Generalized Reverse Derivations On Closed Lie Ideals

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Abstract: *In this study, we investigate commutavity of prime ring R with generalized reverse derivations F and G. Also, we proved that if L is a square closed Lie ideal, then L is contained in center $Z(R)$ under given conditions in theorems.*

Keywords: *Prime ring, Lie ideal, Reverse derivation, Generalized reverse derivation.*
2010 AMS Subject Classification: Primary 16N60; Secondary 16U80, 16W25

1. Introduction

Let R be a ring with center $Z(R)$. Recall that R is prime if for any $x, y \in R$, $xRy = (0)$ implies $x = 0$ or $y = 0$. An additive mapping d form R into R is called derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In [3], Bresar generalized concept of derivation as the following: An additive mapping F from R into R is called generalized derivation with associated derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. In [4], Bresar and Vukman introduced reverse derivation and in [1], Abuabakar and Gonzalez introduced generalized reverse derivation. Let d from R into R be an additive mapping. If $d(xy) = d(y)x + yd(x)$ holds for all $x, y \in R$, then d is called right reverse derivation. Let F from R into R be an additive mapping. If $F(xy) = d(y)x + yF(x)$ holds for all $x, y \in R$, then F is called right generalized reverse derivation with associated reverse derivation d.

For any $x, y \in R$ denote the notation $[x, y]$ for commutator $xy - yx$ and $x \circ y$ for anti-commutator $xy + yx$. We use the following basic identities.

- $[xy,z] = x[y,z] + [x,z]y$
- $[x,yz] = [x,y]z + y[x,z]$
- $(xy) \circ z = x(yoz) - [x,z]y = (x \circ z)y + x[y,z]$
- $x \circ (yz) = (x \circ y)z - y[x,z] = y(x \circ z) + [x,y]z$

Let L be an additive subgroup of R. L is said to be a Lie ideal of R if $[L,R] \subseteq R$. A Lie ideal L is said to be a square closed Lie ideal if $x^2 \in L$ for all $x \in L$.

In [5], Posner showed that two important properties of prime rings with derivation. In a prime ring R with $char R \neq 2$, if the iterate of two derivations is a derivation, then one of them is zero, and if d is a derivation and $[a, d(a)] \in Z(R)$ for all $a \in R$, then either R is commutative or d is zero. After that, several authors have proved commutativity theorems for prime rings with derivation and generalized derivation. Also many researchers have generalized results to ideals and Lie ideals of ring. In [2], Al-Omary and Rehman showed that if L is a square closed Lie ideal of prime ring with generalized derivation, then $L \subseteq Z(R)$ under several conditions.

In this study, we generalize previous studies on prime rings with reverse derivation. Let R be a prime ring with $char R \neq 2$, $F : R \to R$ be a nonzero right generalized reverse derivation with associated right reverse derivation $d : R \to R$ and L be a nonzero square closed Lie ideal of R such that $d(Z(L)) \neq (0)$. We study following conditions and prove $L \subseteq Z(R)$. (i) $[F(x), x] \in Z(R)$ for all $x \in L$. (ii) $F(x) \circ x \in Z(R)$ for all $x \in L$. (iii) $F(x \circ y) - [x,y] \in Z(R)$ for all $x,y \in L$. (iv) $F[x,y] - x \circ y \in Z(R)$ for all $x,y \in L$. (v) $[F(x), d(y)] - [x,y] \in Z(R)$ for all $x,y \in L$. (vi) $[F(x), F(y)] - [x,y] \in Z(R)$ for all $x,y \in L$. (vii) $F(x) \circ F(y) - x \circ y \in Z(R)$ for all $x,y \in L$. $(viii)$ $[F(x), F(y)] - x \circ y \in Z(R)$ for all $x,y \in L$. (ix) $F(x) \circ F(y) - [x,y] \in Z(R)$ for all $x,y \in L$. (x) $[F(x), F(y)] - F[x,y] \in Z(R)$ for all $x,y \in L$. (xi) $F(x) \circ F(y) - F(x \circ y) \in Z(R)$ for all $x,y \in L$. (xii) $F[x,y] - [F(x), y] \in Z(R)$ for all $x,y \in L$. $(xiii)$ $F[x,y] + [F(x), y] - [F(x), F(y)] \in Z(R)$ for all $x,y \in L$. (xiv) $F[x,y] - F(x) \circ y - [d(y), x] \in Z(R)$ for all $x,y \in L$.

In addition, we investigate commutative property for two nonzero right generalized reverse derivations $F, G : R \to R$ with associated right reverse derivations $d, g : R \to R$ respectively. We study following conditions and prove $L \subseteq Z(R)$. (i) $[F(x), G(y)] - [x,y] \in Z(R)$ for all $x,y \in L$. (ii) $[F(x), x] - [x, G(x)] \in Z(R)$ for all $x,y \in L$. (iii) $F(x) \circ x - x \circ G(x) \in Z(R)$ for all $x,y \in L$. (iv) $F[x,y] - [y, G(x)] \in Z(R)$ for all $x,y \in L$. (v) $F(x \circ y) - y \circ G(x) \in Z(R)$ for all $x,y \in L$.

2. Preliminaries

Well-known fact about prime rings:

Remark 2.1. *Let R be a prime ring. For an elements $a \in Z(R)$ and $b \in R$, if $ab \in Z(R)$, then $b \in Z(R)$ or $a = 0$.*

Remark 2.2. *Let R be a prime ring with char$R \neq 2$ and L be a square closed Lie ideal of R. Then $2ab \in L$ for all $a, b \in L$.*

Lemma 2.3. [6, Lemma 2.6] *Let R be a $2-$torsion free semiprime ring and L be a nonzero Lie ideal of R. If L is a commutative Lie ideal of R, i. e., $[x, y] = 0$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Lemma 2.4. [7, Lemma 2.5] *Let R be a $2-$torsion free semiprime ring and L be a nonzero Lie ideal of R. Then $Z(L) \subseteq Z(R)$.*

3. Results

Lemma 3.1. *Let R be a prime ring with char $(R) \neq 2$ and L be a nonzero square closed Lie ideal of R. If $[x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof. Let $[x, y] \in Z(R)$ for all $x, y \in L$. Then $[r, [x, y]] = 0$ for all $x, y \in L$, $r \in R$. Replacing x by $2xy$, we get $0 = [r, [2xy, y]] = 2[r, [xy, y]]$ and using *char* $(R) \neq 2$, we have $[x, y][r, y] = 0$. Replacing r by rs for any $s \in R$, we find $[x, y] r [s, y] = 0$ for all $x, y \in L$, $r, s \in R$. Since R is a prime ring, we obtain

$$[x, y] = 0 \text{ or } [s, y] = 0 \text{ for all } x, y \in L, s \in R.$$

If $[s, y] = 0$, then $y \in Z(R)$ and satisfy condition $[x, y] = 0$. So, $[x, y] = 0$ for all $x, y \in L$ in both cases. From the Lemma 2.3 we get $L \subseteq Z(R)$. □

Lemma 3.2. *Let R be a prime ring with char $(R) \neq 2$ and L be a nonzero square closed Lie ideal of R. If $x \circ y \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof. Let $x \circ y \in Z(R)$ for all $x, y \in L$. Then $[r, x \circ y] = 0$ for all $x, y \in L$, $r \in R$. Replacing x by $2xy$, we get $0 = [r, 2xy \circ y] = 2[r, xy \circ y]$ and using *char* $(R) \neq 2$, we have $(x \circ y)[r, y] = 0$. Replacing r by rs for any $s \in R$, we find $(x \circ y) r [s, y] = 0$ for all $x, y \in L$, $r, s \in R$. Since R is a prime ring, we obtain

$$x \circ y = 0 \text{ or } [s, y] = 0 \text{ for all } x, y \in L, \; s \in R.$$

Let $A = \{y \in L \mid x \circ y = 0 \text{ for all } x \in L\}$ and $B = \{y \in L \mid [s, y] = 0 \text{ for all } s \in R\}$. A and B are additive subgroups of L whose $L = A \cup B$, but a group can not be written as a union of two proper subgroups of its and hence $L=A$ or $L=B$. If $L=A$, then $x\circ y=0$ for all $x\in L$. Replacing y by $2yz$ for any $z\in L$ and using $char\,(R)\neq 2$, we get $[x,y]\,z=0$ for all $x,y,z\in L$. In this equation, replacing z by $[z,r]$ for any $r\in R$ we find $[x,y]\,[z,r]=0$ for all $x,y,z\in L$, $r\in R$. Again replacing r by rs for any $s\in R$, we get $[x,y]\,r\,[z,s]=0$ for all $x,y,z\in L$, $r,s\in R$. Since R is a prime ring, we obtain

$$[x,y]=0 \text{ or } [z,s]=0 \text{ for all } x,y,z\in L, s\in R.$$

If $[x,y]=0$, then from the Lemma 2.3 we get $L\subseteq Z\,(R)$. If $[z,s]=0$, then $z\in Z\,(R)$ for all $z\in L$ and $L\subseteq Z\,(R)$. If $L=B$, then $[s,y]=0$ for all $s\in R$, $y\in L$. Hence, we obtain $y\in Z\,(R)$ for all $y\in L$ and $L\subseteq Z\,(R)$. □

Lemma 3.3. *Let R be a prime ring with $char\,(R)\neq 2$, $0\neq F:R\longrightarrow R$ be a right generalized reverse derivation with associated right reverse derivation d and L be a nonzero square closed Lie ideal of R such that $d\,(Z\,(L))\neq(0)$. If $[F\,(x)\,,x]\in Z(R)$ for all $x,y,z\in L$, then $L\subseteq Z(R)$.*

Proof. Let $[F\,(x)\,,x]\in Z(R)$ for all $x,y,z\in L$. Replacing x by $x+y$, we get

$$[F\,(x)\,,y]+[F\,(y)\,,x]\in Z(R) \text{ for all } x,y\in L \tag{1}$$

Since $d\,(Z\,(L))\neq(0)$, we choose fixed element $0\neq z\in Z\,(L)$ which $d\,(z)\neq 0$. Also, $z,d\,(z)\in Z\,(R)$ from the Lemma 2.4. Replacing x by $2xz$ in Equation (1) and using $char\,(R)\neq 2$, we get

$$d\,(z)\,[x,y]+[d\,(z)\,,y]\,x+z\,[F\,(x)\,,y]+[z,y]\,F\,(x)+[F\,(y)\,,x]\,z+x\,[F\,(y)\,,z]\in Z\,(R)$$

In this expression, using $z,d\,(z)\in Z\,(R)$ and Equation (1), we have

$$d\,(z)\,[x,y]\in Z\,(R) \text{ for all } x,y\in L$$

Hence, using $0\neq d\,(z)\in Z\,(R)$ and Remark 2.1, we obtain $[x,y]\in Z\,(R)$ for all $x,y\in L$. From the Lemma 3.1 we get $L\subseteq Z\,(R)$. □

Lemma 3.4. *Let R be a prime ring with $char\,(R)\neq 2$, $0\neq F:R\longrightarrow R$ be a right generalized reverse derivation with associated right reverse derivation d and L be a nonzero square closed Lie ideal of R such that $d\,(Z\,(L))\neq(0)$. If $F\,(x)\circ x\in Z(R)$ for all $x,y,z\in L$, then $L\subseteq Z(R)$.*

Proof. Let $F\,(x)\circ x\in Z(R)$ for all $x,y,z\in L$. Replacing x by $x+y$, we get

$$F\,(x)\circ y+F\,(y)\circ x\in Z(R) \text{ for all } x,y\in L \tag{2}$$

Since $d(Z(L)) \neq (0)$, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing x by $2xz$ in Equation (2) and using $char(R) \neq 2$, we get

$$(d(z)x) \circ y + (zF(x)) \circ y + F(y) \circ (xz) \in Z(R) \text{ for all } x, y \in L$$

In this expression, using $z, d(z) \in Z(R)$ and Equation (2), we have

$$d(z)(x \circ y) \in Z(R) \text{ for all } x, y \in L$$

Hence, using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we obtain $x \circ y \in Z(R)$ for all $x, y \in L$. From the Lemma 3.2 we get $L \subseteq Z(R)$. □

Theorem 3.5. *Let R be a prime ring with $char(R) \neq 2$, $0 \neq F : R \longrightarrow R$ be a right generalized reverse derivation with associated right reverse derivation d and L be a nonzero square closed Lie ideal of R such that $d(Z(L)) \neq (0)$. If one of the following conditions is satisfy, then $L \subseteq Z(R)$.*

i) $F(x \circ y) - [x, y] \in Z(R)$ for all $x, y \in L$.
ii) $F[x, y] - x \circ y \in Z(R)$ for all $x, y \in L$.
iii) $[F(x), d(y)] - [x, y] \in Z(R)$ for all $x, y \in L$.

Proof. *i*) By hypothesis,

$$F(x \circ y) - [x, y] \in Z(R) \text{ for all } x, y \in L. \tag{3}$$

Since $d(Z(L)) \neq (0)$, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing x by $2xz$ in Equation (3) and using $char(R) \neq 2$, we get

$$F((x \circ y)z + x[z, y]) - x[z, y] - [x, y]z \in Z(R) \text{ for all } x, y \in L$$

In this expression, using $z, d(z) \in Z(R)$ and Equation (3), we have

$$d(z)(x \circ y) \in Z(R) \text{ for all } x, y \in L$$

Hence, using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we obtain $x \circ y \in Z(R)$ for all $x, y \in L$. From the Lemma 3.2 we get $L \subseteq Z(R)$.

ii) By hypothesis,

$$F[x, y] - x \circ y \in Z(R) \text{ for all } x, y \in L. \tag{4}$$

Since $d(Z(L)) \neq (0)$, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing x by $2xz$ in Equation (4) and using

$char\,(R) \neq 2$, we get

$$F\left(x\left[z,y\right]+\left[x,y\right]z\right)-\left(x\circ y\right)z-x\left[z,y\right]\in Z(R) \text{ for all } x,y\in L$$

In this expression, using $z, d\left(z\right) \in Z\left(R\right)$ and Equation (4), we obtain

$$d\left(z\right)\left[x,y\right]\in Z\left(R\right) \text{ for all } x,y\in L$$

Hence, using $0 \neq d\left(z\right) \in Z\left(R\right)$ and Remark 2.1, we have $\left[x,y\right] \in Z\left(R\right)$ for all $x, y \in L$. From the Lemma 3.1 we get $L \subseteq Z\left(R\right)$.

iii) By hypothesis,

$$\left[F\left(x\right),d\left(y\right)\right]-\left[x,y\right]\in Z(R) \text{ for all } x,y\in L. \tag{5}$$

Since $d\left(Z\left(L\right)\right) \neq (0)$, we choose fixed element $0 \neq z \in Z\left(L\right)$ which $d\left(z\right) \neq 0$. Also, $z, d\left(z\right) \in Z\left(R\right)$ from the Lemma 2.4. Replacing y by $2yz$ in Equation (5) and using $char\,(R) \neq 2$, we get

$$\left[F\left(x\right),d\left(z\right)y+zd\left(y\right)\right]-\left[x,y\right]z-y\left[x,z\right]\in Z(R) \text{ for all } x,y\in L$$

In this expression, using $z, d\left(z\right) \in Z\left(R\right)$ and Equation (5), we get

$$d\left(z\right)\left[F\left(x\right),y\right]\in Z\left(R\right) \text{ for all } x,y\in L$$

By using $0 \neq d\left(z\right) \in Z\left(R\right)$ and Remark 2.1, we have

$$\left[F\left(x\right),y\right]\in Z\left(R\right) \text{ for all } x,y\in L$$

Replacing y by x in above expression, we obtain $\left[F\left(x\right),x\right] \in Z\left(R\right)$ for all $x, y \in L$. From the Lemma 3.3 we get $L \subseteq Z\left(R\right)$. □

Theorem 3.6. *Let R be a prime ring with $char\,(R) \neq 2$, $0 \neq F : R \longrightarrow R$ be a right generalized reverse derivation with associated right reverse derivation d and L be a nonzero square closed Lie ideal of R such that $d\left(Z\left(L\right)\right) \neq (0)$. If one of the following conditions is satisfy, then $L \subseteq Z(R)$.*

i) $\left[F\left(x\right),F\left(y\right)\right]-\left[x,y\right]\in Z(R)$ for all $x,y\in L$.
ii) $F\left(x\right)\circ F\left(y\right)-x\circ y\in Z(R)$ for all $x,y\in L$.
iii) $\left[F\left(x\right),F\left(y\right)\right]-x\circ y\in Z(R)$ for all $x,y\in L$.
iv) $F\left(x\right)\circ F\left(y\right)-\left[x,y\right]\in Z(R)$ for all $x,y\in L$.

Proof. i) By assumption,

$$\left[F\left(x\right),F\left(y\right)\right]-\left[x,y\right]\in Z(R) \text{ for all } x,y\in L. \tag{6}$$

Since $d(Z(L)) \neq (0)$, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing x by $2xz$ in Equation (6) and using $char(R) \neq 2$, we have

$$[d(z)x + zF(x), F(y)] - x[z, y] - [x, y]z \in Z(R) \text{ for all } x, y \in L$$

In this expression, using $z, d(z) \in Z(R)$ and Equation (6), we obtain

$$d(z)[x, F(y)] \in Z(R) \text{ for all } x, y \in L$$

By using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we have

$$[x, F(y)] \in Z(R) \text{ for all } x, y \in L$$

Replacing y by x in above expression, we get $[x, F(x)] \in Z(R)$ for all $x, y \in L$. From the Lemma 3.3 we get $L \subseteq Z(R)$.

ii) By assumption,

$$F(x) \circ F(y) - x \circ y \in Z(R) \text{ for all } x, y \in L. \tag{7}$$

Since $d(Z(L)) \neq (0)$, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing x by $2xz$ in Equation (7) and using $char(R) \neq 2$, we get

$$(d(z)x + zF(x)) \circ F(y) - (x \circ y)z - x[z, y] \in Z(R) \text{ for all } x, y \in L$$

In this expression, using $z, d(z) \in Z(R)$ and Equation (7), we obtain

$$d(z)(x \circ F(y)) \in Z(R) \text{ for all } x, y \in L$$

By using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we have

$$x \circ F(y) \in Z(R) \text{ for all } x, y \in L$$

Replacing y by x in above expression, we get $x \circ F(x) \in Z(R)$ for all $x, y \in L$. From the Lemma 3.4 we get $L \subseteq Z(R)$.

iii) By assumption,

$$[F(x), F(y)] - x \circ y \in Z(R) \text{ for all } x, y \in L. \tag{8}$$

Since $d(Z(L)) \neq (0)$, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing x by $2xz$ in Equation (8) and using $char(R) \neq 2$, we obtain

$$[d(z)x, F(y)] + [zF(x), F(y)] - xz \circ y \in Z(R) \text{ for all } x, y \in L$$

In this expression, using $z, d(z) \in Z(R)$ and Equation (8), we get

$$d(z)[x, F(y)] \in Z(R) \text{ for all } x, y \in L$$

By using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we have

$$[x, F(y)] \in Z(R) \text{ for all } x, y \in L$$

Replacing y by x in above expression, we have $[x, F(x)] \in Z(R)$ for all $x, y \in L$. From the Lemma 3.3 we get $L \subseteq Z(R)$.

iv) By assumption,

$$F(x) \circ F(y) - [x, y] \in Z(R) \text{ for all } x, y \in L. \tag{9}$$

Since $d(Z(L)) \neq (0)$, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing x by $2xz$ in Equation (9) and using $char(R) \neq 2$, we obtain

$$(d(z)x \circ F(y)) + (zF(x) \circ F(y)) - x[z, y] - [x, y]z \in Z(R) \text{ for all } x, y \in L$$

In this expression, using $z, d(z) \in Z(R)$ and Equation (9), we get

$$d(z)(x \circ F(y)) \in Z(R) \text{ for all } x, y \in L$$

By using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we have

$$x \circ F(y) \in Z(R) \text{ for all } x, y \in L$$

Replacing y by x in above expression, we have $x \circ F(x) \in Z(R)$ for all $x, y \in L$. From the Lemma 3.4 we get $L \subseteq Z(R)$. □

Theorem 3.7. *Let R be a prime ring with $char(R) \neq 2$, $0 \neq F : R \longrightarrow R$ be a right generalized reverse derivation with associated right reverse derivation d and L be a nonzero square closed Lie ideal of R such that $d(Z(L)) \neq (0)$. If one of the following conditions is satisfy, then $L \subseteq Z(R)$.*

i) $[F(x), F(y)] - F[x, y] \in Z(R)$ for all $x, y \in L$.
ii) $F(x) \circ F(y) - F(x \circ y) \in Z(R)$ for all $x, y \in L$.
iii) $F[x, y] - [F(x), y] \in Z(R)$ for all $x, y \in L$.

Proof. *i*) For all $x, y \in L$, let

$$[F(x), F(y)] - F[x, y] \in Z(R) \tag{10}$$

By hypothesis, $d(Z(L)) \neq (0)$. Then, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing x by $2xz$ in Equation (10) and using $char(R) \neq 2$, we get

$$[d(z)x, F(y)] + [zF(x), F(y)] - F([x,y]z) \in Z(R) \text{ for all } x, y \in L$$

By using the fact that $z, d(z) \in Z(R)$ and Equation (10), we get

$$d(z)([x, F(y)] - [x,y]) \in Z(R) \text{ for all } x, y \in L$$

In this expression, using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we have

$$[x, F(y)] - [x,y] \in Z(R) \text{ for all } x, y \in L$$

Replacing x by $2d(z)y$ in above expression and using $char(R) \neq 2$, we obtain

$$d(z)[y, F(y)] \in Z(R) \text{ for all } x, y \in L$$

Again, using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we get

$$[y, F(y)] \in Z(R) \text{ for all } x, y \in L$$

From the Lemma 3.3 we obtain $L \subseteq Z(R)$.

$ii)$ For all $x, y \in L$, let

$$F(x) \circ F(y) - F(x \circ y) \in Z(R) \tag{11}$$

By hypothesis, $d(Z(L)) \neq (0)$. Then, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing x by $2xz$ in Equation (11) and using $char(R) \neq 2$, we get

$$(d(z)x + zF(x)) \circ F(y) - F((x \circ y)z + x[z,y]) \in Z(R) \text{ for all } x, y \in L$$

By using the fact that $z, d(z) \in Z(R)$ and Equation (11), we have

$$d(z)(x \circ F(y) - x \circ y) \in Z(R) \text{ for all } x, y \in L$$

In this expression, using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we obtain

$$x \circ F(y) - x \circ y \in Z(R) \text{ for all } x, y \in L$$

Replacing y by $2yz$ in above expression and using $char(R) \neq 2$, we get

$$d(z)(x \circ y) \in Z(R) \text{ for all } x, y \in L$$

Again, using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we obtain

$$x \circ y \in Z(R) \text{ for all } x, y \in L$$

From the Lemma 3.2 we get $L \subseteq Z(R)$.

iii) For all $x, y \in L$, let

$$F[x, y] - [F(x), y] \in Z(R) \tag{12}$$

By hypothesis, $d(Z(L)) \neq (0)$. Then, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing y by $2yz$ in Equation (12) and using $char(R) \neq 2$, we get

$$F([x, y]z + y[x, z]) - [F(x), y]z - y[F(x), z] \in Z(R) \text{ for all } x, y \in L$$

By using the fact that $z, d(z) \in Z(R)$ and Equation (12), we have

$$d(z)[x, y] \in Z(R) \text{ for all } x, y \in L$$

In this expression, using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we obtain

$$[x, y] \in Z(R) \text{ for all } x, y \in L$$

From the Lemma 3.1 we get $L \subseteq Z(R)$. □

Theorem 3.8. *Let R be a prime ring with $char(R) \neq 2$, $0 \neq F : R \longrightarrow R$ be a right generalized reverse derivation with associated right reverse derivation d and L be a nonzero square closed Lie ideal of R such that $d(Z(L)) \neq (0)$. If one of the following conditions is satisfy, then $L \subseteq Z(R)$.*

i) $F[x, y] + [F(x), y] - [F(x), F(y)] \in Z(R)$ for all $x, y \in L$.
ii) $F[x, y] - F(x) \circ y - [d(y), x] \in Z(R)$ for all $x, y \in L$.

Proof. *i*) By assumption,

$$F[x, y] + [F(x), y] - [F(x), F(y)] \in Z(R) \text{ for all } x, y \in L. \tag{13}$$

Since $d(Z(L)) \neq (0)$, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing y by $2yz$ in Equation (13) and using $char(R) \neq 2$, we get

$$F[x, yz] + [F(x), yz] - [F(x), F(yz)] \in Z(R) \text{ for all } x, y \in L$$

In this expression, using $z, d(z) \in Z(R)$ and Equation (13), for all $x, y \in L$ we obtain

$$d(z)[x,y] + zF[x,y] + [F(x), y]z - d(z)[F(x), y] - z[F(x), F(y)] \in Z(R)$$

and from this

$$d(z)([x,y] - [F(x), y]) \in Z(R) \text{ for all } x, y \in L$$

By using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we have

$$[x,y] - [F(x), y] \in Z(R) \text{ for all } x, y \in L$$

Replacing y by $2d(z)x$ in above expression and using $z, d(z) \in Z(R)$ and $char(R) \neq 2$, we have

$$d(z)[F(x), x] \in Z(R) \text{ for all } x, y \in L$$

Again, using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we get

$$[F(x), x] \in Z(R) \text{ for all } x, y \in L$$

From the Lemma 3.3 we get $L \subseteq Z(R)$.

$ii)$ By assumption,

$$F[x,y] - F(x) \circ y - [d(y), x] \in Z(R) \text{ for all } x, y \in L. \tag{14}$$

Since $d(Z(L)) \neq (0)$, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing y by $2yz$ in Equation (14) and using $char(R) \neq 2$, we have

$$F[x, yz] - F(x) \circ yz - [d(yz), x] \in Z(R) \text{ for all } x, y \in L$$

In this expression, using $z, d(z) \in Z(R)$ and Equation (13), for all $x, y \in L$ we get

$$d(z)[x,y] + zF[x,y] - (F(x) \circ y)z - d(z)[y,x] - z[d(y), x] \in Z(R)$$

and from this

$$2d(z)[x,y] \in Z(R) \text{ for all } x, y \in L$$

By using $char(R) \neq 2$, $0 \neq d(z) \in Z(R)$ and Remark 2.1, we obtain

$$[x,y] \in Z(R) \text{ for all } x, y \in L$$

From the Lemma 3.1 we get $L \subseteq Z(R)$. □

Theorem 3.9. *Let R be a prime ring with $char(R) \neq 2$, $0 \neq F, G : R \longrightarrow R$ are right generalized reverse derivations with associated right reverse derivation d and* *g respectively, L be a nonzero square closed Lie ideal of R such that $d(Z(L)) \neq (0)$ and $g(Z(L)) \neq (0)$. If one of the following conditions is satisfy, then $L \subseteq Z(R)$.*

i) $[F(x), G(y)] - [x, y] \in Z(R)$ *for all* $x, y \in L$.

ii) $F[x, y] - [y, G(x)] \in Z(R)$ *for all* $x, y \in L$.

iii) $F(x \circ y) - y \circ G(x) \in Z(R)$ *for all* $x, y \in L$.

Proof. $i)$ For all $x, y \in L$, let

$$[F(x), G(y)] - [x, y] \in Z(R) \tag{15}$$

By hypothesis, $d(Z(L)) \neq (0)$. Then, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing x by $2xz$ in Equation (15) and using $char(R) \neq 2$, we get

$$[d(z)x, G(y)] + [zF(x), G(y)] - x[z, y] - [x, y]z \in Z(R) \text{ for all } x, y \in L$$

By using the fact that $z, d(z) \in Z(R)$ and Equation (15), we obtain

$$d(z)[x, G(y)] \in Z(R) \text{ for all } x, y \in L$$

In this expression, using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we have

$$[x, G(y)] \in Z(R) \text{ for all } x, y \in L$$

Replacing y by x in above expression, we have $[x, G(x)] \in Z(R)$ for all $x, y \in L$. From the Lemma 3.3 we get $L \subseteq Z(R)$.

$ii)$ For all $x, y \in L$, let

$$F[x, y] - [y, G(x)] \in Z(R) \tag{16}$$

By hypothesis, $d(Z(L)) \neq (0)$. Then, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing y by $2yz$ in Equation (16) and using $char(R) \neq 2$, we get

$$F([x, y]z + y[x, z]) - y[z, G(x)] - [y, G(x)]z \in Z(R) \text{ for all } x, y \in L$$

By using the fact that $z, d(z) \in Z(R)$ and Equation (16), we get

$$d(z)[x, y] \in Z(R) \text{ for all } x, y \in L$$

In this expression, using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we have

$$[x, y] \in Z(R) \text{ for all } x, y \in L$$

From the Lemma 3.1 we obtain $L \subseteq Z(R)$.

iii) For all $x, y \in L$, let

$$F(x \circ y) - y \circ G(x) \in Z(R) \tag{17}$$

By hypothesis, $d(Z(L)) \neq (0)$. Then, we choose fixed element $0 \neq z \in Z(L)$ which $d(z) \neq 0$. Also, $z, d(z) \in Z(R)$ from the Lemma 2.4. Replacing y by $2yz$ in Equation (17) and using $char(R) \neq 2$, we get

$$F((x \circ y)z - y[x,z]) - (y \circ G(x))z - y[z, G(x)] \in Z(R) \text{ for all } x, y \in L$$

By using the fact that $z, d(z) \in Z(R)$ and Equation (17), we obtain

$$d(z)(x \circ y) \in Z(R) \text{ for all } x, y \in L$$

In this expression, using $0 \neq d(z) \in Z(R)$ and Remark 2.1, we have

$$x \circ y \in Z(R) \text{ for all } x, y \in L$$

From the Lemma 3.2 we get $L \subseteq Z(R)$. □

Example 3.10. *Let* $R = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x, y \in \mathbb{Z} \right\}$ *and* $L = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z} \right\}$, *where* $\mathbb{Z}$ *is the set of all integers. We define the mappings* $F, d : R \to R$ *as following:*

$$F\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = \begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix} \quad , \quad d\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$$

It is easy to show that, L is square closed Lie ideal of ring R, d is right reverse derivation and F is right generalized reverse derivation with associated d. Moreover, since $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in Z(L)$ *and* $d\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *for any* $0 \neq a \in \mathbb{Z}$, *condition* $d(Z(L)) \neq (0)$ *is satisfied.*

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