On the Commutativity of a Prime $*$-Ring with a $*$-α-Derivation

Gülay BOSNALI, Neşet AYDIN, and Selin TÜRKMEN

Çanakkale Onsekiz Mart University, Department of Mathematics
e-mail: glaybosnal@gmail.com
Çanakkale Onsekiz Mart University, Department of Mathematics
e-mail: neseta@comu.edu.tr
Çanakkale Onsekiz Mart University, Lapseki Vocational School
e-mail: selinvurkac@gmail.com

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Abstract: *Let R be a prime $*$-ring where $*$ be an involution of R, α be an automorphism of R, T be a nonzero left α-$*$-centralizer on R and d be a nonzero $*$-α-derivation on R. The aim of this paper is to prove the commutativity of a $*$-ring R with the followings conditions: i) if T is a homomorphism (or an anti-homomorphism) on R,ii) if $d([x,y]) = 0$ for all $x, y \in R$, iii) if $[d(x), y] = [\alpha(x), y]$ for all $x, y \in R$, iv) if $d(x) \circ y = 0$ for all $x, y \in R$, v) if $d(x \circ y) = 0$ for all $x, y \in R$.*

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1. Introduction

Let R be a ring and $Z(R)$ be the center of R. $x, y \in R$ such that $xy - yx$, $xy + yx$ are denoted by $[x, y]$ and $x \circ y$ respectively and the followings are hold for all $x, y \in R$

- $[x, yz] = [x, y]z + y[x, z]$
- $[xy, z] = [x, z]y + x[y, z]$
- $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$
- $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$.

R is called a *prime* (*resp.* semiprime) *ring* if $a, b \in R$ such that $aRb = (0)$ then either $a = 0$ or $b = 0$ (*resp.* If $aRa = (0)$ then $a = 0$). $* : R \to R$ is an additive mapping such that $(xy)^* = y^*x^*$ and $(x^*)^* = x$ is called an *involution* and a ring equipped with an involution is called a *$*$-ring*. If a $*$-ring is prime (*resp.* semiprime) then it is called a *prime* (*resp.* semiprime) *$*$-ring*.

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An additive mapping d of R is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. The authors have been trying to decide that whether a ring is commutative or not with the help of derivation that is defined over the ring. First study was made on this subject by Posner in [4]. Bresar and Vukman in [5] defined a $*$-*derivation* on a $*$-ring as follows: an additive mapping d of R is called a derivation if $d(xy) = d(x)y^* + xd(y)$ for all $x, y \in R$. Kim and Lee showed that in [2] the ring is commutative using some identities with a $*$-derivation which is defined on a prime $*$-ring and semiprime $*$-ring where $*$ is an involution. Firstly, inspired by the definition of $*$-derivation, it is given that d is a $*$-α-*derivation* if $d(xy) = d(x)y^* + \alpha(x)d(y)$ for all $x, y \in R$ where α is a homomorphism on R. Same results are obtained using similar hypothesis in Kim and Lee's paper with $*$-α-derivation which is defined on a prime $*$-ring and semiprime $*$-ring in this study.

In 1957, the *reverse derivation* is defined by Herstein in [6] as follows: the reverse derivation is an additive mapping d of R such that $d(xy) = d(y)x + yd(x)$ for all $x, y \in R$. After this definition, Breaser and Vukman defined the reverse $*$-derivation in [5] as follows: the reverse $*$-derivation is an additive mapping d of R such that $d(xy) = d(y)x^* + yd(x)$ for all $x, y \in R$. Inspired by the definition of reverse $*$-derivation, it is given that d is called a *reverse* $*$-α-*derivation* if $d(xy) = d(y)x^* + \alpha(y)d(x)$ for all $x, y \in R$ where $\alpha : R \longrightarrow R$ is a homomorphism. Kim and Lee showed in [2] that if d is a reverse $*$-derivation of semiprime $*$-ring then it holds $[d(x), z] = 0$ for all $x, z \in R$. This result is given for reverse $*$-α-derivation in this study.

Zalar defined in [7] the *left centralizer* (*etc.* right centralizer) as follows: the left centralizer is an additive mapping T on R such that $T(xy) = T(x)y$ for all $x, y \in R$. Ali and Fosner in [8] defined the left $*$-centralizer on a $*$-ring where $*$ is an involution as follows: a left $*$-centralizer (*etc.* right $*$-centralizer) is an additive mapping T such that $T(xy) = T(x)y^*$ for all $x, y \in R$. In [9], Koç and Gölbaşı said to a *left* α-$*$-*centralizer* (*etc.* right α-$*$-centralizer) that T is an additive mapping such that $T(xy) = T(x)\alpha(y^*)$ for all $x, y \in R$ where α is a homomorphism. Kim et al. proved that in [2] if R is a semiprime $*$-ring and $T : R \to R$ is a left $*$-centralizer then $T : R \to Z(R)$. Rehman et al showed that in [3] if R is a 2-torsion free semiprime $*$-ring and T is both a Jordan $*$-centralizer and a homomorphism on R then $T : R \to Z(R)$. Furthermore, if R is a 2-torsion free prime $*$-ring and T is a nonzero Jordan $*$-centralizer then $T = *$. In the following part of this study, based upon the results are proved by Kim and Lee in [2] and Rehman et al.in [3], if a left

α-$*$-centralizer defined over a prime $*$-ring where α is an automorphism, is also a homomorphism (or an anti-homomorphism), then the ring is commutative.

Throughout this paper, R is a prime or semiprime $*$-ring where $*$ is an involution, $\alpha : R \to R$ is an automorphism, d is a nonzero $*$-α-derivation of R and T is a left α-$*$-centralizer on R.

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2. Results

Lemma 2.1. *[1,Lemma 1.1.4] Suppose that R is semi-prime and that $a \in R$ is such that $a(ax - xa) = 0$ for all $x \in R$. Then $a \in Z(R)$, the center of R.*

Theorem 2.2. *Let R be a $*$-ring where $* : R \to R$ be an involution, α be an automorphism of R and T be a nonzero left α-$*$-centralizer on R.*

i) *If R is semiprime then the mapping T is R into $Z(R)$.*

ii) *If R is prime and T is a homomorphism (or an anti-homomorphism) on R, then R is commutative.*

Proof. i) Let R be semiprime. If it is observed $T(xz^*y^*)$ for $x, y, z \in R$, it is obtained respectively for all $x, y, z \in R$

$$T(xz^*y^*) = T(x(z^*y^*)) = T(x)\alpha((z^*y^*)^*) = T(x)\alpha(yz) = T(x)\alpha(y)\alpha(z) \tag{1}$$

and

$$T(xz^*y^*) = T((xz^*)y^*) = T(xz^*)\alpha((y^*)^*) = T(x)\alpha((z^*)^*)\alpha(y) = T(x)\alpha(z)\alpha(y) \tag{2}$$

Combining the equation (1) and (2), it holds that

$$T(x)[\alpha(y), \alpha(z)] = 0 \text{ for all } x, y, z \in R.$$

Since α is onto mapping, replacing $\alpha(y)$ by $T(x)$ in last equation, it holds

$$T(x)[T(x), \alpha(z)] = 0 \text{ for all } x, y, z \in R.$$

Since α is onto mapping, this means that

$$T(x)[T(x), z] = 0 \text{ for all } x, y, z \in R.$$

From Lemma 2.1, it gets $T(x) \in Z(R)$ for all $x \in R$ which means that $T : R \to Z(R)$.

ii) Let R be prime and T be a homomorphism of R. Since T is a homomorphism, it holds

$$T(xy) = T(x)T(y) \text{ for all } x, y \in R. \tag{3}$$

Also, since T is a left α-$*$-centralizer, it has

$$T(xy) = T(x)\alpha(y^*) \text{ for all } x, y \in R. \tag{4}$$

Combining equations (3) and (4) it holds

$$T(x)T(y) = T(x)\alpha(y^*) \text{ for all } x, y \in R. \tag{5}$$

Replacing y by y^*z^* where $z \in R$ in equation (5), it is obtained

$$T(x)T(y^*)\alpha(z) = T(x)\alpha(z)\alpha(y) \text{ for all } x, y, z \in R.$$

By using (5) it gets

$$T(x)\alpha(y)\alpha(z) = T(x)\alpha(z)\alpha(y) \text{ for all } x, y, z \in R.$$

And so,

$$T(x)[\alpha(z), \alpha(y)] = 0 \text{ for all } x, y, z \in R$$

is obtained. In the last equation, replacing x by xs^* where $s \in R$ and using that α is an onto mapping, it gets

$$T(x)R[\alpha(z), \alpha(y)] = (0) \text{ for all } x, y, z \in R.$$

Since R is a prime $*$-ring, it implies either $T = 0$ or $[\alpha(z), \alpha(y)] = 0$ for all $y, z \in R$. Since T is nonzero, it implies that R is commutative.

Now let R be prime and T be an anti-homomorphism of R. Since T is an anti-homomorphism, it gets

$$T(xy) = T(y)T(x) \text{ for all } x, y \in R. \tag{6}$$

Moreover, since T is a left α-$*$-centralizer, it has

$$T(xy) = T(x)\alpha(y^*) \text{ for all } x, y \in R. \tag{7}$$

If the equations (6) and (7) are considered together and edited, it follows

$$T(y)T(x) = T(x)\alpha(y^*) \text{ for all } x, y \in R. \tag{8}$$

Replacing x by zx^* and y by y^* where $z \in R$ in the last equation, it holds

$$T(y^*)T(zx^*) = T(zx^*)\alpha((y^*)^*) \text{ for all } x, y, z \in R.$$

The last equation is edited by using the equation (8) , it follows

$$T(z)[\alpha(x), \alpha(y)] = 0 \text{ for all } x, y, z \in R.$$

Replacing z by zt^* where $t \in R$ in the last equation and using α is an onto mapping it gets

$$T(z)R[\alpha(x), \alpha(y)] = (0) \text{ for all } x, y, z \in R.$$

Since R is a prime $*$-ring, it implies that either $T = 0$ or $[\alpha(x), \alpha(y)] = 0$ for all $x, y \in R$. Since α is an onto mapping and T is a nonzero mapping, it gets that R is commutative. □

Theorem 2.3. *Let R be a $*$-ring where $* : R \to R$ be an involution, α be an automorphism of R and d be a nonzero $*$-α-derivation on R.*

i) *If R is semiprime, then d is R into $Z(R)$.*
ii) *If R is prime and d acts as a homomorphism on R, then $d = \alpha$.*
iii) *If R is prime and d acts as an anti-homomorphism, then $d = *$.*

Proof.
i) Let R be semiprime. If it is observed $d(xy^*z^*)$ for $x, y, z \in R$ by using that d is a nonzero $*$-α-derivation, it is obtained

$$d(xy^*z^*) = d(x(y^*z^*)) = d(x)zy + \alpha(x)d(y^*)z + \alpha(xy^*)d(z^*) \quad (9)$$

and

$$d(xy^*z^*) = d((xy^*)z^*) = d(x)yz + \alpha(x)d(y^*)z + \alpha(xy^*)d(z^*). \quad (10)$$

Combining the equations (9) and (10), it implies

$$d(x)[z, y] = 0 \text{ for all } x, y, z \in R.$$

Replacing z by $d(x)$ in last equation, by using the Lemma 2.1 the desired result is obtained.

ii) Let R be prime and d be a homomorphism. Since d is both a homomorphism and a $*$-α-derivation, it holds

$$d(xy) = d(x)y^* + \alpha(x)d(y) = d(x)d(y) \text{ for all } x, y \in R.$$

Replacing x by xz where $z \in R$ in the last equation and by using that d is a homomorphism, it implies for all $x, y, z \in R$

$$d(x)d(z)y^* + \alpha(x)\alpha(z)d(y) = d(x)d(z)d(y) = d(x)d(zy)$$

is obtained.

$$d(x)d(z)y^* + \alpha(x)\alpha(z)d(y) = d(x)d(z)y^* + d(x)\alpha(z)d(y) \text{ for all } x, y, z \in R.$$

Since α is onto mapping, it follows

$$(\alpha(x) - d(x))Rd(y) = (0) \text{ for all } x, y \in R.$$

Since R is a prime $*$-ring and d is a nonzero mapping, it is obtained that $d = \alpha$.

iii) Let R be prime and d be an anti-homomorphism. Since d is both an anti-homomorphism and a $*$-α-derivation,

$$d(xy) = d(x)y^* + \alpha(x)d(y) = d(y)d(x).$$

Replacing y by xy^* in last equation and by using that d is an anti-homomorphism, it follows

$$d(x)yx^* + \alpha(x)d(y^*)d(x) = d(x)yd(x) + \alpha(x)d(y^*)d(x).$$

So, it implies

$$d(x)R(d(x) - x^*) = (0) \text{ for all } x \in R.$$

Since R is prime $*$-ring, it implies that either $d(x) = x^*$ or $d(x) = 0$. We set that $A = \{x \in R \mid d(x) = x^*\}$ and $B = \{x \in R \mid d(x) = 0\}$. Then A and B are both additive subgroups of R and R is the union A and B but a group can not be set union of its two proper subgroups. Hence, R equals that either A or B. Assume that $B = R$ which means that $d = 0$ which is a contradiction. So it follows that $A = R$ which means that $d = *$. □

Theorem 2.4. *Let R be a prime $*$-ring where $* : R \to R$ be an involution, α be an automorphism of R and d be a nonzero $*$-α-derivation on R. If $d([x, y]) = 0$ for all $x, y \in R$, then R is commutative.* 56

Proof. Replacing x by xy in the hypothesis and by using that d is a $*$-α-derivation, it holds

$$\alpha([x, y])d(y) = 0 \text{ for all } x, y \in R.$$

Replacing x by xs where $s \in R$ in last equation and using that α is an onto mapping, it hold

$$\alpha([x, y])Rd(y) = (0) \text{ for all } x, y \in R.$$

Since R is a prime $*$-ring, it implies that either $\alpha([x, y]) = 0$ or $d(y) = 0$ for all $x, y \in R$. Since d is nonzero and α is onto, it follows that R is commutative by using the similar method in the proof of (iii) of Theorem 2.3. □

Theorem 2.5. *Let R be a prime $*$-ring where $*: R \to R$ be an involution, $\alpha : R \to R$ be an automorphism and $d : R \to R$ be a nonzero $*$-α-derivation. If $[d(x), y] = [\alpha(x), y]$ for all $x, y \in R$, then R is commutative.*

Proof. Replacing x by xz where $z \in R$ and by using that d is a $*$-α-derivation, it holds

$$(11) \quad [d(x)z^*, y] + [\alpha(x)d(z), y] = \alpha(x)[\alpha(z), y] + [\alpha(x), y]\alpha(z) \text{ for all } x, y, z \in R.$$

Replacing y by $\alpha(x)$ in hypothesis, it holds

$$[d(x), \alpha(x)] = 0.$$

Furthermore, replacing y by $\alpha(x)$ in (11) and by using that $[d(x), \alpha(x)] = 0$, it implies

$$d(x)[z^*, \alpha(x)] = 0 \text{ for all } x, z \in R.$$

Replacing z by $(zr)^*$ where $r \in R$ and by using the last equation, it holds

$$d(x)R[r, \alpha(x)] = (0) \text{ for all } x, r \in R.$$

Since R is prime $*$-ring, it implies that either $d(x) = 0$ or $[r, \alpha(x)] = 0$ for all $x, r \in R$. Since d is nonzero and α is onto, it follows that R is commutative by using the similar method in the proof of (iii) of Theorem 2.3. ☐

Theorem 2.6. *Let R be a prime $*$-ring where $*: R \to R$ be an involution, α be an automorphism and d be a nonzero $*$-α-derivation on R. If $a \in R$ such that $[d(x), \alpha(a)] = 0$ for all $x \in R$ then $d(a) = 0$ or $a \in Z(R)$.* 57

Proof. Replacing for x by xy where $y \in R$ in the hypothesis and by using that d is a $*$-α-derivation, it implies

$$d(x)[y^*, \alpha(a)] + [\alpha(x), \alpha(a)]d(y) = 0 \text{ for all } x, y \in R.$$

Replacing x by a in the last equation

$$d(a)[y^*, \alpha(a)] = 0 \text{ for all } y \in R.$$

Replacing y by $(yr)^*$where $r \in R$ in the last equation, it implies

$$d(a)R[r, \alpha(a)] = (0) \text{ for all } r \in R.$$

Since R is a prime $*$- ring and α is an onto mapping, it follows that either $d(a) = 0$ or $a \in Z(R)$. ☐

Theorem 2.7. *Let R be a semiprime $*$-ring where $*: R \to R$ be an involution and α be an automorphism of R. If d is a nonzero reverse $*$-α-derivation on R, the mapping d is R into $Z(R)$.*

Proof. Since d is a reverse $*$-α-derivation, it holds

$$d(xy) = d(y)x^* + \alpha(y)d(x) \text{ for all } x, y \in R.$$

Replacing x by xz and y by zy where $z \in R$ in the last equation respectively, it gets that for all $x, y, z \in R$

$$d((xz)y) = d(y)z^*x^* + \alpha(y)d(z)x^* + \alpha(y)\alpha(z)d(x). \tag{12}$$

and

$$d(x(zy)) = d(y)z^*x^* + \alpha(y)d(z)x^* + \alpha(z)\alpha(y)d(x). \tag{13}$$

Combining equations (12) and (13), it implies

$$[\alpha(y), \alpha(z)]d(x) = 0 \text{ for all } x, y, z \in R. \tag{14}$$

Replacing y by yr where $r \in R$ in the last equation, it holds

$$[\alpha(y), \alpha(z)]\alpha(r)d(x) = 0 \text{ for all } x, y, z, r \in R. \tag{15}$$

On the other hand, the equation (14) multiplies by $\alpha(r)$ from right side, it holds

$$[\alpha(y), \alpha(z)]d(x)\alpha(r) = 0 \text{ for all } x, y, z, r \in R. \tag{16}$$

Combining equations(15) and (16), it implies

$$[\alpha(y), \alpha(z)][\alpha(r), d(x)] = 0 \text{ for all } x, y, z, r \in R.$$

Since α is onto, it holds

$$[y, z][r, d(x)] = 0 \text{ for all } x, y, z, r \in R. \tag{17}$$

Replacing y by r and z by $d(x)s$ where $s \in R$ in the last equation and by using the equation (17), it follows

$$[r, d(x)]R[r, d(x)] = (0) \text{ for all } r, x \in R.$$

Since R is a semiprime $*$-ring, d is R into $Z(R)$ which means that $d: R \to Z(R)$. □

Theorem 2.8. *Let R be a prime $*$-ring where $*: R \to R$ be an involution, α be an automorphism and d be a nonzero $*$-α-derivation on R. If $d(x) \circ y = 0$ for all $x, y \in R$, then R is commutative.*

Proof. Replacing x by xz where $z \in R$ in the hypothesis, it holds

$$d(x)[z^*, y] - [\alpha(x), y]d(z) = 0 \text{ for all } x, y, z \in R.$$

Replacing y by $\alpha(x)$ in last equation,

$$d(x)[z^*, \alpha(x)] = 0 \text{ for all } x, y \in R$$

is obtained. Replacing z by $(rz)^*$ where $r \in R$ in the last equation and by using α is an onto mapping with the last equation, it gets

$$d(x)R[z, \alpha(x)] = (0) \text{ for all } x, z \in R.$$

Since R is a prime $*$-ring, it follows that either $d(x) = 0$ or $[z, \alpha(x)] = 0$ for all $z, x \in R$. Since d is nonzero and α is onto, it follows that R is commutative by using the similar method in the proof of (iii) of Theorem 2.3. □

Theorem 2.9. *Let R be a prime $*$-ring, where $* : R \to R$ be an involution, α be an automorphism and d be a nonzero $*$-α-derivation on R. If $d(x \circ y) = 0$ for all $x, y \in R$, then R is commutative.*

Proof. Replacing x by xy in hypothesis, it holds

$$d((x \circ y)y) = \alpha(x \circ y)d(y) = 0 \text{ for all } x, y \in R.$$

Furthermore replacing x by xz where $z \in R$ in the last equation and by using that α is an onto mapping 59

$$\alpha([x, y])Rd(y) = (0) \text{ for all } x, y \in R$$

is obtained. Since R is a prime $*$-ring, it implies that either $\alpha([x, y]) = 0$ or $d(y) = 0$ for all $x, y \in R$. Since d is nonzero and α is onto, it follows that R is commutative by using the similar method in the proof of (iii) of Theorem 2.3. □

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