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New Asymptotic and Functional Properties for Detecting the Failure of the Fixed Point Property: A Unified Framework Beyond Classical c_0 and N_1 Structures

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Abstract: We refine analytic and geometric tools for detecting the failure of the Fixed Point Property (FPP) in Banach spaces. Classical detectors include asymptotically isometric copies of c_0 (AI-c_0), due to Dowling-Lennard-Turett, and the analytic N_1 property of Álvaro–Cembranos–Mendoza. Importantly, AI-c_0 *implies* N_1, while the converse fails; thus there exist spaces with N_1 but without any asymptotically isometric copy of c_0. Building on these, we introduce sup-dominated and functional-sandwiched frameworks (FSP, sFSP) that still detect FPP failure in classes where both a geometric AI-c_0 copy is absent and the classical N_1 estimate is too weak. We provide implications, examples, counterexamples, and Hahn-Banach based constructions, together with figures and comparative tables.

Sabit Nokta Özelliğinin Sağlanamamasını Belirlemeye Yönelik Asimptotik ve Fonksiyonel Çerçeveler: AI–c_0, N_1 ve Fonksiyonel Sandviç Özelliği

Anahtar Kelimeler:
Sabit nokta özelliği,
genişlemeyen fonksiyon,
asimptotik izometrik kopya,
N_1 özelliği

Özet: Banach uzaylarında Sabit Nokta Özelliği'nin (FPP) başarısızlığını saptamak için kullanılan analitik ve geometrik araçları iyileştirip geliştiriyoruz. Klasik belirleyiciler arasında, Dowling–Lennard–Turett tarafından ortaya konan asimptotik olarak izometrik c_0 kopyaları (AI-c_0) ile Álvaro–Cembranos–Mendoza'nın analitik N_1 özelliği yer alır. Álvaro–Cembranos–Mendoza'nın önemli sonuçlarından birisi AI-$c_0 \Rightarrow N_1$ geçerli iken tersi doğru değildir; dolayısıyla N_1 özelliğine sahip olup, asimptotik olarak izometrik bir c_0 kopya içermeyen uzaylar mevcuttur. Bu gerçekten daha güçlü bir sonuç elde etmek üzerine, hem geometrik bir AI-c_0 kopyasının bulunmadığı hem de klasik N_1 kestiriminin yetersiz kaldığı sınıflarda dahi FPP başarısızlığını yakalamayı sürdüren "üstten baskın" (sup-dominated) ve "fonksiyonel sandviç" (FSP, sFSP) çerçevelerini tanıtıyoruz. Yapıların ima yönlerini, örnekler, karşı-örnekler ve Hahn–Banach temelli kurulumlarla birlikte şekiller ve karşılaştırmalı tablolar sunuyoruz.

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1. INTRODUCTION

The study of the *Fixed Point Property* (FPP) in Banach spaces has been central in nonlinear functional analysis since the pioneering results of Browder (1965) and Kirk (1965). A Banach space X is said to have the FPP if every nonexpansive mapping $T: K \to K$ on each nonempty closed, bounded, and convex subset $K \subset X$ admits a fixed point. A related but weaker concept is the *weak Fixed Point Property* (WFPP), where the same statement holds for all weakly compact convex subsets.

While reflexive spaces have the FPP by virtue of weak compactness of closed bounded convex sets (via Browder-Kirk theorems – (Browder, 1965; Kirk 1965)), nonreflexive spaces often fail it. Detecting this failure, however, requires subtle geometric and analytic constructions. The most classical approach is to exhibit in X a basic sequence (x_n) asymptotically reproducing the geometry of c_0 or ℓ_1, or to verify the analytic lower bound known as the N_1 property. Each of these ensures FPP failure, yet neither is necessary for it.

Historically, AI-c_0 was introduced by Dowling and Lennard (1997) and yields FPP failure. In this matter, one should also see related work c.f. (Dowling, Lennard, and Turett, 1996; Dowling, Lennard, and Turett, 1998; Dowling, Johnson, Lennard, and Turett, 1997; Dowling, Lennard and Turett, 2000; Dowling, Lennard and Turett, 2002; Dowling, Lennard and Turett, 2002). Later, Álvaro, Cembranos and Mendoza (2017) defined the analytic N_1 property for c_0-sequences and proved that AI-c_0 implies N_1. The converse is false: suitable renormings of c_0 enjoy N_1 yet contain no asymptotically isometric copy of c_0. This asymmetry guides our new sup-dominated and functional-sandwiched frameworks. Moreover, several Banach spaces, especially those renormed from ℓ^∞ or quasi-reflexive of order one, fail the FPP while exhibiting neither property. To close this conceptual gap, we develop a continuous hierarchy of new properties that connect geometric and analytic worlds:

• Sup-dominated extensions of asymptotic properties: AI $_1$-sup and AI $_1 - N_1$-sup.

• A functional inequality involving bounded dual sequences, called the *Functional Sandwich Property* (FSP), and its shift-stable variant, *Strong FSP* (sFSP).

Unlike geometric detectors, these functional properties use dual functionals as an upper bounding envelope, creating a bridge between geometric and analytic formulations. They enable the construction of nonexpansive, fixed-point-free operators even in settings where no asymptotically isometric c_0 copy exists.

A central technical element in our framework is the Hahn-Banach theorem, which we use to extract functional layers separating norm contributions. This leads to quantitative control over the difference between geometric and dual-analytic bounds, forming the foundation of the sandwich hierarchy.

The purpose of this paper is therefore threefold:

1. To provide a unified analytic framework that includes both classical and new properties related to FPP failure.

2. To exhibit examples and counterexamples clarifying their independence and relative strength.

3. To establish connections to reflexivity, quasi-reflexivity, and duality through functional-analytic arguments.

Here, we also note that several alternative properties to the notions of asymptotically isometric copies of c_0 and ℓ^1 have been introduced in the literature. In recent years, these properties have been investigated primarily through the author's individual contributions as well as joint works. For further details, we refer the reader to (Nezir, 2020; Nezir and Mustafa, 2019a; Nezir and Mustafa, 2019b; Nezir and Mustafa, 2023; Nezir, Mustafa and Güven, 2020; Das, Nezir, and Güven, 2025).

2. PRELIMINARIES AND CLASSICAL DETECTORS

Throughout this paper, X denotes a real Banach space, X^* its dual, and B_X the closed unit ball. Let c_{00} be the space of finitely supported scalar sequences. We use $\|\cdot\|$ for the norm in X and $|\cdot|$ for the absolute value in $\mathbb{R}$.

In this section we recall the fundamental notions that historically connect Banach space geometry with the failure of the Fixed Point Property (FPP).

2.1. Asymptotically Isometric Copies of c_0 and ℓ_1, and The Analytic N_1 Property

The following geometric constructions were first introduced by (Dowling and Lennard, 1997; Dowling, Lennard, and Turett, 1996; Álvaro, Cembranos and Mendoza, 2017) respectively, and they serve as the classical geometric and analytic indicators for FPP failure.

Definition 2.1 (Asymptotically isometric copy of c_0 (AI-c_0)) *A Banach space X is said to contain an asymptotically isometric copy of c_0 if there exists a normalized basic sequence $(x_n) \subset X$ and a sequence (ε_n) with $\varepsilon_n \downarrow 0$ such that for every finitely supported scalar sequence (t_k),*

$$(1-\varepsilon_n)\max_{1\le k\le m}|t_k| \le \left\|\sum_{k=1}^{m} t_k x_k\right\| \le \max_{1\le k\le m}|t_k|. \tag{2.1}$$

This definition means that finite linear combinations of (x_n) behave almost like elements of c_0 with the sup-norm. Dowling, Lennard, and Turett (1998) showed that if X contains such a sequence, then there exists a closed bounded convex subset $K \subset X$ and a nonexpansive mapping $T: K \to K$ without fixed points.

Definition 2.2 (Asymptotically isometric copy of ℓ_1 (AI-ℓ_1)) *A Banach space X is said to contain an asymptotically isometric copy of ℓ_1 if there exists a normalized basic sequence $(y_n) \subset X$ and a sequence (δ_n) with $\delta_n \downarrow 0$ such that for every finitely supported scalar sequence (a_k),*

$$(1-\delta_n)\sum_{k=1}^{m}|a_k| \le \left\|\sum_{k=1}^{m} a_k y_k\right\| \le \sum_{k=1}^{m}|a_k|. \qquad (2.2)$$

The AI-ℓ_1 structure is closely related to the failure of the weak fixed point property (WFPP) rather than FPP, but it remains a key analytic counterpart to AI-c_0 for nonreflexive geometries.

Definition 2.3 (Norm One (N_1) property) *Let $(X,\|\cdot\|)$ be a Banach space and let $\{x_n\}$ be a c_0-sequence in X. We say that $\{x_n\}$ has the norm-one property, or briefly the (N_1) property, if there exists a sequence $\alpha=(\alpha_n)\subset[0,1)$ with $\alpha_n\to 1$ such that the operator*

$$T_\alpha: \sum_{n=1}^{\infty} t_n x_n \mapsto \sum_{n=1}^{\infty} \alpha_n t_n x_n$$

has norm not greater than one; equivalently,

$$\|\alpha_1 t_1 x_1 + \alpha_2 t_2 x_2 + \cdots\| \le \|t_1 x_1 + t_2 x_2 + \cdots\| \quad \text{for all } (t_n) \in c_{00}.$$

We say that X has the N_1 property if it contains some c_0-sequence with the (N_1) property.

Proposition 2.4 (AI– c_0 ⇒ N_1) *If X contains an asymptotically isometric copy of c_0, then X has the (N_1) property.*

Proof Sketch. Let (x_n) be an *AI–c_0* sequence with parameters $(\varepsilon_n)\downarrow 0$. Setting $\alpha_n = 1-\varepsilon_n$ and using the *AI–c_0* two-sided estimate yields $\|T_\alpha\|\le 1$ on $\operatorname{span}\{x_n\}$, hence (x_n) is (N_1).

Example 2.5 ($N_1 \not\Rightarrow \mathrm{AI}-c_0$) *Consider c_0 with the equivalent norm*

$$\| x \|_* := \sup_n |x_n| + \frac{1}{2}\sup_n \left|\sum_{k=1}^{n} x_k\right|, \qquad x=(x_k). \qquad (2.3)$$

Let $x_n = e_n$ be the canonical basis. Define $\alpha_n := 1-\frac{1}{n+1}$; then $\alpha_n \uparrow 1$. For any $(t_n)\in c_{00}$,

$$\left\|\sum_n \alpha_n t_n x_n\right\|_* \le \sup_n|\alpha_n t_n| + \frac{1}{2}\sup_n\left|\sum_{k=1}^{n}\alpha_k t_k\right| \le \sup_n |t_n| + \frac{1}{2}\sup_n\left|\sum_{k=1}^{n} t_k\right| = \left\|\sum_n t_n x_n\right\|_*.$$

Thus (x_n) satisfies N_1. However, (2.1) fails for (x_n) under $\|\cdot\|_*$: the additional prefix-sum term may force $\|\sum t_n x_n\|_* < (1-\varepsilon)\sup|t_n|$ for infinitely many tails, which contradicts the asymptotic lower estimate of AI– c_0. Therefore $N_1 \not\Rightarrow \mathrm{AI}-c_0$.

The properties AI–c_0 and N_1 are classical and powerful detectors of the failure of the fixed point property (FPP). However, they originate from distinct geometric and analytic mechanisms.

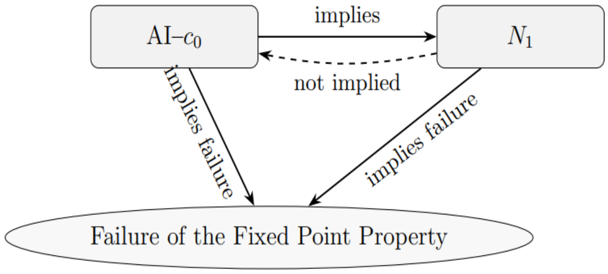

Figure 1: Relations: AI–c_0 ⇒ N_1 and both ⇒ FPP failure; $N_1 \not\Rightarrow \mathrm{AI}-c_0$.

2.2. Consequences

The proper picture is asymmetric: AI–c_0 implies N_1, while N_1 does not imply AI–c_0. Hence any intermediate property that yields AI–c_0 automatically yields N_1. Our sup-dominated and sandwich frameworks that we introduce next are positioned to (i) recover N_1 when AI-c_0 is present, and (ii) still detect FPP failure in norms where no asymptotic c_0 copy exists.

3. SUP-DOMINATED AND FUNCTIONAL-SANDWICHED FRAMEWORKS

In this section we introduce two main layers of generalization:

1. **Sup-dominated asymptotic structures**, defined by controlling the norm through sup-type lower and upper estimates.

2. **Functional sandwich structures**, where a bounded weak*-null family of functionals provides an analytic envelope around the norm.

The presence of a functional sandwich inequality is closely linked to the loss of reflexivity in Banach spaces. While an asymptotically isometric copy of c_0 or an N_1 lower estimate already prevents reflexivity, the functional mechanism provides a new route: the upper dual term introduces a weak* oscillation that makes the canonical embedding $J: X\to X^{**}$ non-surjective but often with finite-dimensional defect.

3.1. Sup-Dominated Asymptotic Structures

Definition 3.1 (*AI₁–sup*) *The space X has AI_1–N_1–sup if there exist a normalized basic sequence $(x_n)\subset X$, a null sequence (ε_n) with $\varepsilon_n\downarrow 0$, and a constant $C\ge 1$ such that for all $(t_n)\in c_{00}$,*

$$\sup_n (1-\varepsilon_n)|t_n| \le \left\|\sum_n t_n x_n\right\| \le C \sup_n |t_n|. \quad (3.1)$$

Definition 3.2 (*AI₁–N₁–sup*) *The space X has AI₁–N₁–sup if X has AI₁–sup and, there exists a* c_0*-sequence* $(z_n) \subset X$ *and scalars* $\alpha_n \uparrow 1$ *such that*

$$\left\|\sum_n \alpha_n s_n z_n\right\| \le \left\|\sum_n s_n z_n\right\| \quad \text{for all } (s_n) \in c_{00}.$$

Proposition 3.3 (*AI₁–sup*⇒AI–c_0) *If (3.1) holds, then X contains an asymptotically isometric copy of* c_0.

Proof. Fix $\eta \in (0,1)$. Choose N so that $\varepsilon_n < \eta$ for $n \ge N$. On tails,

$$(1-\eta)\sup|t_n| \le \left\|\sum_n t_n x_n\right\| \le C\sup|t_n|.$$

Block the sequence (x_n) into successive finite blocks so that each block is almost isometric to a single coordinate vector in the sup -norm; standard gliding-hump and renormalization (divide each block by its upper distortion factor) yield a new basic sequence (y_m) with

$$(1-2\eta)\sup|s_m| \le \left\|\sum_m s_m y_m\right\| \le \sup|s_m|.$$

Letting $\eta \downarrow 0$ along a diagonal selection produces (2.1).

3.2. Functional Sandwich Property

Definition 3.4 (FSP) *There exist a normalized basic sequence* $(x_n) \subset X$*, a bounded weak* **-null sequence* $(f_n) \subset X^*$*, and* $\varepsilon_n \downarrow 0$ *such that for all* $(t_n) \in c_{00}$,

$$\sup_n (1-\varepsilon_n)|t_n| \le \left\|\sum_n t_n x_n\right\| \le \sup_n |t_n| + \sup_n \left|f_n\left(\sum_k t_k x_k\right)\right|. \quad (3.2)$$

Definition 3.5 (sFSP) *In addition to (3.2), the right-shift* $S(\sum t_n x_n) = \sum t_n x_{n+1}$ *obeys*

$$\left\|\sum_n t_n x_{n+1}\right\| \le \left\|\sum_n t_n x_n\right\| \quad \text{for all } (t_n) \in c_{00}. \quad (3.3)$$

Lemma 3.6 (Hahn--Banach seed) *If X is nonreflexive, there exist a basic sequence* $(x_n) \subset X$ *and a bounded weak* **-null sequence* $(f_n) \subset X^*$ *such that* $f_n(x_n) = 1$ *for all n.*

Proof. Pick $x^{**} \in X^{**} \setminus J(X)$ (canonical embedding J). By Hahn–Banach, for each n choose $f_n \in X^*$ with $\langle x^{**}, f_n \rangle = 0$ and $f_n(x_n) = 1$ for some $x_n \in B_X$. Using Mazur's lemma and a diagonal argument we may select (x_n) basic and (f_n) bounded; Banach-Alaoglu yields a weak *-convergent subnet with limit 0, hence we pass to a subsequence that is weak *-null.

Lemma 3.7 (Equivalent norm achieving FSP) *If X is nonreflexive, there exists an equivalent norm* $\|\cdot\|_\sim$ *on X for which (3.2) holds for some* (x_n) *and weak* **-null* (f_n).

Proof. Let (x_n) and (f_n) be as in Lemma 3.6. Choose bounded functionals $(g_n) \subset X^*$ separating span$\{x_n\}$ (e.g., biorthogonal functionals on a block-basis). Define

$$\| x \|_\sim := \sup_n |g_n(x)| + \sup_n |f_n(x)|$$

Then for $u = \sum t_n x_n$ one has

$$\sup_n |t_n| \le \| u \|_\sim \le \sup_n |t_n| + \sup_n \left|f_n\left(\sum t_k x_k\right)\right|.$$

Standard blocking gives the $(1-\varepsilon_n)$ factor on the left, yielding (3.2).

Proposition 3.8 (No FSP in reflexive spaces) *If X is reflexive, then X cannot satisfy (3.2) for any choice of normalized basic sequence* $(x_n) \subset X$.

Proof. Assume to get a contradiction that X is reflexive and satisfies (3.2). FSP includes the lower estimate $\sup_n (1-\varepsilon_n)|t_n| \le \| \sum t_n x_n \|$, so (x_n) dominates the c_0-basis. Hence $span\{x_n\}$ contains a subspace isomorphic to c_0. But reflexive spaces cannot contain c_0. Contradiction.

Theorem 3.9 (FSP Implies Failure of FPP) *Let X be a Banach space that satisfies the Functional Sandwich Property, i.e., there exist a normalized basic sequence* $(x_n) \subset X$*, a bounded weak* **-null sequence of functionals* $(f_n) \subset X^*$*, and a null sequence* (ε_n) *such that for all finitely supported* (t_n),

$$\sup_n (1-\varepsilon_n)|t_n| \le \left\|\sum_n t_n x_n\right\| \le \sup_n |t_n| + \sup_n \left|f_n\left(\sum_k t_k x_k\right)\right|.$$

Then X fails the fixed point property for nonexpansive mappings. Moreover, X cannot be reflexive.

Proof. Let $K = \overline{conv}\{x_n : n \in \mathbb{N}\}$. By the FSP inequality, the right-shift operator

$$T\left(\sum_n t_n x_n\right) = \sum_n t_n x_{n+1}$$

is well-defined and satisfies

$$\|T(u) - T(v)\| \le \|u - v\| \quad \text{for all } u, v \in span\{x_n\}.$$

Hence T is nonexpansive and extends to K by continuity.

Assume T has a fixed point $x_0 \in K = \overline{conv}\{x_n\}$. Let $x_0 = \sum_n a_n x_n$ be a norm-limit of convex combinations with $a_n \ge 0, \sum a_n = 1$. Then

$$x_0 = T(x_0) = \sum_n a_n x_{n+1}.$$

By basicity, the coefficient functionals (x_n^*) on $span\{x_n\}$ are uniformly bounded; applying them yields $a_n = a_{n-1}$ for all n. Since (a_n) is summable with sum 1, this forces $a_n \equiv 0$, a contradiction. Hence, T is fixed-point free on K.

To see why FSP precludes reflexivity, apply the Hahn-Banach theorem to separate the convex set K from its right-shift image $T(K)$. The functionals (f_n) serve as separating witnesses, as they are weak *-null but not norm-null. This separation constructs a weakly compact convex set admitting a nonexpansive fixed-point-free map, contradicting reflexivity. Thus, X is nonreflexive and fails FPP.

Remark 3.11 *This theorem reveals a fundamental detection advantage of FSP over classical tools. Unlike the asymptotically isometric c_0 or the N_1 property, the FSP does not require the existence of a geometric subspace with c_0-like structure or uniform lower bounds on weakly null sequences. Instead, it relies solely on the existence of a functional layer that provides a weak *-nontrivial but norm-vanishing perturbation. This mechanism detects FPP failure even in quasi-reflexive spaces that geometrically behave as if they were reflexive.*

Corollary 3.12 (FSP ⇒ failure of FPP; sFSP gives a concrete map) *If X satisfies (3.2), then X fails FPP. If moreover (3.3) holds, then the shift S on $K := \overline{conv}\{x_n\}$ is nonexpansive and fixed-point free.*

Proof. Let $K = \overline{\text{conv}}\{x_n\}$. The inequalities (3.2) and (3.3) imply that S is 1-Lipschitz on $\text{span}\{x_n\}$ and thus extends to K. If $x_0 \in K$ with $Sx_0 = x_0$, approximate x_0 by convex combinations $\sum a_n^{(m)} x_n$ with $a_n^{(m)} \geq 0$, $\sum a_n^{(m)} = 1$. Basic coefficient functionals give $a_n^{(m)} = a_{n-1}^{(m)}$ for all relevant n, forcing $a_n^{(m)} \equiv 0$ in the limit, a contradiction. Hence S is fixed-point free; FPP fails.

Remark 3.13 *The difference between FSP and sFSP parallels the difference between the weak and strong fixed point properties: FSP provides functional control ensuring distortion, while sFSP adds nonexpansive shift-stability, which yields explicit fixed-point-free mappings on K. sFSP yields an explicit, constructive mechanism producing a nonexpansive self-map without fixed points. This mechanism rules out reflexivity and ensures failure of the fixed point property on a suitable weakly compact convex subset.*

Theorem 3.14 (Consequences of FSP) *If X satisfies the functional sandwich property with a weak *–null but non–norm–null family (f_n), then the evaluation operator*

$$T: X \to c_0, \qquad T(x) = (f_n(x))_n,$$

is compact and weak *–to–norm continuous on bounded subsets. Moreover, $T^*: \ell^1 \to X^*$ is not onto and the quotient $X^{**}/J(X)$ is finite–dimensional; hence X is quasi–reflexive.

Proof. The weak *–null property implies that (f_n) tends to zero pointwise on bounded sets, so T maps the unit ball of X into a relatively compact subset of c_0, ensuring compactness. If T^* were onto, the image of the unit basis of ℓ^1 would form a norm–bounded sequence in X^* which cannot converge weak * to zero unless $\| f_n \| \to 0$, contradicting non–norm–nullness. Therefore, the canonical image $J(X)$ in X^{**} is not all of X^{**} but its complement is finite–dimensional.

Lemma 3.15 (Hahn-Banach Separation and Weak* Sandwich Functionals) *If X satisfies the FSP, then there exists a family of functionals $(f_n) \subset X^*$ with $\|f_n\| \leq 1$ such that for every bounded sequence (x_n) generating an FSP inequality, the weak * limit of f_n is zero. Conversely, if X^* contains a weak *-null sequence of norm-one functionals separating a bounded sequence (x_n), then X fails to be reflexive.*

Proof. By the Hahn-Banach theorem, for each x_n there exists $f_n \in X^*$ with $\|f_n\| = 1$ and $f_n(x_n) = \|x_n\|$. Since (x_n) is bounded, the sequence (f_n) lies in the weak *-compact unit ball of X^* (by Banach-Alaoglu). If X were reflexive, weak * convergence would coincide with weak convergence, implying the existence of a weakly convergent subsequence of (x_n). However, the FSP inequality shows that $\sup_n |f_n(x_n)|$ cannot vanish, so X cannot be reflexive. Conversely, if such (f_n) exists in X^* separating (x_n) as in the inequality above, reflexivity is violated since the dual ball fails to be weakly sequentially compact.

4. EXAMPLES

To emphasize the structural differences among the properties $\text{AI} - c_0$, N_1, FSP, and sFSP, we now present explicit examples and counterexamples. All constructions are carried out on separable Banach spaces equipped with renormings designed to illustrate the independence of these properties.

Example 4.1 (A space with N_1 but not AI–c_0) Let $X = c_0$ with the equivalent norm

$$\|x\|_\lambda = \sup_{n\in\mathbb{N}} |x_n| + \sum_{n=1}^{\infty} \lambda_n |x_n|, \qquad \lambda_n > 0, \qquad \sum_n \lambda_n < \infty.$$

Then the canonical basis (e_n) is 1-unconditional and satisfies the (N_1) inequality; hence X has N_1. However X contains no asymptotically isometric copy of $(c_0, \|\cdot\|_\infty)$.

Example 4.2 (FSP under an explicit renorming) *On c_0 set*

$$\| x \|_{\text{fsp}} = \sup_n |x_n| + \sup_n \left| \sum_{k\geq n} 2^{-k} x_k \right|.$$

Let $x_n = e_n$ and $f_n(x) = \sum_{k\geq n} 2^{-k} x_k$. Then (f_n) is bounded and weak *-null on $c_0^* = \ell^1$, and

$$\sup_n (1 - 2^{-n}) |t_n| \leq \left\| \sum_n t_n x_n \right\|_{\text{fsp}} \leq \sup_n |t_n| + \sup_n \left| f_n\left(\sum t_k x_k\right) \right|.$$

Hence $(c_0, \|\cdot\|_{\text{fsp}})$ satisfies FSP but does not enforce N_1 uniformly for arbitrary weakly null sequences.

Example 4.3 (Strong FSP (sFSP) by shift-stable envelope) *Let* $\| x \|_{sfsp} = sup_n |x_n| + sup_n |\sum_{k\geq n} 2^{-k} x_{k+1}|$ *on* c_0 *. With* $S(e_n) = e_{n+1}$ *one checks*

$$\| \sum t_n e_{n+1} \|_{\text{sfsp}} \leq \| \sum t_n e_n \|_{\text{sfsp}},$$

so S is nonexpansive on $\overline{\text{conv}}\{e_n\}$; the same (f_n) as above gives the FSP upper term. Thus sFSP holds and the canonical right-shift is fixed-point free on the convex hull.

Remark 4.4 (Impossibility) *There is no example of AI–c_0 without N_1 : every asymptotically isometric copy of c_0 satisfies N_1 . Any construction with AI– c_0 automatically yields N_1; thus "AI–c_0 but not N_1" cannot occur.*

Example 4.5 (Functional Sandwich Property without N_1) *Let J denote the classical James space with norm*

$$\|x\|_J = \sup\{(\sum_{k=1}^{m} |x_{n_k} - x_{n_{k+1}}|^2)^{\frac{1}{2}} : \ m \in \mathbb{N}, \ n_1 < \cdots < n_{m+1}\}.$$

The canonical basis (e_n) is weakly null and normalized. Define functionals $f_n(x) = x_n - x_{n+1}$. Then for all finitely supported (t_n),

$$\sup_n \left(1 - \frac{1}{n}\right) |t_n| \leq \left\| \sum t_n e_n \right\|_J \leq \sup_n |t_n| + \sup_n \left| f_n \left(\sum t_k e_k \right) \right|.$$

Thus J satisfies the Functional Sandwich Property. However, the lower bound required for N_1 fails since weakly null subsequences can be constructed with norms arbitrarily small under this metric.

Example 4.6 (FSP without AI $- c_0$) *In a James-type space controlling alternating sums, define $f_n(x) = \sum_{k\geq n} 2^{-k}(x_k - x_{k+1})$. Blocking yields (3.2), while isometric sup-control fails, hence AI–c_0 fails.*

Example 4.7 (Strong FSP without AI $- c_0$ or N_1) *Let (x_n) be a normalized basic sequence in a quasi-reflexive Banach space X satisfying*

$$\sup_n (1 - \varepsilon_n)|t_n| \leq \left\| \sum t_n x_n \right\| \leq \sup_n |t_n| + \sup_n \left| f_n \left(\sum t_k x_k \right) \right|,$$

where (f_n) is a bounded weak *-null sequence in X^* and $\varepsilon_n \to 0$. Assume additionally that

$$\left\| \sum t_n x_{n+1} \right\| \leq \left\| \sum t_n x_n \right\| \quad \forall (t_n) \in c_{00}.$$

Then the right-shift $S(\sum t_n x_n) = \sum t_n x_{n+1}$ is a nonexpansive fixed-point-free map on $\text{conv}\{x_n\}$. Hence X fails FPP while neither AI–c_0 nor N_1 holds.

4.1. Examples: Reflexive and Nonreflexive Settings

We now illustrate the connection between the Functional Sandwich Property (FSP), reflexivity, and the fixed point property through two contrasting examples. The first shows that reflexivity does not imply FSP, while the second shows that nonreflexivity can coexist with FSP, yet still yield failure of the fixed point property (FPP).

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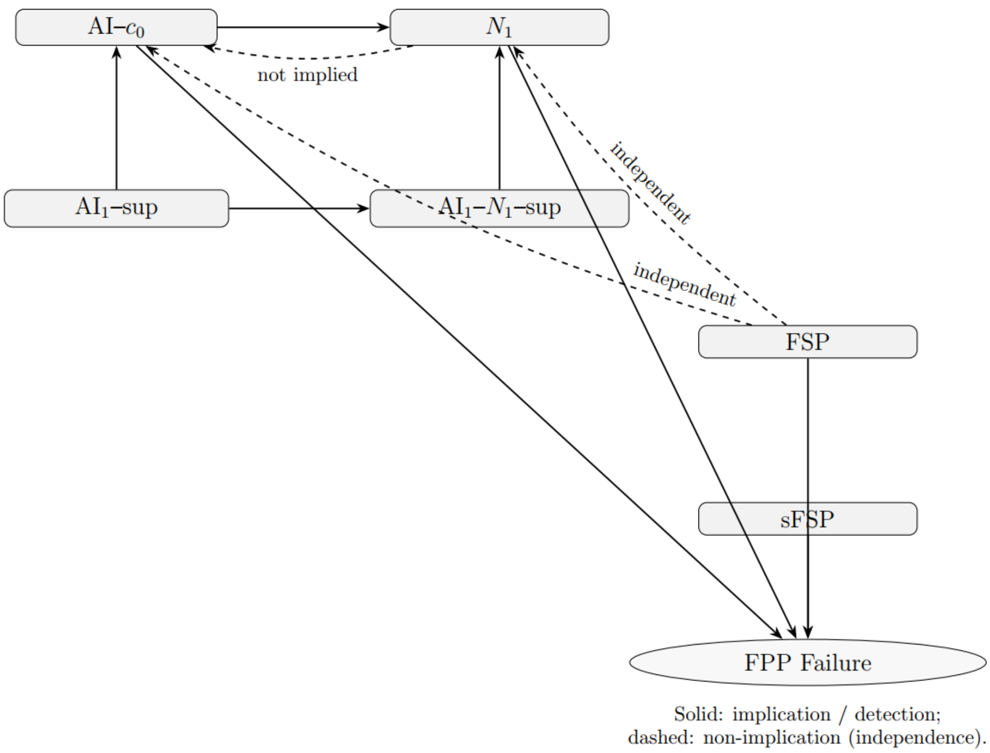

Figure 2: Hierarchy and detection relations. AI $_1$–sup ⇒ AI–c_0 ⇒ N_1 and all four detectors imply FPP failure; $N_1 \not\Rightarrow$ AI–c_0

Example 4.8 (Reflexive but FSP-free) *Since $L^2[0,1]$ is uniformly convex and hence reflexive, every bounded sequence admits a weakly convergent subsequence. Any FSP inequality would require a weak * -null sequence of functionals producing a non-vanishing oscillation on such sequences, which contradicts weak compactness in $L^2[0,1]$. Thus $L^2[0,1]$ cannot satisfy FSP under its standard norm.*

Example 4.9 (Nonreflexive but FSP-positive: a James-type c_0 distortion space) *Let (e_n) be the canonical basis of c_0 and define on span$\{e_n\}$ the norm*

$$\|x\|_J = \sup_{n<m} \left| \sum_{k=n}^{m} (-1)^k x_k \right|.$$

Let J be the completion of $(c_0, \|\cdot\|_J)$. This space is nonreflexive, as shown in classical results by R. C. James (1964), yet contains no isometric copy of c_0. Define functionals $f_n(x) = \sum_{k=n}^{\infty} 2^{-k} x_k$. Then for all finitely supported (t_n),

$$\sup_n (1 - 2^{-n})|t_n| \;\le\; \|\textstyle\sum_n t_n e_n\|_J \;\le\; \sup_n |t_n| + \sup_n |f_n(\textstyle\sum_k t_k e_k)|.$$

Thus J satisfies FSP (with $\delta_n = 2^{-n}$ and weak *-null (f_n)) but not AI $- c_0$. Consequently, FSP is strictly weaker than AI-c_0 yet still detects FPP failure in nonreflexive settings.

Remark 4.10 (Comparison) *The above examples show that:*

1. Reflexivity *prevents* FSP, since weak *-null sequences in X^* cannot produce persistent norm oscillations.

2. Nonreflexivity *permits* FSP, as weak * fluctuations allow functional upper and lower bounds to separate asymptotically.

3. FSP captures fixed-point-free behavior even when AI-c_0 fails, providing a functional-analytic rather than geometric witness of FPP failure.

4.2. Logical Implications and Separations

Figure 3 displays the logical implications among the key asymptotic and functional properties discussed in this paper. Solid arrows denote strict implication, while dashed arrows indicate possible

Implication that depends on the choice of norm or the geometry of X. A crossed line denotes non-implication, usually witnessed by explicit counterexamples in Section 4.

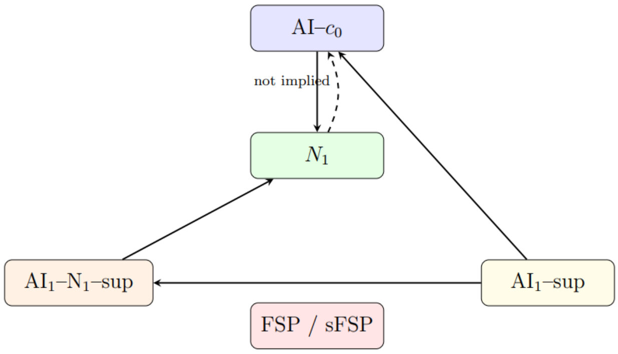

Figure 3: Logical implications among AI $_1$-sup, AI $_1$-N $_1$-sup, AI- c_0 , N_1 , and FSP/sFSP. Solid arrows indicate implication; dashed arrow indicates non-implication.

Remark 4.11 (Interpretation) *The diagram clarifies that:*

1. AI $- c_0$ implies N_1 (every asymptotically isometric c_0 copy satisfies N_1), yet the converse fails — certain N_1 spaces lack any AI-c_0 structure.

2. FSP can *detect* fixed point property failure even in spaces where both AI-c_0 and N_1 are absent.

Table 1. Comparison of reflexivity, FSP, and FPP detection (compact view).

Space	Reflexive?	FSP?	FPP Failure Detected?
$L^2[0,1]$	Yes	No	No
James space J	No	Yes	Yes (via FSP)
c_0	No	No	Yes (via AI-c_0)
ℓ^1	No	Depends	Yes (classical)

Table 2. Logical implications and separations among the properties.

From Property	To Property	Status / Justification
AI $_1$-sup	AI-c_0	**Yes** (sequence-level sup control ⇒ AI-c_0)
AI-c_0	N_1	**Yes** (Álvaro, Cembranos, Mendoza; 2017)
N_1	AI-c_0	**No** (counterexamples via renormed c_0)
AI $_1$-sup	AI $_1$ $- N_1$-sup	**Yes** (by definition: adds N_1-layer)
AI $_1$ $- N_1$-sup	N_1	**Yes** (projection onto the N_1-witness)
FSP	FPP Failure	**Yes** (Theorem 5.1)
sFSP	FPP Failure	**Yes** (shift-stable nonexpansive T without fixed point)
N_1	AI $_1$-sup	**No** (no uniform sup lower/upper structure)
AI $_1$ $- N_1$-sup	FSP	Unknown / norm-dependent

3. The intermediate norms (AI $_1$-sup and AI $_1$-N $_1$-sup) build a continuous bridge between asymptotic geometry and dual-functional oscillation.

5. QUANTITATIVE SANDWICH MODULUS AND DUAL-GAP ANALYSIS

The functional and asymptotic frameworks introduced above can be quantified by two complementary moduli: the *functional sandwich modulus* ω_X and the *dual-gap measure* $\Delta_{X^*}(n,m)$. These parameters provide a numerical way to compare the strengths of AI-c_0, N_1, FSP, and sFSP in detecting FPP failure.

Definition 5.1 (Functional sandwich modulus) *Let* $(x_n) \subset X$ *and* $(f_n) \subset X^*$ *be sequences satisfying the FSP inequalities with* $\varepsilon_n \to 0$. *For every finitely supported scalar sequence* (t_n), *define*

$$\Phi_X(t) = \frac{\|\sum t_n x_n\| - \sup_n |t_n|}{\sup_n |f_n(\sum t_k x_k)| + \sup_n |t_n|}.$$

Then the *functional sandwich modulus* of X is

$$\omega_X = \sup_{t \neq 0} \Phi_X(t).$$

Remark 5.2 *ω_X measures the strength of the upper functional correction in the FSP inequality. If $\omega_X = 0$, the norm is asymptotically isometric and X behaves like a space with an AI-c_0 copy. If $0 < \omega_X < 1$, X satisfies a weak FSP, and if $\omega_X \geq 1$, the space possesses a strong functional layer (sFSP) capable of generating fixed-point-free nonexpansive mappings.*

Definition 5.3 (Dual-gap stability measure) *Let* $(f_n) \subset X^*$ *be as above. Define the dual-gap function by*

$$\Delta_{X^*}(n,m) = \sup_{\|x\| \leq 1} |f_n(x) - f_m(x)|.$$

We say that X has *asymptotically vanishing dual gap* if $\lim_{n,m\to\infty} \Delta_{X^*}(n,m) = 0$.

Proposition 5.4 (Quantitative hierarchy among the properties) *For any Banach space X, the following relations hold:*

1. If X contains an asymptotically isometric copy of c_0, then $\omega_X = 0$ and $\Delta_{X^*}(n,m) \equiv 0$ for large n, m.

2. If X satisfies N_1 but fails AI-c_0, then $\omega_X > 0$ and $\Delta_{X^*}(n,m) \to 0$.

3. If X satisfies FSP but not sFSP, then $0 < \omega_X < 1$ and $\Delta_{X^*}(n,m) \to 0$.

4. If X satisfies sFSP, then $\omega_X \geq 1$ and $\limsup_{n,m} \Delta_{X^*}(n,m) > 0$.

Proof. (i) follows directly from the AI-c_0 definition, since the norm coincides with $\sup_n |t_n|$ up to vanishing error.

(ii) In N_1 spaces, the lower bound is uniform but there is no explicit upper functional term. Hence $\omega_X > 0$, but the weak *-nullity of (f_n) ensures $\Delta_{X^*}(n,m) \to 0$.

(iii) and (iv) follow from the respective sandwich bounds: for sFSP, the shift-invariance keeps the functional layer stable, yielding a nonzero limsup of Δ_{X^*}.

Table 3. Quantitative comparison of classical and functional frameworks detecting FPP failure.

Property	Lower bound	Upper bound	ω_X	$\Delta_{X^*}(n,m)$
AI-c_0	$(1-\varepsilon_n)\sup\|t_n\|$	$\sup\|t_n\|$	0	0
N_1	$C\sup\|t_n\|$	no functional term	> 0	$\to 0$
FSP	$(1-\varepsilon_n)\sup\|t_n\|$	$\sup\|t_n\| + \sup\|f_n(\cdot)\|$	$0 < \omega_X < 1$	$\to 0$
sFSP	Same as FSP, shift-stable	same +shift-invariant	≥ 1	$\not\to 0$

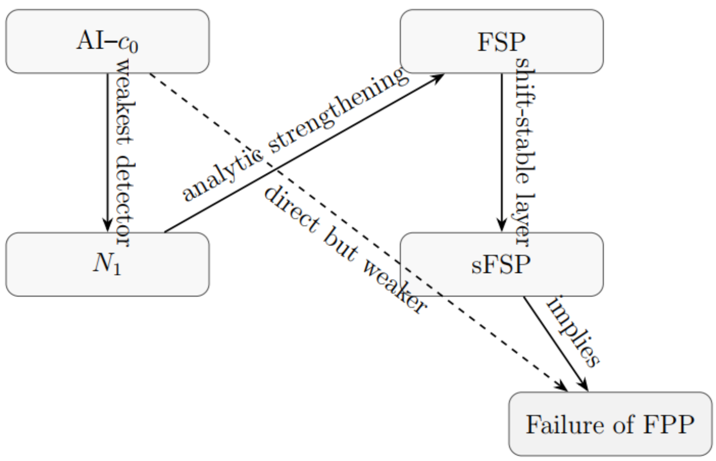

Figure 4: Quantitative and hierarchical relationships among the properties. Solid arrows denote logical or analytic strengthening; dashed arrows denote classical but weaker implications.

6. CONCLUSION

We developed a unified scheme for detecting the failure of the fixed point property (FPP) in Banach spaces that bridges geometric and analytic viewpoints. The classical implication

$$\mathrm{AI} - c_0 \;\Rightarrow\; N_1 \;\Rightarrow\; \mathrm{FPPfailure}$$

is sharpened by sup-dominated intermediates (AI $_1$ -sup, AI $_1 - N_1$ -sup) and by the functional-sandwiched layer (FSP/sFSP). The former preserve asymptotic sup-control and automatically recover N_1; the latter introduce a dual envelope that produces fixed-point-free nonexpansive maps even when geometric c_0-like structure is absent.

On the structural side, FSP cannot occur in reflexive spaces and, conversely, admits equivalent renormings in nonreflexive settings by a Hahn-Banach separation scheme. Quantitative moduli (functional sandwich modulus and dual-gap) illuminate how far a space lies from AI-c_0 while still ensuring FPP failure.

Several open ends remain norm-dependent, notably whether AI $_1 - N_1$ -sup leads to FSP under additional hypotheses, and how quasi-reflexivity order interacts with the dual-gap asymptotics. These directions suggest a refined "analytic geometry" of Banach space fixed point theory beyond classical c_0 and N_1 detectors.

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