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Persistence and boundedness in a logistics chemotaxis system including one-species, two-chemicals, and singularity

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Abstract: Chemotaxis systems describe how biological populations respond to chemical gradients, yet singular sensitivities often lead to blow-up phenomena that challenge mathematical stability. This study addresses a parabolic–elliptic–elliptic chemotaxis model involving one species and two interacting chemicals with a logistic growth term under homogeneous Neumann boundary conditions. Motivated by the need to identify mechanisms preventing blow-up, we establish rigorous conditions ensuring persistence and global boundedness of classical solutions. Specifically, it is proven that when the logistic damping effect is sufficiently strong, the system admits a unique global solution that remains positive and uniformly bounded over time. These findings advance previous theoretical results by clarifying how logistic regulation stabilizes chemotactic aggregation even under singular sensitivity.

Tek Tür, İki Kimyasal ve Tekillik İçeren Lojistik Bir Kemotaksi Sisteminde Süreklilik ve Sınırlılık

Anahtar Kelimeler
Kemotaksi,
Çok türlü sistem,
Tekil duyarlılık,
Küresel sınırlılık,
Süreklilik

Öz: Kemotaksi sistemleri, biyolojik popülasyonların kimyasal gradyanlara tepkisini tanımlar; ancak tekil duyarlılıklar, matematiksel kararlılığı zorlaştıran patlama (blow-up) olgularına yol açabilmektedir. Bu çalışma, tek tür ve iki etkileşen kimyasal madde içeren, lojistik büyüme terimiyle desteklenmiş parabolik–eliptik–eliptik tipte bir kemotaksi modelini homojen Neumann sınır koşulları altında ele almaktadır. Patlamayı önleyen mekanizmaların belirlenmesi gerekliliğinden hareketle, klasik çözümlerin sürekliliği ve küresel sınırlılığı için yeterli koşullar ortaya konmuştur. Özellikle, lojistik sönüm etkisi yeterince güçlü olduğunda sistemin zamana bağlı olarak pozitifliğini koruyan ve küresel olarak sınırlı bir çözüme sahip olduğu kanıtlanmıştır. Bu bulgular, lojistik düzenlemenin tekil duyarlılığa rağmen kemotaktik birikimi nasıl dengelediğini ortaya koyarak önceki teorik sonuçları geliştirmektedir.

1. Introduction

Chemotaxis models describe the directed movement of motile organisms in response to specific chemical signals. The pioneering framework for these mathematical models was introduced by Keller and Segel in the late 1970s [1, 2]. Since then, this phenomenon has been recognized as a key mechanism in various biological circumstances, such as tumor growth, immune cell migration, embryo development, and population dynamics. In these models, the density of cells or organisms is coupled with the concentration of chemical substances, where organisms tend to migrate towards regions of higher concentration or away from them. Mathematically, Keller-Segel type systems are typically formulated using partial differential equations (PDEs), which describe the spatiotemporal evolution of cell density and chemical concentration through the interaction of diffusion and chemotactic sensitivity. These equations provide a robust framework for understanding the complex dynamics of chemotactic systems. For a comprehensive overview of the field, readers are referred to the surveys in [3-5].

In the specific context of singular sensitivity, which is the focus of this work, the system exhibits critical challenges regarding stability. Motivated by these biological and mathematical considerations, this research paper studies the subsequent chemotaxis model with one-mobile species, two-chemicals, and logistic kinetics:

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$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (\frac{u}{v}\nabla v) + \mathrm{a}u - bu^2, \\ 0 = \Delta v - v + z, \\ 0 = \Delta z - z + u, \end{cases} \tag{1}$$

with $x \in S \subset \mathbb{R}^n$ $(n \geq 2)$ being a smooth domain and $a, b, \chi > 0$. Moreover, the initial $u_0(x) := u(0, x; u_0)$ fulfilling,

$$u_0 \in C^0(\bar{S}), \quad u_0 \geq 0, \quad \int_S u_0 > 0, \tag{2}$$

and

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \quad x \in \partial S. \tag{3}$$

A central challenge in the study of chemotaxis models Eq.(1) lies in determining whether solutions undergo finite-time blow-up or persist for all time. In cases where global existence can be ensured, a natural question arises regarding the boundedness of these solutions. Should boundedness hold, it then becomes essential to explore their asymptotic behavior, including properties such as persistence, stability, extinction, and coexistence. In this context, it is both relevant and necessary to summarize the established results in the literature while also identifying the open problems that remain for the system Eq.(1).

Case 1: We first consider system Eq.(1) under the assumptions $z(t, x) = 0$, and $a = b = 0$, which reduces to the following form:

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot \left(\frac{u}{v}\nabla v\right), \\ 0 = \Delta v - v + u, \end{cases} \tag{4}$$

where S denotes a ball. The first investigation of the global existence and boundedness of positive solutions for this system was carried out by Nagai and Senba [6].Their results showed that classical, radially symmetric, and positive solutions are guaranteed if $\chi > 0$ when $n = 2$, or if $\chi < \frac{2}{n-2}$ in the case $n \geq 3$. In contrast, they also established the emergence of radial blow-up solutions whenever $\chi > \frac{2n}{n-2}$ with $n \geq 3$. Subsequent progress was made by Fujie, Winkler, and Yokota [7], who showed that system Eq.(4) admits a global bounded classical solution under the sharper condition $\chi < \frac{2}{n}$ for $n \geq 2$. This result was later refined by Fujie and Senba [8], who demonstrated that, in two dimensions, global existence and boundedness of classical positive solutions cannot be guaranteed for any $\chi > 0$ when $n = 2$. More recently, an improvement of this two-dimensional result was provided in [9], where it was established that equation Eq.3 possesses a uniformly bounded global classical solution whenever

$$\chi < \frac{2}{n} + \frac{2n-1}{2n^3} \cdot \sqrt{\frac{n}{2n+2}} \quad \text{with} \quad n \geq 3.$$

For additional related results and further developments, we refer the reader to [10, 11].

Case 2: We first consider system Eq.(1) under the assumptions $z(t, x) = 0$, $a, b > 0$ and $k > 0$, which reduces to the following form:

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (\frac{u}{v^k}\nabla v) + au - bu^2, \\ 0 = \Delta v - v + u, \end{cases} \tag{5}$$

when $n = 2$ and $k = 1$, Fujie, Winkler, and Yokota [12] demonstrated that system Eq.(5) does not exhibit finite-time blow-up. They further proved that every global positive solution remains uniformly bounded, provided that $a > \frac{\chi^2}{4}$ if $0 < \chi \leq 2$, and $a > \chi - 1$ if $\chi > 2$. Building upon this result, Kurt and Shen [13] extended the analysis to higher dimensions, establishing the global existence and boundedness of classical positive solutions of Eq.(5) under suitable parameter relations involving a, χ and the initial data u_0. Moreover, under the same assumptions, further studies in Eq.(5) investigated the dynamical behavior of positive solutions, including uniform boundedness, persistence, the existence of entire solutions, and periodic solutions. On the other hand, for the case $k \in (0,1)$, Zhao [14] proved that in two dimensions, system Eq.(5) admits globally bounded classical solutions whenever the parameter b is sufficiently large. Later, Le [15] improved this result by removing the largeness condition on b, showing instead that a weaker sub-logistic source of the form

$$-\frac{\mu u^2}{\ln^\gamma(u+e)} \quad \text{with} \ \gamma \in (0,1),$$

is already sufficient to rule out blow-up in two-dimensional settings. Then, Kurt [16] demonstrated that system Eq.(5) admits a unique global classical solution provided that the parameter b is sufficiently large. Furthermore, the global boundedness of solutions was established under this condition.

$$k < \frac{1}{2} + \frac{1}{n} \quad \text{with} \quad n \geq 2.$$

Recently, Le and Kurt in [17] proved the boundedness of classical solutions under a slighter conditions, that is,

$b > \chi^{\frac{1}{1-k}} 2^{\frac{k}{1-k}} + 4^{\frac{2-k}{1-k}} (\chi k)^{\frac{1}{1-k}}$ if $n = 3$;

$b > \chi^{\frac{1}{1-k}} \left(\frac{n}{2}\right)^{\frac{k}{1-k}} + 2(\chi k)^{\frac{1}{1-k}} \left(\frac{2n}{n-2}\right)^{\frac{1}{2}} n^{\frac{1}{1-k}}$ if $n \geq 4$.

Case 3: Beyond single-species models, significant attention has been directed toward two-species singular chemotaxis systems involving Lotka-Volterra competitive kinetics. We refer to the following system where two biological species, u and v, compete for resources while interacting with a single chemical signal w:

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot \left(\frac{u}{w}\nabla w\right) + u(a_1 - b_1 u - c_1 v), \\ v_t = \Delta v - \chi_2 \nabla \cdot \left(\frac{v}{w}\nabla w\right) + v(a_2 - b_2 v - c_2 u), \\ 0 = \Delta w - w + u + v, \end{cases}$$

where χ_1, χ_2 represent the chemotactic sensitivities. In this context, Kurt and Shen [19-20] provided a comprehensive analysis regarding the global dynamics of such systems. They rigorously established the global existence, boundedness, and persistence of solutions under explicit assumptions concerning the sensitivity parameters. Expanding on these findings, the same authors investigated the asymptotic stability of the system in [20]. Their analysis demonstrated that, subject to certain conditions, any positive constant solution of the system is globally stable. These studies highlight the complexity introduced by competitive interactions in singular chemotaxis models, paving the way for further multi-species investigations.

Despite the extensive literature summarized above regarding sub-systems and simplified cases, the full system Eq. (1) involving singular sensitivity, two interacting chemicals, and logistic growth remains less explored. This paper aims to bridge this gap by establishing rigorous conditions for the global existence and asymptotic behavior of the solutions. Specifically, we prove that if the logistic damping effect is sufficiently strong, the system avoids blow-up and admits a unique, globally bounded, and persistent classical solution. These findings contribute to a deeper theoretical understanding of how logistic regulation stabilizes chemotactic aggregation in complex, multi-component environments. The readers are directed to see the research papers [18- 26] for additional results on large time behaviors of classical bounded solutions in one and multi species scenarios.

2. Material and Method

The local existence and uniqueness of solutions to Eq.(1), subject to initial data u_0 satisfying Eq.(2) are stated below. As the underlying computations closely follow the arguments presented in [7, Proposition 3.1], we refrain from providing the detailed proof here.

Lemma 2.1: For any initial function u_0 satisfying Eq.(2), there exists $T_{\max}(u_0) \in (0, \infty]$ such that system Eq.(1) admits a unique classical solution (u, v, z) on $(0, T_{\max})$ with the initial condition $u(0, x; u_0) = u_0(x)$. The solution satisfies

$$u \in C((0, T_{\max}) \times \bar{S}) \cap C^{2,1}((0, T_{\max}) \times \bar{S}) \quad \text{and}$$

$$v, z \in C^{2,0}((0, T_{\max}) \times \bar{S})).$$

Moreover, if $T_{\max} < \infty$, then blow-up occurs in the sense that either

$$\limsup_{t \nearrow T_{\max}} \|u(t,\cdot)\|_{L^\infty(\bar{S})} = \infty \quad \text{or} \quad \liminf_{t \nearrow T_{\max}} \inf_{x \in S} v(t, x) = 0.$$

The following estimate is direct result of [16, Proposition 3.1].

Lemma 2.2: If $\rho \geq 3$ and $2 < k < 2\rho - 2$, then

$$\int_S \frac{|\nabla v|^{2\rho}}{v^k} \leq \left(\frac{4(\rho-1)^2}{2\rho - k - 2}\right)^{\rho} \cdot \left(\frac{2(k-1)}{k-2}\right)^{\frac{\rho}{2}} \int_S \frac{z^\rho}{v^{k-\rho}} + C_{\rho,k} \int_S v^{2\rho - k},$$

for all $t \in (0, T_{\max})$.

Known Results: Note that in our previous work, we have established the global existence if b is sufficiently large. Namely, by Theorem 3.2 in [27], we have that $T_{\max} = \infty$.

Idea of Persistence of classical solutions: We first let recall the lower bound of $u(t, x)$. Note that, by the second and third equations of Eq.(1) as well as [7, Lemma 2.1], we get that

$$v(t, x) \geq \delta_0 \int_S z(t, x)dx = \delta_0 \int_S u(t, x)dx, \tag{6}$$

for all $(t, x) \in (0, T_{\max}) \times S$. Note also by Hölder inequality that

$$\int_S u \geq |S|^{\frac{\rho+1}{\rho}} \left(\int_S u^{-\rho}\right)^{-\frac{1}{\rho}}.$$

Hence,

$$v \geq \delta_0 |S|^{\frac{\rho+1}{\rho}} \left(\int_S u^{-\rho}\right)^{-\frac{1}{\rho}}, \tag{7}$$

where $\delta_0 > 0$. Hence, to prove the persistency of global solutions, it is enough to prove that

$$\int_S u^{-\rho} \geq m^* \quad \forall t \in (0, T_{\max}), \quad \forall t > 0. \tag{8}$$

Idea of Boundedness of classical solutions: After having the positivity of classical solutions for $t > 0$, we apply the classical methods to obtain the global bounds for solutions.

3. Results

Persistence

The idea of the following estimate comes from the argument of [13, Lemma 3.2].

Lemma 3.1: For every $\rho, \chi, \beta > 0$,

$$\chi \int_S \frac{u^{-\rho-1}}{v} \nabla u \cdot \nabla v \leq \int_S u^{-\rho-2} |\nabla u|^2 + \frac{\beta\mu}{\rho} \int_S u^{-\rho} + \left(\frac{(\chi-\beta)^2}{4} - \frac{\beta}{\rho} \right) \int_S u^{-\rho} \frac{|\nabla v|^2}{v^2}.$$

Proof. Note that, for every $\beta > 0$,

$$\chi \int_S \frac{u^{-\rho-1}}{v} \nabla u \cdot \nabla v = (\chi - \beta) \int_S \frac{u^{-\rho-1}}{v} \nabla u \cdot \nabla v + \beta \int_S \frac{u^{-\rho-1}}{v} \nabla u \cdot \nabla v.$$

By Young's inequality, we have

$$(\chi - \beta) \int_S \frac{u^{-\rho-1}}{v} \nabla u \cdot \nabla v \leq \int_S u^{-\rho-2} |\nabla u|^2 + \frac{(\chi-\beta)^2}{4} \int_S u^{-\rho} \frac{|\nabla v|^2}{v^2}. \tag{9}$$

Integrating by parts over S, we get

$$\beta \int_S \frac{u^{-\rho-1}}{v} \nabla u \cdot \nabla v \leq \frac{\beta\mu}{\rho} \int_S u^{-\rho} - \frac{\beta}{\rho} \int_S u^{-\rho} \frac{|\nabla v|^2}{v^2} - \frac{\beta}{\rho} \int_S \frac{u^{-\rho} z}{v}. \tag{10}$$

Therefore Eq.(9) and Eq.(10) completes the proof.

Lemma 3.2: Assume that

$$a > \begin{cases} \frac{\chi^2}{4}, & 0 < \chi \leq 2 \\ \chi - 1, & \chi > 2 \end{cases} \tag{11}$$

holds. Then

$$\int_S u^{-\rho} \leq \tilde{C}, \quad \forall t > \tau > 0, \tag{12}$$

for some $\rho > 0$ and $\tilde{C} > 0$.

Proof. Multiplying the equation Eq.(1) $_1$ by $u^{-\rho-1}$ and integrating over S and by Lemma 3.1, we get

$$\frac{1}{\rho} \cdot \frac{d}{dt} \int_S u^{-\rho} = -(\rho + 1) \int_S u^{-\rho-2} |\nabla u|^2 + (\rho + 1)\chi \int_S \frac{u^{-\rho-1}}{v} \nabla u \cdot \nabla v - a \int_S u^{-\rho} + b \int_S u^{-\rho+1}$$

$$\leq (\rho + 1) \left(\frac{(\chi-\beta)^2}{4} - \frac{\beta}{\rho} \right) \int_S u^{-\rho} \frac{|\nabla v|^2}{v^2} + \left(\frac{(\rho+1)\beta\mu}{\rho} - a \right) \int_S u^{-\rho} + b \int_S u^{-\rho+1}.$$

Let define

$$\beta_{\pm} = \chi - 2 \pm 2\sqrt{b + 1 - \chi}. \tag{13}$$

If $\beta \in (\beta_-, \beta_+)$, then one obtain $\frac{(\chi-\beta)^2}{4} - \frac{\beta}{\rho} = 0$ and $\frac{(\rho+1)\beta\mu}{\rho} - a < 0$. Hence there exists $\beta^* = \beta^*(a, \rho, \chi) > 0$ such that

$$\frac{1}{\rho} \cdot \frac{d}{dt} \int_S u^{-\rho} \leq -\beta^* \int_S u^{-\rho} + b \int_S u^{-\rho+1} \leq -\frac{\beta^*}{2} \int_S u^{-\rho} + C.$$

Setting $y(t) = \int_S u^{-\rho}(t,x)dx$ yields

$$y'(t) \leq -\frac{\beta^* \rho}{2} y(t) + Cp,$$

which implies

$$y(t) \leq \max\left\{ \int_S u^{-\rho}(x,\tau)dx, \frac{2Cp}{\beta^* \rho} \right\} := \tilde{C},$$

for all $t > \tau > 0$.

Teorem 3.3 (Persistence of global classical solutions). There is $\delta > 0$ such that

$$v \geq \delta \quad \forall t > 0.$$

Proof. In view of Eq.(7) and Lemma 3.2, we have for $\forall t > 0$,

$$v \geq \delta_0 |S|^{\frac{\rho+1}{\rho}} \left(\int_S u^{-\rho} \right)^{-\frac{1}{\rho}} \geq \delta_0 |S|^{\frac{\rho+1}{\rho}} (\tilde{C})^{-\frac{1}{\rho}} := \delta,$$

The proof is completed.

Boundedness

Lemma 3.4. For all $\rho \geq 1$, we have

$$\int_S v^{\rho} \leq \int_S z^{\rho} \quad \text{and} \quad \int_S z^{\rho} \leq \int_S u^{\rho} \quad \forall t > 0.$$

Proof. It follows from [27, Lemma 2.4].

The following lemma was first established in [27, Lemma 3.1]. Here we extend it for all $0 < t \leq \infty$.

Lemma 3.5 (Global L^ρ-boundedness for $\rho \geq 2$). Assume that u_0 satisfies Eq.(2). Then there is $b > b^*$ such that

$$\int_S u^\rho \leq C \quad \forall t > 0.$$

where $\rho \geq 2$ and

$$b^* := b^*_{\rho,\chi,\delta} := \frac{(\rho-1)^2}{\rho+1}\frac{\chi^2}{4}\left\{\rho + \frac{1}{\delta^{\rho+1}}\left(\left(\frac{32p^5}{(\rho-1)^3}\right)^{\frac{\rho+1}{2}} + C_p\right)\right\}.$$

Proof. First of all, testing the first equation in Eq.(1) by $u^{\rho-1}$ with $\rho \geq 2$ gives that

$$\frac{1}{\rho}\cdot\frac{d}{dt}\int_S u^\rho = -(\rho-1)\int_S u^{\rho-2}|\nabla u|^2 + \chi(\rho-1)\int_S \frac{u^{\rho-1}}{v^k}\nabla u\cdot\nabla v + a\int_S u^\rho - b\int_S u^{\rho+1},$$

for all $t \in (0, T_{\max})$. Applying Young's inequality yields that

$$(\rho-1)\chi\int_S \frac{u^{\rho-1}}{v}\nabla u\cdot\nabla v \leq (\rho-1)\int_S u^{\rho-2}|\nabla u|^2 + \frac{(\rho-1)^2\chi^2}{4}\int_S u^\rho\frac{|\nabla v|^2}{v^2}$$

$$\leq (\rho-1)\int_S u^{\rho-2}|\nabla u|^2 + \frac{\rho(\rho-1)^2\chi^2}{4(\rho+1)}\int_S u^{\rho+1} + \frac{(\rho-1)^2\chi^2}{4(\rho+1)}\int_S \frac{|\nabla v|^{2p+2}}{v^{2p+2}} \quad (14)$$

Note also that by Theorem 3.3, Lemma 3.4, and Young's inequality, we have

$$\frac{(\rho-1)^2\chi^2}{4(\rho+1)}\int_S \frac{|\nabla v|^{2\rho+2}}{v^{2\rho+2}} \leq \frac{(\rho-1)^2\chi^2}{4(\rho+1)}\frac{1}{\delta^{\rho+1}}\int_S \frac{|\nabla v|^{2p+2}}{v^{\rho+1}}$$

$$\leq \frac{(\rho-1)^2\chi^2}{4(\rho+1)}\frac{1}{\delta^{\rho+1}}\left\{\left(\frac{4\rho^2}{\rho-1}\right)^{\rho+1}\cdot\left(\frac{2\rho}{\rho-1}\right)^{\frac{\rho+1}{2}}\int_S z^{\rho+1} + C_\rho\int_S v^{\rho+1}\right\}$$

$$\leq \frac{(\rho-1)^2\chi^2}{4(\rho+1)}\frac{1}{\delta^{\rho+1}}\left\{\left(\frac{4\rho^2}{\rho-1}\right)^{\rho+1}\cdot\left(\frac{2\rho}{\rho-1}\right)^{\frac{\rho+1}{2}} + C_\rho\right\}\int_S z^{\rho+1}$$

$$\leq \frac{(\rho-1)^2\chi^2}{4(\rho+1)}\frac{1}{\delta^{\rho+1}}\left\{\left(\frac{32p^5}{(\rho-1)^3}\right)^{\frac{\rho+1}{2}} + C_p\right\}\int_S u^{\rho+1} \quad (15)$$

Hence, by Eq.(14) and Eq.(15), we get that

$$\chi(\rho-1)\int_S \frac{u^{\rho-1}}{v}\nabla u\cdot\nabla v \leq (\rho-1)\int_S u^{\rho-2}|\nabla u|^2 + b^*_{\rho,\chi,\delta}\int_S u^{\rho+1},$$

where

$$b^*_{\rho,\chi,\delta} := \frac{\rho(\rho-1)^2\chi^2}{4(\rho+1)} + \frac{(\rho-1)^2\chi^2}{4(\rho+1)}\frac{1}{\delta^{\rho+1}}\left\{\left(\frac{32\rho^5}{(\rho-1)^3}\right)^{\frac{\rho+1}{2}} + C_p\right\}.$$

It then follows that

$$\frac{1}{\rho}\cdot\frac{d}{dt}\int_S u^\rho \leq a\int_S u^\rho - (b - b^*_{\rho,\chi,\delta})\int_S u^{\rho+1}$$

$$\leq -\int_S u^\rho + C \quad \forall t > 0,$$

due to b being sufficiently large. This entails that $\int_S u^\rho \leq C$ for all $t > 0$.

Lemma 3.6 (Global L^ρ-boundedness for $\rho \in \max\{\frac{n}{2}, 2\}$). Assume that u_0 satisfies Eq.(2) and Eq.(11) hold. Then for given $\rho \in (\frac{n}{2}, n)$, there is $b > \tilde{b}^*(n) > b^*_{\rho,\chi,\delta}$ such that

$$\int_S u^\rho \leq C \quad \forall t > 0.$$

Proof. Note that for any given $\rho \in (\frac{n}{2}, n)$, one can find $\tilde{b}^*(n) > 0$ fulfilling $b > \tilde{b}^*(n)$ since b is chosen sufficiently large. Hence, the proof follows the similar arguments of Lemma 4.

Now we give our main result.

Theorem 3.7 (Global boundedness of classical solutions). Assume that b is sufficiently large and Eq.(11) holds. Then for any initial data u_0 satisfying Eq.(2), we have for $C > 0$,

$$\|u(t,\cdot)\|_{L^\infty(\bar{S})} \leq C, \qquad \forall t > 0.$$

Proof. Recall that if $\rho > \frac{n}{2}$, then L^{ρ}-boundedness of solutions in time implies the L^{∞}-boundedness in time of solutions. Therefore, this together with Theorem 3, and Lemma 4, as well as the similar arguments in the proof of [12, 13, 27, 28], we conclude that

$$\|u(t,\cdot)\|_{L^{\infty}(\bar{S})} \leq C \ , \qquad \forall t > 0.$$

for some $C > 0$. To prevent the repetition, we omit the proof.

4. Conclusion

In this study, the persistence and global boundedness of classical solutions were rigorously examined within a logistic chemotaxis system involving one mobile species and two interacting chemical substances under singular sensitivity. The analytical framework built upon parabolic–elliptic–elliptic equations demonstrated that, when the logistic damping coefficient is sufficiently large and appropriate parameter constraints are satisfied, the system guarantees a unique, positive, and globally bounded classical solution. This finding confirms that the population density not only avoids blow-up but also remains persistent over time, ensuring stability in the long-term dynamics of the system.

The obtained results extend and refine several previously established criteria in the literature, particularly those by Fujie, Winkler, and Kurt, by relaxing the restrictive assumptions on the parameters governing singular sensitivity. Thus, the present work contributes to the theoretical understanding of how logistic mechanisms regulate chemotactic aggregation, offering deeper insight into the interplay between singular sensitivity and nonlinear growth in reaction–diffusion systems.

However, despite these theoretical advances, several open problems remain. Future research should aim to generalize the current model in multiple directions. One promising avenue is to incorporate nonlinear diffusion mechanisms, which would better represent heterogeneous environments where diffusion rates depend on population density. Another is to include time delays and stochastic perturbations, reflecting more realistic biological and chemical processes. Moreover, extending the analysis to multi-species and higher-dimensional systems could shed light on coexistence phenomena and competitive interactions. Finally, numerical simulations and stability analyses would serve as valuable tools to verify and visualize the theoretical predictions derived in this study.

In conclusion, this work establishes a comprehensive theoretical foundation for understanding the bounded and persistent behavior of chemotactic populations under logistic regulation, and it provides a robust basis for future explorations in both mathematical theory and applied biological modeling.

Declaration of Ethical Code

In this study, we undertake that all the rules required to be followed within the scope of the "Higher Education Institutions Scientific Research and Publication Ethics Directive" are complied with, and that none of the actions stated under the heading "Actions Against Scientific Research and Publication Ethics" are not carried out.

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