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A new family of distributions

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Abstract

In this study, a new family of distributions is introduced which is called alpha log-transformation family. We consider a special case of this family with exponential distribution in details. Several properties of the proposed distribution including the raw moments, moment generating function, quantile function and hazard rate function are obtained. Statistical inference is discussed based on complete and progressive censored samples. Simulation study is also performed to observe the performance of the estimates and approximate confidence intervals. A real data is given to illustrate the capability of ALT- Exponential distribution for modelling real data.

Keywords: Confidence intervals, Estimation, Fisher information, Lifetime distribution, Progressive censoring.

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1. Introduction

Recently, there are several distributional families are introduced by using a transformation of existing well-known distribution functions such as Exponentied and family $\alpha-$power family. The exponentied family is given by

$$F(x) = F_0(x)^{\alpha}, \alpha > 0,$$

where $F_0(x)$ is well-known distribution(besline) functions such as exponential, Weibull and etc. Exponentied distribution family is studied by Mudholkar and Srivastava (1993).

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Mahdavi and Kundu (in press) proposed an another generalized family called $\alpha-$power family which is defined by cumulative distribution function

$$F(x) = \begin{cases} \frac{\alpha^{F_0(x)}-1}{\alpha-1} & , \quad \alpha \in (0,\infty) - \{1\} \\ F_0(x) & \alpha = 1 \end{cases}$$

where $F_0(x)$ is well-known arbitrary distribution function. In this paper, we introduce a new distribution family in Section 2. The exponential case is considered in the introduced family. Some distributional characteristics are studied in Section 3. In Section 4, the statistical inference on distribution parameters are studied by maximum likelihood and least squares methods. A simulation study is performed to compare the estimates in Section 5. In Section 6, a numerical example with real data is also provided.

2. ALT Family

Let $F_0(x)$ be the cumulative distribution function (cdf)—introduced before or well-known— of a continuous random variable X, then the $\alpha-$logarithmic transformation of $F_0(x)$ for $\alpha \in (-1,\infty) - \{0\}$, is defined as follows:

$$F(x) = \frac{\log(1+\alpha F_0(x))}{\log(1+\alpha)}, \; x\epsilon\mathbb{R} \tag{2.1}$$

This family is called $\alpha-$logarithmic transformation (ALT). It is easily seen that F satisfy the property of cdf. If $F_0(x)$ is an absolute continuous distribution function with the probability distribution function (pdf) $f_0(x)$, then $F_{ALT}(x)$ is also absolute continuous distribution function with the pdf

$$f_{ALT}(x) = \frac{\alpha f_0(x)}{\log(1+\alpha)(1+\alpha F_0(x))}. \tag{2.2}$$

3. ALT-Exp Distribution and Its Properties

In this section, we apply the ALT method to a specific class of distribution function, namely to an exponential distribution, and call this new distribution as the two parameter ALT-Exp distribution. Using Exponential distribution function(with mean β) as $F_0(x)$ in (2.1) and (2.2), the cdf and corresponding pdf are given, respectively, by

$$F(x;\alpha,\beta) = \frac{\log(1+\alpha(1-\exp(-x/\beta)))}{\log(1+\alpha)}\mathbb{I}_{\mathbb{R}^+}(x) \tag{3.1}$$

and

$$f(x;\alpha,\beta) = \frac{\alpha\exp(-x/\beta)}{\beta\log(1+\alpha)(1+\alpha(1-\exp(-x/\beta)))}\mathbb{I}_{\mathbb{R}^+}(x), \tag{3.2}$$

where $\mathbb{I}_A(\cdot)$ is the indicator function on set A and the $\alpha \in (-1,\infty)-\{0\}$ and $\beta \in (0,\infty)$ are parameters. The random variable X having pdf (3.2) is said to have a two parameter ALT-Exp distribution denoted by ALT-Exp(α,β). Pdf (3.2) of ALT-Exp are plotted in Figure 1 for different values of α. It can be seen that the pdf (3.2) is decreasing for all $\alpha > -1$. Indeed, for $\alpha > -1$

$$\frac{d\log(f(x;\alpha,\beta))}{dx} = \frac{1+\alpha}{\beta\left(-1-\alpha+\alpha\exp\left(-\frac{x}{\beta}\right)\right)} < 0$$

and it can be said that for all values of the parameters the density (3.2) is strictly decreasing in x and tends to zero as $x \to \infty$. The mode of the distribution is at zero and the modal value is $\alpha/(\beta\log(1+\alpha))$. It is clear that $\lim_{\alpha\to 0} F(x;\alpha,\beta) \to 1-\exp(-x/\beta)$. In other words, ALT-Exp distribution behaves like exponential when α lies around zero.

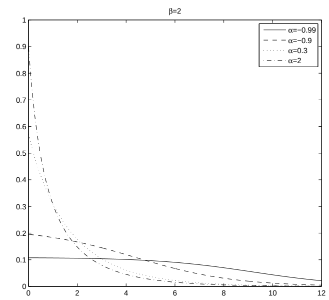

Figure 1. Probability density function

Survival (reliability) function and the hazard rate function for ALT-Exp distribution are obtained, respectively, by

$$S(x;\alpha,\beta)=\frac{\log(1+\alpha)-\log(1+\alpha(1-\exp(-x/\beta)))}{\log(1+\alpha)}$$

and

$$h(x;\alpha,\beta)=\frac{\alpha\exp(-x/\beta)}{\beta(1+\alpha-\alpha\exp(-x/\beta))(\log(1+\alpha)-\log(1+\alpha-\alpha\exp(-x/\beta)))}.$$

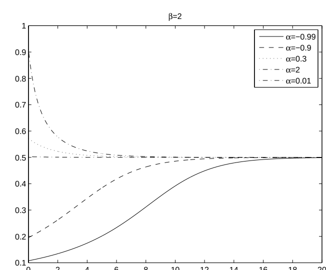

Figure 2. Hazard rate function

Some plots of hazard rate functions are provided in Fig. 2. From Fig. 2, it appears that hazard function is increasing (IFR, increasing failure rate) for $\alpha<0$ and decreasing (DFR, decreasing failure rate) for $\alpha>0$. When $\alpha\to 0$ the hazard function is constant. It can be concluded that the ALT-Exp distribution is flexible to modelling to real data which comes from DFR or IFR distribution. It should be point out that some well-known distributions such as Weibull and Gamma have this property. Note that DFR or IFR property is not discussed here.

The rth raw moments, expected value and variance of ALT-Exp distribution are given, respectively, by

$$E(X^r)=\frac{r!\beta^r\text{poly}\log\left(r+1,\frac{\alpha}{1+\alpha}\right)}{\log(1+\alpha)},\ r\in\mathbb{N}^+ \tag{3.3}$$

$$E(X)=\frac{\beta\text{poly}\log\left(2,\frac{\alpha}{1+\alpha}\right)}{\log(1+\alpha)},$$

and

$$Var\left(X\right)=\frac{2\beta^{2}\mathrm{poly}\log\left(3,\frac{\alpha}{1+\alpha}\right)}{\log\left(1+\alpha\right)}-\left(\frac{\beta\mathrm{poly}\log\left(2,\frac{\alpha}{1+\alpha}\right)}{\log\left(1+\alpha\right)}\right)^{2}.$$

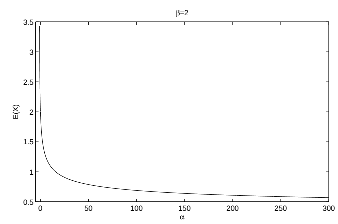

Figure 3. Expected value for different α

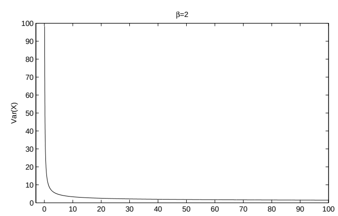

Figure 4. Variance value for different α

Note that $\alpha\to 0$ then $E\left(X\right)\to\beta$ and $\alpha\to\infty$ then $E\left(X\right)\to 0$. Moreover $\alpha\to 0$ then $Var\left(X\right)\to\beta^{2}$ and $\alpha\to\infty$ then $Var\left(X\right)\to 0$. From Figs 3-4, one can conclude that when α increases, the expected value and variance of ALT-Exp distribution decrease.

By using Eq. (3.3), moment generation function of ALT-Exp distribution can be written by

$$M_{X}\left(t\right)=\frac{1}{\log\left(1+\alpha\right)}\sum_{r=0}^{\infty}t^{r}r!\beta^{r}\mathrm{poly}\log\left(r+1,\frac{\alpha}{1+\alpha}\right).$$

The quantile function of the ALT-Exp distribution is obtained by

$$Q\left(u\right)=-\beta\left(\log\left(\alpha+1-\left(1+\alpha\right)^{u}\right)-\log\left(\alpha\right)\right),\ 0<u<1.$$

In a special case, the median of ALT-Exp distribution is also given by

$$Q\left(0.5\right)=-\beta\left(\log\left(\alpha+1-\left(1+\alpha\right)^{0.5}\right)-\log\left(\alpha\right)\right).$$

4. Parameter Estimation

In this section, we discuss the maximum likelihood and least squares estimates of the ALT-Exp parameters based on complete and progressive censored samples.

4.1. Maximum Likelihood Method based on Complete Sample. Let $X_1, X_2, \cdots, X_n$ be a random sample from ALT-Exp(α,β), then the likelihood and log-likelihood function are given, respectively, by

$$\begin{aligned} L(\alpha,\beta) &= \prod_{i=1}^{n}\left(\frac{\alpha\exp(-x_i/\beta)}{\beta\log(1+\alpha)(1+\alpha(1-\exp(-x_i/\beta)))}\right), \\ \ell(\alpha,\beta) &= n\log(\alpha)-n\log(\beta)-\frac{\sum_{i=1}^{n}x_i}{\beta}-n\log(\log(1+\alpha)) \\ &\quad -\sum_{i=1}^{n}\log(1+\alpha(1-\exp(-x_i/\beta))). \end{aligned} \tag{4.1}$$

Hence, the gradients are found to be

$$\begin{aligned} \frac{\partial\ell(\alpha,\beta)}{\partial\alpha} &= \frac{n}{\alpha}-\frac{n}{(1+\alpha)\log(1+\alpha)}-\sum_{i=1}^{n}\frac{(1-\exp(-x_i/\beta))}{\log(1+\alpha(1-\exp(-x_i/\beta)))} \\ \frac{\partial\ell(\alpha,\beta)}{\partial\beta} &= \frac{-n}{\beta}+\frac{\sum\limits_{i=1}^{n}x_i}{\beta^2}+\frac{\sum_{i=1}^{n}\alpha x_i\exp(-x_i/\beta)}{\beta^2}. \end{aligned}$$

Maximum likelihood estimates of α and β can be obtained by using any numerical method. The MLEs of α and β will be denoted by $\widehat{\alpha}$ and $\widehat{\beta}$ later. In our study, **fminsearch** command of Matlab Software is used to maximize the log-likelihood (4.1). **fminsearch** command uses the Nelder-Mead simplex algorithm as described in Lagarias et al. (1998).

4.2. Least-Squares Method based on Complete Sample. Consider the distribution function is given in Eq. (3.1). That is

$$F(x)=\frac{\log(1+\alpha(1-\exp(-x/\beta)))}{\log(1+\alpha)},\ x>0 \tag{4.2}$$

$$F\left(x_{(i)}\right)=\frac{\log\left(1+\alpha\left(1-\exp\left(-x_{(i)}/\beta\right)\right)\right)}{\log(1+\alpha)},\ i=1,2,\ldots,n. \tag{4.3}$$

Empirical distribution function (denoted by $F^*\left(x_{(i)}\right)$) can be used to estimate $F\left(x_{(i)}\right)$. Substituting the Empirical distribution function in Eq. (4.3), following model is obtained:

$$F^*\left(x_{(i)}\right)=\frac{\log\left(1+\alpha\left(1-\exp\left(-x_{(i)}/\beta\right)\right)\right)}{\log(1+\alpha)}+\varepsilon_i,\ i=1,2,\ldots,n,$$

where ε_i is the error term for ith observation. Now, least squares estimate (LSE) of the parameters can be obtained by minimizing the following equation with respect to α and β:

$$Q(\alpha,\beta)=\sum_{i=1}^{n}\varepsilon_i^2=\sum_{i=1}^{n}\left(F^*\left(x_{(i)}\right)-\frac{\log(1+\alpha(1-\exp(-x_i/\beta)))}{\log(1+\alpha)}\right)^2. \tag{4.4}$$

The LSEs of α and β will be denoted by $\widetilde{\alpha}$ and $\widetilde{\beta}$ later. The **fminsearch** command in Matlab Software can be also used to minimize the function $Q(\alpha,\beta)$ given in (4.4).

4.3. Maximum Likelihood Method based on Progressive Censored Sample. The model of progressive type-II right censoring is of importance in the field of reliability and life testing. Suppose n identical units are placed on a lifetime test. At the time of the i-th failure, r_i surviving units are randomly withdrawn from the experiment, $1\le i\le m$. Thus, if m failures are observed then $r_1+\cdots+r_m$ units are progressively censored; hence, $n=m+r_1+\cdots+r_m$. Let $X^{\mathbf{r}}_{1:m:n}\le X^{\mathbf{r}}_{2:m:n}\le\cdots\le X^{\mathbf{r}}_{m:m:n}$ be the progressively censored failure times, where $\mathbf{r} = (r_1, \ldots, r_m)$ denotes the censoring scheme. As a special case, if $\mathbf{r} = (0, \ldots, 0)$ where no withdrawals are made, we obtain the ordinary order statistics Bairamov and Eryilmaz (2006). If $\mathbf{r} = (0, \ldots, 0, n-m)$ the progressive type-II censoring becomes type-II censoring. For more details see Balakrishnan and Aggarwala (2000).

Let $X^{\mathbf{r}}_{1:m:n} < X^{\mathbf{r}}_{2:m:n} < \cdots < X^{\mathbf{r}}_{m:m:n}$ denote a progressive type-II right censored order statistics from ALT-Exp distribution. Then the log-likelihood function is given by

$$\begin{aligned}\ell(\alpha,\beta) &\approx \sum\nolimits_{i=1}^{m} \log\left(f(x_{i:m:n})\right) + \sum\nolimits_{i=1}^{m} r_i \log(1 - F(x_{i:m:n}))\\ &= \sum\nolimits_{i=1}^{m} \log\left(\frac{\alpha \exp\left(-x_i/\beta\right)}{\beta \log\left(1+\alpha\right)\left(1+\alpha\left(1-\exp\left(-x_i/\beta\right)\right)\right)}\right)\\ &\quad + \sum\nolimits_{i=1}^{m} r_i \log\left(1 - \frac{\log\left(1+\alpha\left(1-\exp\left(-x_{(i)}/\beta\right)\right)\right)}{\log\left(1+\alpha\right)}\right). \qquad (4.5)\end{aligned}$$

Let $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ denote the ordered observations from ALT-Exp with parameters α and β.

The **fminsearch** command in Matlab Software can be used to maximize the log-likelihood (4.5). The approximate confidence intervals for α and β can be found by taking $\left(\widehat{\alpha}, \widehat{\beta}\right)$ to be bivariate normally distributed with mean (α, β) and covariance matrix with inverse of Fisher information matrix, where $\left(\widehat{\alpha}, \widehat{\beta}\right)$ is ML estimates of (α, β). Hence for any $0 < \alpha < 1$, $100\,(1-\alpha)\,\%$ approximate confidence intervals for parameters α and β can be obtained using ML estimates and their approximate variances as following, respectively:

$$\left(\widehat{\alpha} - z_{\alpha/2}\sqrt{Var\left(\widehat{\alpha}\right)}, \widehat{\alpha} + z_{\alpha/2}\sqrt{Var\left(\widehat{\alpha}\right)}\right) \text{ and } \left(\widehat{\beta} - z_{\alpha/2}\sqrt{Var\left(\widehat{\beta}\right)}, \widehat{\beta} + z_{\alpha/2}\sqrt{Var\left(\widehat{\beta}\right)}\right),$$

where $Var\left(\widehat{\alpha}\right)$ and $Var\left(\widehat{\beta}\right)$ are the approximate variances of the ML estimates of the α and β, respectively. This values can be estimated by using observed Fisher information matrix. Also z_α be the percentile of standard normal distribution with right-tail probability α. Approximate variances can be obtained using the inverse of Fisher information matrix which is obtained by second derivatives of negative log-likelihood.

5. Simulation Study

In this section, a simulation study is performed to compare the performance of ML and least-square estimates. In the simulation, we have generated 5000 random samples with size of n from the ALT-Exp distribution and then computed the MLEs, LSEs of parameters. We then compared the performances of these estimates in terms of their biases and mean square errors (MSEs). The results are given in Table 1 and 3.

Another simulation study is performed to assess the accuracy of the approximation of the variances of the MLEs determined from the information matrix described above. We have carried out a simulation study for complete sample and different choices of $\mathbf{r}$ given in Table 2. It is noted that the algorithm of Balakrishnan and Sandhu (1995) is used to generate the progressive censored sample. The simulated values of $Var(\widehat{\alpha})$ and $Var(\widehat{\beta})$ as well as the approximate values determined by averaging the corresponding values obtained from the information matrix are presented in Table 4-5. The coverage probabilities(CP) of asymptotic confidence intervals based on fisher information matrix are also given in Table 4-5. The nominal level of CI is taken to be 0.95 in the simulation.

Table 1 indicates that MLEs have better MSEs and bias than LSEs have. It can be also said that MLEs and LSEs are biased but asymptotically unbiased based on complete and censored samples. Furthermore, as the sample size n increases, the bias and MSE of the MLEs and LSEs reduce as expected. According to Table 3, in progressive censoring scheme, when removals are all made in the first stage of experiment, the variances of MLEs are smaller than the other schemes given in Table 2. From Table 4-5, it can be observed that the asymptotic variances of MLEs obtained from Fisher information matrix and simulated variance are almost identical and CPs of asymptotic CI reach to the nominal levels 0.95 for $n \geq 500$. In other words, asymptotic CI based on Fisher information can be used without any doubt for moderate sample size.

Table 1. Bias and MSEs of MLEs and LSEs based on complete sample

α	β	n	$\widehat{\alpha}$		$\widehat{\beta}$		$\widetilde{\alpha}$		$\widetilde{\beta}$	
			MSE	$Bias$	MSE	$Bias$	MSE	$Bias$	MSE	$Bias$
-0.9	2	50	0.0880	0.0958	0.3014	0.0789	0.2709	0.1664	0.7328	-0.0167
		100	0.0203	0.0376	0.1569	0.0110	0.0621	0.0719	0.3624	-0.0138
		200	0.0067	0.0159	0.0820	-0.0051	0.0168	0.0320	0.1808	-0.0131
		500	0.0022	0.0046	0.0352	-0.0092	0.0047	0.0121	0.0728	-0.0052
		1000	0.0011	0.0035	0.0179	-0.0001	0.0021	0.0062	0.0361	-0.0028
-0.3	2	50	1.0938	0.2446	0.2896	-0.0126	2.9829	0.4506	0.5761	-0.0481
		100	0.4134	0.1216	0.1598	-0.0113	0.7512	0.1949	0.2737	-0.0388
		200	0.1620	0.0534	0.0805	-0.0083	0.2769	0.1022	0.1373	-0.0109
		500	0.0570	0.0189	0.0331	-0.0053	0.0976	0.0387	0.0586	-0.0059
		1000	0.0268	0.0089	0.0161	-0.0041	0.0429	0.0225	0.0279	-0.0009
0.5	2	50	3.9275	0.4722	0.2864	0.0036	9.6879	0.7718	0.5643	-0.0512
		100	1.3165	0.2014	0.1519	-0.0122	2.5654	0.3797	0.2786	-0.0147
		200	0.5686	0.0918	0.0790	-0.0077	0.9879	0.1639	0.1387	-0.0129
		500	0.2112	0.0358	0.0313	-0.0045	0.3211	0.0740	0.0540	-0.0041
		1000	0.1048	0.0132	0.0171	-0.0049	0.1601	0.0335	0.0284	-0.0045
2	2	50	15.6787	0.9579	0.3028	0.0260	29.2288	1.2958	0.5478	-0.0469
		100	4.3404	0.3248	0.1493	-0.0132	8.8150	0.6184	0.2834	-0.0259
		200	2.0705	0.1642	0.0799	-0.0117	3.3662	0.3179	0.1410	-0.0095
		500	0.7529	0.0630	0.0337	-0.0030	1.1834	0.1384	0.0566	-0.0012
		1000	0.3536	0.0259	0.0163	-0.0042	0.5153	0.0631	0.0269	-0.0011
2	0.8	50	13.4871	0.7161	0.0513	-0.0059	30.1599	1.4741	0.0920	-0.0078
		100	4.7189	0.3330	0.0250	-0.0065	8.7841	0.6425	0.0440	-0.0059
		200	2.2368	0.2145	0.0136	0.0008	3.3332	0.3085	0.0219	-0.0040
		500	0.7698	0.0646	0.0055	-0.0004	1.1008	0.1354	0.0091	-0.0006
		1000	0.3626	0.0364	0.0027	-0.0004	0.5248	0.0570	0.0044	-0.0008

Table 2. The sample size m and censoring scheme $\mathbf{r}$

Case	m	$R_i, i = 1, 2, \cdots, m$
1	200	1,1, $\cdots$,1
2	200	100,0, $\cdots$,100
3	200	200,0, $\cdots$,0
4	200	0,0, $\cdots$,200
5	300	1,1, $\cdots$,1
6	300	150,0, $\cdots$,150
7	300	300,0, $\cdots$,0
8	300	0,0, $\cdots$,300
9	1000	1,1, $\cdots$,1
10	1000	500,0, $\cdots$,500
11	1000	1000,0, $\cdots$,0
12	1000	0,0, $\cdots$,1000

Table 3. Bias and MSE of MLE based on progressive censored data

(α, β)		$\widehat{\alpha}$		$\widehat{\beta}$	
	Case	*Bias*	*MSE*	*Bias*	*MSE*
$(-0.5, 1)$	1	0.0566	0.1433	-0.0074	0.0442
	2	0.0995	0.2747	-0.0048	0.0724
	3	0.0420	0.0918	-0.0047	0.0202
	4	0.2039	0.7584	0.0031	0.1826
	5	0.0333	0.0877	-0.0073	0.0307
	6	0.0249	0.0610	-0.0058	0.0198
	7	0.0215	0.0538	-0.0054	0.0134
	8	0.1211	0.3576	-0.0055	0.1174
	9	0.0076	0.0238	-0.0040	0.0094
	10	0.0202	0.0404	-0.0008	0.0148
	11	0.0055	0.0145	-0.0015	0.0040
	12	0.0351	0.0792	-0.0015	0.0336
$(1, 3)$	1	0.1652	1.4699	-0.0040	0.4063
	2	0.2761	2.3490	0.0222	0.5595
	3	0.1108	0.9495	-0.0140	0.1818
	4	0.4100	4.4580	0.0369	1.1606
	5	0.0693	0.9044	-0.0252	0.2742
	6	0.1397	1.3249	-0.0123	0.3566
	7	0.0862	0.6199	-0.0077	0.1222
	8	0.2873	2.7478	0.0219	0.7908
	9	0.0172	0.2482	-0.0104	0.0824
	10	0.0430	0.3629	-0.0043	0.1114
	11	0.0292	0.1754	-0.0014	0.0377
	12	0.0818	0.6550	0.0058	0.2318

Table 4. Variances of the MLEs and Coverage Probabilities for selected parameters (Complete Sample Case)

			Simulated		From Information		CP	
α	β	n	$Var(\widehat{\alpha})$	$Var\left(\widehat{\beta}\right)$	$Var(\widehat{\alpha})$	$Var\left(\widehat{\beta}\right)$	$\widehat{\alpha}$	$\widehat{\beta}$
−0.9	2	100	0.0208	0.1613	0.0279	0.1926	0.8832	0.9690
		200	0.0067	0.0845	0.0081	0.0916	0.8932	0.9528
		500	0.0023	0.0346	0.0024	0.0359	0.9172	0.9494
		1000	0.0011	0.0187	0.0011	0.0179	0.9258	0.9432
−0.3	2	100	0.3907	0.1565	0.4805	0.1686	0.8796	0.9368
		200	0.1498	0.0776	0.1717	0.0816	0.9114	0.9436
		500	0.0570	0.0330	0.0586	0.0324	0.9262	0.9418
		1000	0.0269	0.0164	0.0271	0.0160	0.9316	0.9408
0.5	2	100	1.3403	0.1576	1.7190	0.1701	0.8986	0.9408
		200	0.5948	0.0802	0.6523	0.0820	0.9050	0.9402
		500	0.2079	0.0329	0.2151	0.0322	0.9240	0.9388
		1000	0.1020	0.0160	0.1036	0.0161	0.9382	0.9452
2	2	100	4.2442	0.1542	5.4052	0.1688	0.8886	0.9294
		200	1.9017	0.0807	2.1302	0.0831	0.9132	0.9336
		500	0.7397	0.0331	0.7614	0.0331	0.9330	0.9408
		1000	0.3541	0.0169	0.3595	0.0165	0.9398	0.9418
2	0.8	100	5.0876	0.0271	5.8839	0.0277	0.8826	0.9288
		200	1.9386	0.0127	2.2035	0.0134	0.9134	0.9372
		500	0.7478	0.0052	0.7646	0.0053	0.9360	0.9488
		1000	0.3659	0.0027	0.3614	0.0026	0.9390	0.9422

Table 5. Variances of the MLEs and Coverage Probabilities for selected parameters (Progressively censored Sample Case)

		Simulated		From Information		CP	
(α,β)	Case	$Var(\widehat{\alpha})$	$Var\left(\widehat{\beta}\right)$	$Var(\widehat{\alpha})$	$Var\left(\widehat{\beta}\right)$	$\widehat{\alpha}$	$\widehat{\beta}$
$(-0.5,1)$	1	0.1428	0.0447	0.1737	0.0490	0.8848	0.9380
	2	0.2502	0.0699	0.3199	0.0778	0.8690	0.9468
	3	0.0901	0.0202	0.0981	0.0205	0.8976	0.9342
	4	0.6432	0.1765	0.8792	0.1944	0.8468	0.9680
	5	0.0895	0.0314	0.1024	0.0324	0.9028	0.9352
	6	0.1587	0.0498	0.1793	0.0505	0.8830	0.9390
	7	0.0564	0.0134	0.0607	0.0137	0.9188	0.9500
	8	0.3420	0.1156	0.4211	0.1238	0.8582	0.9638
	9	0.0247	0.0099	0.0249	0.0094	0.9282	0.9390
	10	0.0389	0.0346	0.0411	0.0147	0.9218	0.9470
	11	0.0153	0.0041	0.0155	0.0041	0.9386	0.9420
	12	0.0763	0.0334	0.0828	0.0338	0.9086	0.9488
$(1,3)$	1	1.4464	0.4072	1.6557	0.4256	0.8882	0.9294
	2	2.2808	0.5522	2.6782	0.5928	0.8936	0.9330
	3	0.9682	0.1811	1.0689	0.1862	0.9132	0.9410
	4	5.0169	1.2506	6.0785	1.3235	0.8640	0.9320
	5	0.8667	0.2632	0.9746	0.2784	0.9104	0.9338
	6	1.3784	0.3695	1.5150	0.3802	0.9004	0.9364
	7	0.6129	0.1225	0.6423	0.1223	0.9166	0.9372
	8	2.4604	0.7688	2.9648	0.8161	0.8832	0.9332
	9	0.2472	0.0797	0.2555	0.0817	0.9418	0.9516
	10	0.3616	0.1116	0.3710	0.1111	0.9342	0.9472
	11	0.1634	0.1360	0.1714	0.0365	0.9406	0.9488
	12	0.6228	0.2284	0.6576	0.2308	0.9276	0.9420

6. Real Data Application

In this section, we fit the ALT model to a real data set and show that the ALT-Exp distribution is more flexible in analyzing of the data than of the Beta-Pareto (BP)(Akinsete et al. 2008), Generalized Exponential(GE)(Gupta and Kundu,1999), Exponential Poisson (EP) (Kus, 2007), Beta Generalized Half-Normal (BGHN) (Pescim et. al. 2009) and Generalized Half-Normal (GHN)(Cooray and Ananda, 2008) distributions. In order to compare the models, we used following four criteria: Akaike InformationCriterion(AIC), Bayesian Information Criterion (BIC), log-likelihood values, where the lower values of AIC, BIC and the upper value of log-likelihood values for models indicate that these models could be chosen as the best model to fit the data. The data set is given in Feigl and Zelen (1965) for the patients who died of acute myelogenous leukemia. Feigl and Zelen (1965) represent observed survival times (weeks) for AG negative. The data set is: 56, 65, 17, 17, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43. Torabi and Montazeri (2012) used this data and K-S values are given in their paper for selected models. The data analysis is given in Table 6 according to Torabi and Montazeri (2012).

Table 6. Estimates of the model parameters for leukemia data and the measures AIC, BIC log-likelihood and K-S

Model	Parameters	AIC	BIC	ℓ	K-S
ALT-Exp	$\widehat{\alpha} = 1.935, \widehat{\beta} = 24.198$	128.9	130.5	-62.4	0.135
BGHN	$\widehat{a} = 148.23, \widehat{b} = 94.77, \widehat{\alpha} = 0.06, \widehat{\theta} = 136.5$	131.9	134.98	-83.4	0.23
GHN	$\widehat{\alpha} = 0.74, \widehat{\theta} = 22.79$	130.2	131.8	-86.3	0.22
GE	$\widehat{\alpha} = 0.097, \widehat{\theta} = 0.053$	129.5	131.0	-86.8	0.24
EP	$\widehat{\alpha} = 1.01, \widehat{\theta} = 0.04$	129.1	130.6	-87.3	0.211
BP	$\widehat{a} = 1.53, \widehat{b} = 9.88, \widehat{\alpha} = 1.86, \widehat{\theta} = 0.09$	129.7	132.8	-90.8	0.22

From the Table 6 and Fig. 5, ALT-Exp is quite effectively to provide better fit to data than the others.

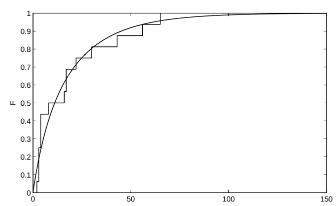

Figure 5. Emprical and fitted distribution function based on leukemia data

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