $U(\alpha, \beta,)$ of the unknown parameters α and β given by

$$\tilde{R}^B_{r,k} = k\binom{n_2}{k}\sum_{l=0}^{k-1}\binom{k-1}{l}\frac{(-1)^l}{B} + k\binom{n_2}{k}\sum_{j=r}^{n_1-1}\sum_{i=1}^{j}\sum_{l=0}^{k-1}\binom{n_1}{j}\binom{j}{i}\binom{k-1}{l}(-1)^{l+i}$$
$$\times\{\frac{\hat{\beta}_{Mod}}{\hat{\alpha}_{Mod}A + B\hat{\beta}_{Mod}} + \frac{\hat{\alpha}^2_{Mod}\hat{\beta}_{Mod}A^2}{(n_1+\mu-1)(\hat{\alpha}_{Mod}A + \hat{\beta}_{Mod}B)^3} - \frac{\hat{\alpha}_{Mod}\hat{\beta}^2_{Mod}AB}{(n_2+\upsilon-1)(\hat{\alpha}_{Mod}A + \hat{\beta}_{Mod}B)^3}$$
$$-\frac{\hat{\alpha}_{Mod}\hat{\beta}_{Mod}A}{(n_1+\mu-1)(\hat{\alpha}_{Mod}A + \hat{\beta}_{Mod}B)^2} + \frac{\hat{\alpha}_{Mod}\hat{\beta}_{Mod}A}{(n_2+\upsilon-1)(\hat{\alpha}_{Mod}A + \hat{\beta}_{Mod}B)^2}\}. \quad (3.20)$$

Here also we have not a closed form for the risk of $\tilde{R}^B_{r,k}$ as $\hat{R}^B_{r,k}$, hence numerical computations are needed. Notic that $\tilde{R}^B_{r,k}$ is easy calculable with respect to $\hat{R}^B_{r,k}$.

4. Numerical studies and conclusions

In this numerical study, we have considered the sample sizes of $n_1 = n_2 = 5, 10, 30$, which are representative of small, moderate and large data sets. We intend to observe the behavior of UMVUE, MLE and Bayes estimator for different parameters and for different sample sizes in the case of series system. To compare the performance of MLE and UMVUE, the parameters α and β are chosen in such away that the reliability parameter $R_{n_1,1}$ in series system equals a given values based on (2.7). The algorithm used to compute the MSE of UMVU and ML (δ_i) estimate for series system are as follows:

1. For given α and β, we compute $R_{n_1,1}$ from (2.7) so that it takes the values: 0.01 to 0.99.
2. For given n_1 and n_2, generate a sample size n_1 from (2.1) and n_2 from (2.2) with given α and β, respectively.
3. The estimate δ_i(MLE or UMVUE) is computed using (3.3) or (3.10).
4. Steps 2-3 are repeated $N = 10^4$ times and MSE's and Biases are calculated and are given by $MSE(\delta) = \frac{1}{N}\sum_{i=1}^{N}(\delta_i - R_{n_1,1})^2$ and $Bias(\delta) = \frac{1}{N}\sum_{i=1}^{N}(\delta_i - R_{n_1,1})$.

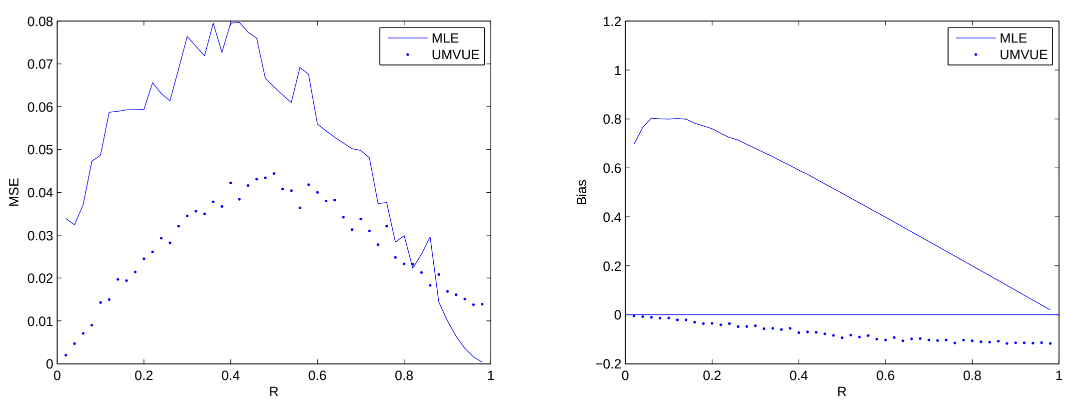

FIGURE 1. MSE and Bias for $n_1 = n_2 = 5$.

Using N=10^4 replications, the Figures 1, 2 and 3 show the MSEs and Biases of $\hat{R}^U_{n_1,1}$ and $\tilde{R}^M_{n_1,1}$ corresponding to different sample sizes $n_1 = n_2 = 5, 10, 30$. From the Figures 1, 2 and 3, it is observed that when $R_{n_1,1}$ is around 0.5 the MSEs are large and when $R_{n_1,1}$ is small or large, the MSE for both estimators take small values. We expect the MSEs and Biases of estimators such as UMVUE or MLE decrease when sample sizes increase. In our case for UMVU, ML and Bayes estimators when n_1, n_2 increase the average Biases and the MSEs decrease. For large sample sizes

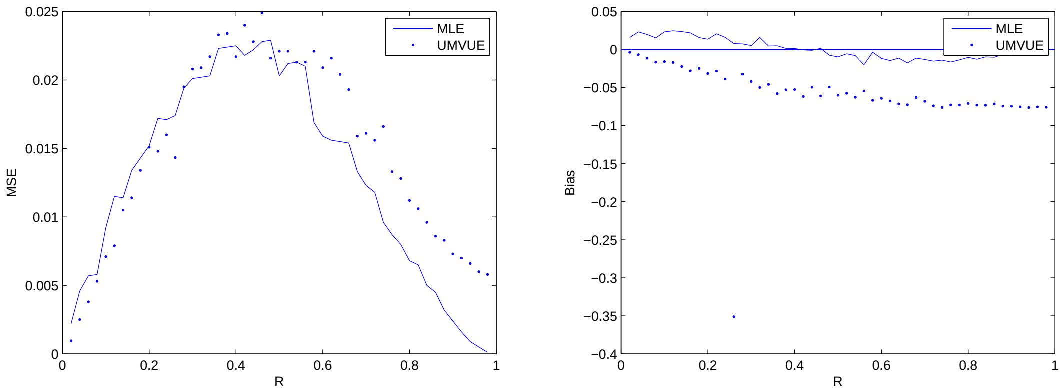

FIGURE 2. MSE and Bias for $n_1 = n_2 = 10$.

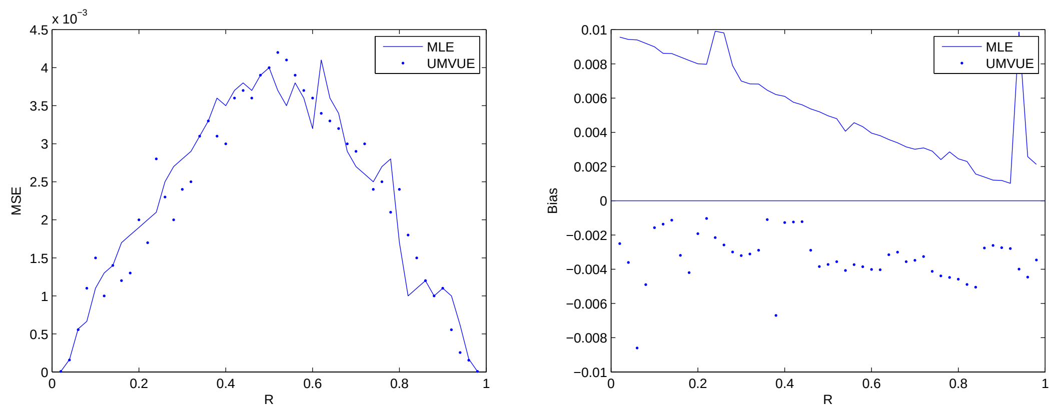

FIGURE 3. MSE and Bias for $n_1 = n_2 = 30$.

the performance of the MLE and UMVUE is similar and in this case we prefer MLE. Since MLE is easiest to obtain computationally, it has been proposed to use the MLE in practice, when the sample sizes are sufficiently large.

Also, we study sensitivity the Estimated Risks (ER) of Bayes and approximation Bayes estimators with respect to prior parameters. First for a given vector of parameters $(\mu, \gamma, \upsilon, \lambda)$ which includes least informative, informative and most informative, for N=10^4 we generate α_i and β_i from the prior distribution in (3.11) and (3.12), respectively. Then we put $\alpha_0 = \frac{1}{N}\sum_{i=1}^{N}\alpha_i$ and $\beta_0 = \frac{1}{N}\sum_{i=1}^{N}\beta_i$. Sample of sizes $n_1 = 5, 10, 30$ and $n_2 = 5, 10, 30$ are generated from (2.1) with $\alpha = \alpha_0$ and (2.2) with $\beta = \beta_0$, respectively. For the given vector of parameters $(\mu, \gamma, \upsilon, \lambda)$, for N=$10^4$ replication the Bayes and approximation Bayes estimates based on SEL function are computed. These are presented in Table 1.

TABLE 1. The values of ERs and Bias for estimators in (3.17) and (3.20).

$n_1 = n_2$	μ	γ	υ	λ	E(α)=E(β)	Var(α)=Var(β)	ER($\tilde{R}^B_{r,k}$)	Bias($\tilde{R}^B_{r,k}$)	ER($\hat{R}^B_{r,k}$)	Bias($\hat{R}^B_{r,k}$)
	1	20	1	20	0.05	0.0025	0.4081	0.6388	0.4328	0.6579
5	5	100	5	100	0.05	0.0005	0.3438	0.5864	0.3565	0.5971
	10	200	10	200	0.05	0.00025	0.3258	0.5708	0.3338	0.5777
	1	20	1	20	0.05	0.0025	0.3338	-0.5778	0.3481	-0.5900
10	5	100	5	100	0.05	0.0005	0.2764	-0.5257	0.2853	-0.5341
	10	200	10	200	0.05	0.00025	0.2499	-0.4999	0.2559	-0.5059
	1	20	1	20	0.05	0.0025	0.2539	-0.5038	0.2580	-0.5079
30	5	100	5	100	0.05	0.0005	0.2284	-0.4779	0.2316	-0.4813
	10	200	10	200	0.05	0.00025	0.1862	-0.4315	0.1885	-0.4342

From Table 1, by an empirical evidence, it is observed that the ER's is sensitive with respect to prior parameters, and also is decrease as the sample size increases. Moreover, it is observed that the estimated risks of $\tilde{R}^B_{r,k}$ is less than that of $\hat{R}^B_{r,k}$.

5. Summary

In this paper, we have studied the problem of estimating $R_{r,k} = P(X_{r:n_1} < Y_{k:n_2})$ for the exponential distribution. We obtain different point estimators, namely MLE, UMVUE and Bayes estimator. It is observed that in series system for large sample sizes the performance of the MLE and UMVUE is similar and in this case the MLE is preferred. Since MLE is easiest to obtain computationally, it has been proposed to use the MLE in practice, when the sample sizes are sufficiently large. The Baysian estimator of $R_{r,k}$ is obtained by using series expansion and Lindley's approximation methods. It is observed that the estimated risks in Lindley's approximation is less than Bayes estimator. Therefore, Lindley's approximation is a better alternative for the case in which the Bayes estimator of $R_{r,k}$ cannot be obtained in explicit forms. Finally, we emphasize here that the probability $P(X_{r:n_1} < Y_{k:n_2})$ generalizes various stress-strength reliability models for particular selection of r and k.

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References

[1] Abramowitz, M. and Stegun, I. A. (1992), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Reprint of the 1972 edition, Dover Publications, New York.

[2] Ahmad, K. E., Fakhry, M. E. and Jaheen Z. F. (1997), Empirical Bayes estimation of $P(X < Y)$ and characterization of Burr type-X model, *Journal of Statistical Planing and Inference*, **64**, 297–308.

[3] Basu, D. (1955), On statistics independent of a complete sufficient statistic, *Sankhya*, **15**, 377–380.

[4] Bhattacharyya, G. K. and Johnson, R. A.(1974), Estimation of reliability in a multicomponent stress-strength model, *Journal of the American Statistical Association*, **69**, 966–970.

[5] Chao, A. (1982), On comparing estimators of $P(X < Y)$ in the exponential case, *IEEE Transactions on Reliability*, **31**, 389–392.

[6] David, H. A. and Nagaraja, H. N. (2003), *Order Statistics*, John Wiley & Sons, New York.

[7] DasGupta, A. (2008), *Asymptotic Theory of Statistics and Probability*, Springer, New York.

[8] Enis, P. and Geisser, S. (1971), Estimation of probability that $Y < X$, *Journal of the American Statistical Association*, **66**, 162–168.

[9] Eryilmaz, S. (2008a), Consecutive k-out-of-n: G system in stress strength set up, *Communication in Statistics-Simulation and Computation*, **37**, 579–589.

[10] Eryilmaz, S. (2008b), Multivariate stress-strength reliability model and its evaluation for coherent structures, *Journal of Multivariate Analysis*, **99**, 1878–1887.

[11] Eryilmaz, S. (2010), On system reliability in stress-strength setup, *Statistics and Probability Letters*, **80**, 834–839.

[12] Kelley, G. D., Kelley, J. A. and Schuncandy, W. R. (1976), Efficient estimation of $P(X < Y)$ in the exponential case, *Technometrics*, **18**, 395–404.

[13] Kotz, S., Lumelskii, Y., Pensky, M. (2003) *The Stress-Strength Model and its Generalizations: Theory and Applications*, World Scientific, Singapore.

[14] Kundu, D. and Gupta, R. D. (2005), Estimation of $P(X < Y)$ for the generalized exponential distribution, *Metrika*, **61**, 291–308.

[15] Lindley, D. V. (1980), Approximation Bayesian methods, *Trabajos de Estadistica*, **21**, 223–237.

[16] Pandey, M., Uddin, M. B. and Ferdous, J. (1992), Reliability estimation of an s-out-of-k system with non-identical component strengths: the Weibull case, *Reliability Engineering and System Safety*, **36**, 109–116.

[17] Rao, C. R. (1973), Linear Statistical Inference and Its Applications, *John Wiley and Sons*, New York.

[18] Reiser, B. and Guttman, I. (1987), A comparison of three estimators for $P(Y < X)$ in the normal case, *Computational Statistics and Data Analysis*, **5**, 59–66.

[19] Saraçoğlu, B. and Kaya, M. F. (2007), Maximum likelihood estimation and confidence intervals of system reliability for Gompertz distribution in stress strength models, *Selçuk Journal of Applied Mathematics*, **8**, 25–36.

[20] Shah, S. P. and Sathe, Y. S. (1981), On estimating $P(X > Y)$ for the exponential distribution, *Communication in Statistics-Theory and Methods*, **10**, 39–47.

[21] Tong, H. (1974), A note on the estimation of $P(X < Y)$ in the exponential case, *Technometrics*, **16**, 617–625.