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Some Types of Regularity and Normality Axioms in Čech Fuzzy Soft Closure Spaces

Lina Hussein Maibed <linah8324@gmail.com>
Rasha Naser Majeed* <rashanm6@gmail.com>

Department of Mathematics, Faculty of Education for Pure Sciences Abn Al-Haitham, University of Baghdad, Baghdad, Iraq.

Abstract – In the present paper, some types of regularity and normality axioms in Čech fuzzy soft closure spaces and their associative fuzzy soft topological spaces are defined and studied. Basic properties of these axioms, hereditary and topological properties are discussed.

***Keywords* –** Fuzzy soft set, fuzzy soft point, Čech fuzzy soft closure space, quasi regular, semi-regular, regular, semi-normal, normal, completely normal.

1 Introduction

Many problems in medical science, engineering, environments, economics etc. have several uncertainties. To skip these uncertainties, some types of theories were given like theory of fuzzy sets [19], rough sets [15], intuitionistic fuzzy sets [1], i.e., which we can use as mathematical tools for dealing with uncertainties. Most of these present for computing and formal modeling are crisp. In 1999, Molodsov [14] introduced the concept of soft set theory, which is a completely new approach for modeling vagueness and uncertainty. The concept of fuzzy soft sets was defined by Maji et al. [12] as a fuzzy generalizations of soft sets. Then in 2011, Tanay and Kandemir [17] were gave the concept of topological structure based on fuzzy soft sets.

In 1966, Čech [3], was introduced the concept closure spaces and then various notions in general topology have been extended to closure spaces. After Zadeh introduced the concept of fuzzy sets, in 1985 Mashhour and Ghanim [13] present the concept of Čech fuzzy closure spaces. They exchanged sets by fuzzy sets in the definition of Čech closure space. In 2014, Gowri and Jegadeesan [4] and Krishnaveni and Sekar [7], used the concept of soft sets to introduced and investigate the notion of soft Čech closure spaces. Recently, motivated by the concept of fuzzy soft set and fuzzy soft topology Majeed [9] was defined the concept of Čech fuzzy soft closure spaces. After that, Majeed and Maibed [10] introduced some structures of Čech fuzzy soft closure spaces, they show that every Čech fuzzy soft closure space gives a parameterized family of Čech fuzzy closure spaces.

The separation axioms in closure spaces were introduced by Čech [3]. Gowri and Jegadeesan [4, 5] studied separation axioms in soft Čech closure spaces. In our previous paper [11] we have introduced and discussed some properties of lower separation axioms in Čech fuzzy soft closure spaces. In the current work, we introduced and studied a new types of higher separation axioms like quasi regular, semi-regular, pseudo regular, regular, semi-normal, pseudo normal, normal and completely normal in Čech fuzzy soft closure spaces.

2 Preliminaries

In this section, we review some basic definitions and their results of fuzzy soft theory and Čech fuzzy soft closure spaces that are helpful for subsequent discussions, and we expect the reader be familiar with the basic notions of fuzzy set theory. Throughout paper, X refers to the initial universe, $I = [0,1]$, $I_0 = (0,1]$, I^X be the family of all fuzzy sets of X, and K the set of parameters for X.

Definition 2.1 [16, 18] A fuzzy soft set (fss, for short) λ_A on X is a mapping from K to I^X, i.e., $\lambda_A : K \to I^X$, where $\lambda_A(h) \neq \bar{0}$ if $h \in A \subseteq K$ and $\lambda_A(h) = \bar{0}$ if $h \in K - A$, where $\bar{0}$ is the empty fuzzy set on X. The family of all fuzzy soft sets over X denoted by $\mathcal{F}_{ss}(X, K)$.

Definition 2.2 [18] Let $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$. Then,

1. λ_A is called a fuzzy soft subset of μ_B, denoted by $\lambda_A \subseteq \mu_B$, if $\lambda_A(h) \leq \mu_B(h)$, for all $h \in K$.
2. λ_A and μ_B are said to be equal, denoted by $\lambda_A = \mu_B$ if $\lambda_A \subseteq \mu_B$ and $\mu_B \subseteq \lambda_A$.
3. the union of λ_A and μ_B, denoted by $\lambda_A \cup \mu_B$ is the fss $\sigma_{(A \cup B)}$ defined by $\sigma_{(A \cup B)}(h) = \lambda_A(h) \vee \mu_B(h)$, for all $h \in K$.
4. the intersection of λ_A and μ_B, denoted by $\lambda_A \cap \mu_B$ is the fss $\sigma_{(A \cap B)}$ defined by $\sigma_{(A \cap B)}(h) = \lambda_A(h) \wedge \mu_B(h)$, for all $h \in K$.
5. the complement of a fss $\lambda_A \in \mathcal{F}_{ss}(X, K)$, denoted $\bar{1}_K - \lambda_A$, is the fss defined by $(\bar{1}_K - \lambda_A)(h) = \bar{1} - \lambda_A(h)$, for each $h \in K$. It is clear that $\bar{1}_K - (\bar{1}_K - \lambda_A) = \lambda_A$.

Definition 2.3 [18] The null fss, denoted by $\bar{0}_K$, is a fss defined by $\bar{0}_K(h) = \bar{0}$, for all $h \in K$, where $\bar{0}$ is the empty fuzzy set of X.

Definition 2.4 [18] The universal fss, denoted by $\bar{1}_K$, is a fss defined by $\bar{1}_K(h) = \bar{1}$, for all $h \in K$, where $\bar{1}$ is the universal fuzzy set of X.

Definition 2.5 [8] Two fss's λ_A, $\mu_B \in \mathcal{F}_{ss}(X, K)$ are said to be disjoint, denoted by $\lambda_A \cap \mu_B = \bar{0}_K$, if $\lambda_A(h) \cap \mu_B(h) = \bar{0}$ for all $h \in K$.

Definition 2.6 [2] A fss $\lambda_A \in \mathcal{F}_{ss}(X, K)$ is called fuzzy soft point, denoted by $x_t{}^h$, if there exist $x \in X$ and $h \in K$ such that $\lambda_A(h)(x) = t$ $(0 < t \leq 1)$ and 0 otherwise for all $y \in X - \{x\}$.

Definition 2.7 [2] The fuzzy soft point $x_t{}^h$ is said to be belongs to the fss λ_A, denoted by $x_t{}^h \tilde{\in} \lambda_A$ if for the element $h \in K, t \leq \lambda_A(h)(x)$.

Definition 2.8 [17, 18] A fuzzy soft topological space (fsts, for short) (X, τ, K) where X is a non-empty set with a fixed set of parameter and τ is a family of fuzzy soft sets over X satisfying the following properties:

1. $\bar{0}_K, \bar{1}_K \in \tau$,
2. If $\lambda_A, \mu_B \in \tau$, then $\lambda_A \cap \mu_B \in \tau$,
3. If $(\lambda_A)_i \in \tau$, then $\bigcup_{i \in J \in}(\lambda_A)_i \in \tau$.

τ is called a topology of fuzzy soft sets on X. Every member of τ is called an open fuzzy soft set (open-fss, for short). The complement of an open-fss is called a closed fuzzy soft set (closed-fss, for short).

Definition 2.9 [9] An operator $\theta: \mathcal{F}_{ss}(X, K) \to \mathcal{F}_{ss}(X, K)$ is called Čech fuzzy soft closure operator ($\check{C}$-fsco, for short) on X, if the following axioms are satisfied.

(C1) $\theta(\bar{0}_K) = \bar{0}_K$,
(C2) $\lambda_A \subseteq \theta(\lambda_A)$, for all $\lambda_A \in \mathcal{F}_{ss}(X, K)$,
(C3) $\theta(\lambda_A \cup \mu_B) = \theta(\lambda_A) \cup \theta(\mu_A)$, for all $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$.

The triple (X, θ, K) is called a Čech fuzzy soft closure space ($\check{C}\mathcal{F}$-fscs, for short).
A fss λ_A is said to be closed-fss in (X, θ, K) if $\lambda_A = \theta(\lambda_A)$. And a fss λ_A is said to be an open-fss if $\bar{1}_K - \lambda_A$ is a closed-fss.

Proposition 2.10 [9] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs, and $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$. Then,

1. If $\lambda_A \subseteq \mu_B$, then $\theta(\lambda_A) \subseteq \theta(\mu_B)$.
2. $\theta(\lambda_A \cap \mu_B) \subseteq \theta(\lambda_A) \cap \theta(\mu_A)$.

Definition 2.11 [9] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs, and let $\lambda_A \in \mathcal{F}_{ss}(X, K)$. The interior of λ_A, denoted by $Int(\lambda_A)$ is defined as $Int(\lambda_A) = \bar{1}_K - (\theta(\bar{1}_K - \lambda_A))$.

Proposition 2.12 [9] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs, and let $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$. Then,

1. $\mathrm{Int}(\bar{0}_K) = \bar{0}_K$ and $\mathrm{Int}(\bar{1}_K) = \bar{1}_K$.
2. $\mathrm{Int}(\lambda_A) \subseteq \lambda_A$.
3. $\mathrm{Int}(\lambda_A \cap \mu_B) = \mathrm{Int}(\lambda_A) \cap \mathrm{Int}(\mu_B)$.
4. If $\lambda_A \subseteq \mu_B$, then $\mathrm{Int}(\lambda_A) \subseteq \mathrm{Int}(\mu_B)$.
5. λ_A is an open-fss $\Leftrightarrow \mathrm{Int}(\lambda_A) = \lambda_A$.
6. $\mathrm{Int}(\lambda_A) \cup \mathrm{Int}(\mu_B) \subseteq \mathrm{Int}(\lambda_A \cup \mu_B)$.

Definition 2.13 [9] Let V be a non-empty subset of X. Then $\bar{V}_K$ denotes the fuzzy soft set V_K over X for which $V(h) = \bar{1}_V$ for all $h \in K$, (where $\bar{1}_V: X \to I$ such that $\bar{1}_V(x) = 1$ if $x \in V$ and $\bar{1}_V(x) = 0$ if $x \notin V$).

Theorem 2.14 [9] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs, $V \subseteq X$ and let $\theta_V : \mathcal{F}_{ss}(V, K) \to \mathcal{F}_{ss}(V, K)$ defined as $\theta_V(\lambda_A) = \bar{V}_K \cap \theta(\lambda_A)$. Then θ_V is a $\check{C}\mathcal{F}$-sco. The triple (V, θ_V, K) is called $\check{C}$ech fuzzy soft closure subspace ($\check{C}\mathcal{F}$-sc subspace, for short) of (X, θ, K).

Proposition 2.15 [9] Let (V, θ_V, K) be a closed $\check{C}\mathcal{F}$-sc subspace of $\check{C}\mathcal{F}$-scs (X, θ, K) and λ_A be a closed-fss in (V, θ_V, K). Then λ_A is a closed-fss in (X, θ, K).

Proposition 2.16 [11] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs and let (V, θ_V, K) be a closed $\check{C}\mathcal{F}$-sc subspace of (X, θ, K). If λ_A is an open-fss of (X, $\theta, K)$. Then $\lambda_A \cap \bar{V}_K$ is also open-fss in (V, θ_V, K).

Theorem 2.17 [9] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs and let $\tau_\theta \subseteq \mathcal{F}_{ss}(X, K)$, defined as follows

$$\tau_\theta = \{\bar{1}_K - \lambda_A : \theta(\lambda_A) = \lambda_A\}.$$

Then τ_θ is a fuzzy soft topology on X and (X, τ_θ, K) is called an associative fsts of (X, θ, K).

Next the definition of fuzzy soft closure (respectively, interior) of a fss in the associative fsts of (X, τ_θ, K) is given.

Definition 2.18 [11] Let (X, τ_θ, K) be an associative fuzzy soft topological space of (X, θ, K) and let $\lambda_A \in \mathcal{F}_{ss}(X, K)$. The fuzzy soft topological closure of λ_A with respect to θ, denoted by $\tau_\theta\text{-}cl(\lambda_A)$, is the intersection of all closed fuzzy soft supersets of λ_A . i.e.,

$$\tau_\theta\text{-}cl(\lambda_A) = \cap \{\rho_C : \lambda_A \subseteq \rho_C \text{ and } \theta(\rho_C) = \rho_C \}. \tag{2.1}$$

And, The fuzzy soft topological interior of λ_A with respect to θ, denoted by $\tau_\theta\text{-}int(\lambda_A)$ is the union of all open fuzzy soft subset of λ_A. i.e.,

$$\tau_\theta\text{-}int(\lambda_A) = \cup\{\rho_C : \rho_C \subseteq \lambda_A \text{ and } \theta(\bar{1}_K - \rho_C) = \bar{1}_K - \rho_C\}. \tag{2.2}$$

From Theorem 2.17, it is clear that $\tau_\theta\text{-}cl(\lambda_A)$ (respectively, $\tau_\theta\text{-}int(\lambda_A)$) is the smallest (respectively, largest) closed- (respectively, open-)fss over X which contains (respectively, contained in) λ_A .

Proposition 2.19 [11] Let (X, τ_θ, K) be an associative fsts of (X, θ, K) and let $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$. Then,

1. $\tau_\theta\text{-}cl(\bar{0}_K) = \bar{0}_K$ and $\tau_\theta\text{-}cl(\bar{1}_K) = \bar{1}_K$.
2. $\lambda_A \subseteq \tau_\theta\text{-}cl(\lambda_A)$.
3. if $\lambda_A \subseteq \mu_B$, then $\tau_\theta\text{-}cl(\lambda_A) \subseteq \tau_\theta\text{-}cl(\mu_B)$.
4. $\tau_\theta\text{-}cl(\lambda_A \cup \mu_B) = \tau_\theta\text{-}cl(\lambda_A) \cup \tau_\theta\text{-}cl(\mu_B)$.
5. $\tau_\theta\text{-}cl(\tau_\theta\text{-}cl(\lambda_A)) = \tau_\theta\text{-}cl(\lambda_A)$.
6. λ_A is a closed-fss if and only if $\lambda_A = \tau_\theta\text{-}cl(\lambda_A)$.

Proposition 2.20 [11] Let (X, τ_θ, K) be an associative fsts of (X, θ, K) and let $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$. Then,

1. $\tau_\theta\text{-}int(\bar{0}_K) = \bar{0}_K$ and $\tau_\theta\text{-}int(\bar{1}_K) = \bar{1}_K$.

2. τ_θ- $int(\lambda_A) \subseteq \lambda_A$.
3. if $\lambda_A \subseteq \mu_B$, then τ_θ- $int(\lambda_A) \subseteq \tau_\theta$- $int(\mu_B)$.
4. τ_θ-$int(\lambda_A \cap \mu_B) = \tau_\theta$-$int(\lambda_A) \cap \tau_\theta$-$int(\mu_B)$.
5. τ_θ-$int(\tau_\theta$-$int(\lambda_A)) = \tau_\theta$-$int(\lambda_A)$.
6. λ_A is an open fuzzy soft set if and only if $\lambda_A = \tau_\theta$-$int(\lambda_A)$.

The next theorem gives the relationships between the Čech fuzzy soft closure operator θ (respectively, interior operator Int) and the fuzzy soft topological closure τ_θ-cl (respectively, interior τ_θ-int).

Theorem 2.21 [11] Let (X, θ, K) be $\check{C}\mathcal{F}$-scs and (X, τ_θ, K) be an associative fuzzy soft topological space of (X, θ, K). Then for any $\lambda_A \in \mathcal{F}_{ss}(X, K)$

$$\tau_\theta\text{-}int(\lambda_A) \subseteq Int(\lambda_A) \subseteq \lambda_A \subseteq \theta(\lambda_A) \subseteq \tau_\theta\text{-}cl(\lambda_A). \tag{2.3}$$

Definition 2.22 [18] Let $\mathcal{F}_{ss}(X, K)$ and $\mathcal{F}_{ss}(Y, R)$ be a families of fuzzy soft sets over X and Y, respectively. Let $u: X \to Y$ and $p: K \to R$ be two functions. Then $f_{up}: \mathcal{F}_{ss}(X, K) \to \mathcal{F}_{ss}(Y, R)$ is called fuzzy soft mapping.

1. If $\lambda_A \in \mathcal{F}_{ss}(X, K)$, then the image of λ_A under the fuzzy soft mapping f_{up} is the fuzzy soft set over Y defined by $f_{up}(\lambda_A)$, where $\forall\, r \in p(K), \forall y \in Y$,

$$f_{up}(\lambda_A)(r)(y) = \begin{cases} \bigvee_{u(x)=y} \left(\bigvee_{p(h)=r} (\lambda_A(h)) \right)(x) & if\ x \in u^{-1}(y), \\ 0 & otherwise. \end{cases}$$

2. If $\mu_B \in \mathcal{F}_{ss}(Y, R)$, then the pre-image of μ_B under the fuzzy soft mapping f_{up} is the fuzzy soft set over X defined by $f_{up}^{-1}(\mu_B)$, where $\forall h \in p^{-1}(R), \forall x \in X$,

$$f_{up}^{-1}(\mu_B)(h)(x) = \begin{cases} \mu_B(p(h))(u(x)) & for\ p(h) \in B, \\ 0 & otherwise. \end{cases}$$

The fuzzy soft mapping f_{up} is called surjective (respectively, injective) if u and p are surjective (respectively, injective), also it is said to be constant if u and p are constant.

Theorem 2.23 [6] Let X and Y crisp sets $\lambda_A, (\lambda_A)_i \in \mathcal{F}_{ss}(X, K)$ and $\mu_B, (\mu_B)_i \in \mathcal{F}_{ss}(Y, R)$ for all $i \in J$, where J is an index set. Then,

1. If $(\lambda_A)_1 \subseteq (\lambda_A)_2$, then $f_{up}((\lambda_A)_1) \subseteq f_{up}((\lambda_A)_2)$.
2. If $(\mu_B)_1 \subseteq (\mu_B)_2$, then $f_{up}^{-1}((\lambda_A)_1) \subseteq f_{up}^{-1}((\lambda_A)_2)$.
3. If $\lambda_A \subseteq f_{up}^{-1}(f_{up}(\lambda_A))$, the equality holds if f_{up} is injective.
4. If $f_{up}(f_{up}^{-1}(\mu_B)) \subseteq \mu_B$, the equality holds if f_{up} is surjective.

Definition 2.24 [9] Let (X, θ, K) and (Y, θ^*, R) be two $\check{C}\mathcal{F}$-*scs's*. A fuzzy soft mapping f_{up}: $(X, \theta, K) \to (Y, \theta^*, R)$ is said to be Čech fuzzy soft continuous ($\check{C}\mathcal{F}s$-continuous, for short) mapping, if $f_{up}(\theta(\lambda_A)) \subseteq \theta^*(f_{up}(\lambda_A))$, for every fuzzy soft set λ_A of $\mathcal{F}_{ss}(X, K)$.

Theorem 2.25 [9] Let (X,θ,K) and (Y,θ^*,R) be two $\check{C}\mathcal{F}$-*scs*'s. If f_{up}: $(X,\theta,K) \to (Y,\theta^*,R)$ is a $\check{C}\mathcal{F}c$-continuous mapping, then $f_{up}^{-1}(\lambda_A)$ is an open (respectively, closed)-fss of (X,θ,K) for every open (respectively, closed)-fss fuzzy soft set λ_A of (Y,θ^*,R).

Definition 2.26 [9] Let (X,θ,K) and (Y,θ^*,R) be two $\check{C}\mathcal{F}$-*scs*'s. A fuzzy soft mapping f_{up}: $(X,\theta,K) \to (Y,\theta^*,R)$ is said to be Čech fuzzy soft open ($\check{C}\mathcal{F}s$-open, for short) mapping, if $f_{up}(\lambda_A)$ is an open fuzzy soft set of (Y,θ^*,R) whenever λ_A is an open fuzzy soft set of (X,θ,K).

3 Regularity in Čech Fuzzy Soft Closure Spaces

This section is devoted to introduce and study some new types of regularity axioms, namely quasi regular, semi-regular, pseudo regular, regular in both $\check{C}\mathcal{F}$-scs's and their associative fsts and study the relationships between them. We show that in all these types of axioms hereditary property satisfies under closed $\check{C}\mathcal{F}$-sc subspace of (X,θ,K).

Definition 3.1 A $\check{C}\mathcal{F}$-scs (X,θ,K) is said to be a quasi regular-$\check{C}\mathcal{F}$-scs, if for every fuzzy soft point x_t^h disjoint from a closed-fss ρ_C there exists an open-fss λ_A such that $x_t^h \tilde{\in} \lambda_A$ and $\theta(\lambda_A) \cap \rho_C = \bar{0}_K$.

Example 3.2 Let $X=\{a,b\}$, $K=\{h_1,h_2\}$. Define θ: $\mathcal{F}_{ss}(X,K) \to \mathcal{F}_{ss}(X,K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ \{(h_1,a_1),(h_2,a_1 \vee b_1)\} & if\ \lambda_A \subseteq \{(h_1,a_1),(h_2,a_1 \vee b_1)\}, \\ \{(h_1,b_1)\} & if\ \lambda_A \subseteq \{(h_1,b_1)\}, \\ \bar{1}_K & otherwise. \end{cases}$$

To show that (X,θ,K) is a quasi regular-$\check{C}\mathcal{F}$-scs, we must find all closed-fss's in (X,θ,K) and all fuzzy soft points which are disjoint from these closed-fss's. Thus, we have the following cases:

1- $\lambda_A=\{(h_1,a_1),(h_2,a_1 \vee b_1)\}$ is a closed-fss and $\{b_s^{h_1},\ s > 0\}$ be the set of all fuzzy soft points which is disjoint from λ_A. For any $s > 0$, there exists an open-fss $\rho_C = \{(h_1,b_1)\}$ such that $b_s^{h_1} \tilde{\in} \rho_C$ and $\theta(\rho_C) \cap \lambda_A=\bar{0}_K$.

2- $\lambda_A=\{(h_1,b_1)\}$ is a closed-fss and the fuzzy soft points which is disjoint from λ_A are: $\{a_{t_1}^{h_1},\ t_1 > 0\}$, $\{a_{t_2}^{h_2},\ t_2 > 0\}$ and $\{b_{s_2}^{h_2}, s_2 > 0\}$. For all these fuzzy soft points there exists an open-fss $\rho_C = \{(h_1,a_1),(h_2,a_1 \vee b_1)\}$ such that $a_{t_1}^{h_1} \tilde{\in} \rho_C$ and $\theta(\rho_C) \cap \lambda_A=\bar{0}_K$, $a_{t_2}^{h_2} \tilde{\in} \rho_C$ and $\theta(\rho_C) \cap \lambda_A=\bar{0}_K$ and $b_{s_2}^{h_2} \tilde{\in} \rho_C$ and $\theta(\rho_C) \cap \lambda_A=\bar{0}_K$. Hence, (X,θ,K) is quasi regular-$\check{C}\mathcal{F}$-scs.

Theorem 3.3 Every closed $\check{C}\mathcal{F}$-sc subspace (V,θ_V,K) of a quasi regular-$\check{C}\mathcal{F}$-scs (X,θ,K) is a quasi regular-$\check{C}\mathcal{F}$-sc subspace.

Proof. Let x_t^h be a fuzzy soft point in (V, θ_V, K) and ρ_C be a closed-fss in (V, θ_V, K) such that $x_t^h \cap \rho_C = \bar{0}_K$, this implies $x_t^h \tilde{\notin} \rho_C$. By Proposition 2.15, we have ρ_C be a closed-fss in (X, θ, K) not contains x_t^h. But (X, θ, K) is a quasi regular-$\check{C}\mathcal{F}$-scs. This yield, there exists an open-fss λ_A such that $x_t^h \tilde{\in} \lambda_A$ and $\theta(\lambda_A) \cap \rho_C = \bar{0}_K$. From Proposition 2.16, $\lambda_A \cap \bar{V}_K$ is an open-fss in (V, θ_V, K) and $x_t^h \tilde{\in} \lambda_A \cap \bar{V}_K$. That is mean we found an open-fss $\lambda_A \cap \bar{V}_K$ in V contains x_t^h. Now, it remain only to show $\theta_V(\lambda_A \cap \bar{V}_K) \cap \rho_C = \bar{0}_K$.

$$\begin{aligned} \theta_V(\lambda_A \cap \bar{V}_K) \cap \rho_C &= \bar{V}_K \cap \theta(\lambda_A \cap \bar{V}_K) \cap \rho_C && \text{(By Theorem 2.16)} \\ &\subseteq \bar{V}_K \cap \theta(\lambda_A) \cap \theta(\bar{V}_K) \cap \rho_C && \text{(By Proposition 2.11(2))} \\ &= \bar{V}_K \cap \theta(\lambda_A) \cap \rho_C \\ &= \bar{0}_K. \end{aligned}$$

Hence, (V, θ_V, K) is a quasi regular- $\check{C}\mathcal{F}$-sc subspace. ∎

Definition 3.4 An associative fsts (X, τ_θ, K) of (X, θ, K) is said to be quasi regular-fsts, if for every fuzzy soft points x_t^h disjoint from a closed-fss ρ_C in (X, τ_θ, K), there exists an open-fss λ_A in (X, τ_θ, K) such that $x_t^h \tilde{\in} \lambda_A$ and $\tau_\theta\text{-}cl(\lambda_A) \cap \rho_C = \bar{0}_K$.

Theorem 3.5 If (X, τ_θ, K) is a quasi regular-fsts, then (X, θ, K) is also a quasi regular-$\check{C}\mathcal{F}$-scs.

Proof. Let x_t^h be a fuzzy soft point disjoint from a closed-fss ρ_C in (X, θ, K). That means $x_t^h \tilde{\notin} \rho_C$. Since ρ_C is a closed-fss in (X, θ, K). Then ρ_C is a closed-fss in (X, τ_θ, K). But (X, τ_θ, K) is a quasi regular-fsts. Therefore, there exists τ_θ-open-fss λ_A such that $x_t^h \tilde{\in} \lambda_A$ and $\tau_\theta\text{-}cl(\lambda_A) \cap \rho_c = \bar{0}_K$. From Theorem 2.21, we get $\theta(\lambda_A) \cap \rho_C = \bar{0}_K$. Hence, (X, θ, K) is a quasi regular-$\check{C}\mathcal{F}$-scs. ∎

Definition 3.6 A $\check{C}\mathcal{F}$-scs (X, θ, K) is said to be semi-regular-$\check{C}\mathcal{F}$-scs, if for every fuzzy soft points x_t^h disjoint from a closed-fss ρ_C, there exists an open-fss λ_A such that $\rho_C \subseteq \lambda_A$ and $x_t^h \tilde{\notin} \theta(\lambda_A)$.

Example 3.7 Let $X=\{a, b\}$, $K=\{h_1, h_2\}$, and let $(\lambda_A)_i \in \mathcal{F}_{ss}(X, K), i = 1,2,3,4$, such that

$$(\lambda_A)_1=\{(h_1, a_{0.5}), (h_2, a_1 \vee b_1)\}, \quad (\lambda_A)_2=\{(h_1, a_1), (h_2, a_1 \vee b_1)\},$$

$$(\lambda_A)_3=\{(h_1, b_1)\} \text{ and } \quad (\lambda_A)_4=\{(h_1, a_{0.5} \vee b_1), (h_2, a_1 \vee b_1)\}.$$

Define $\theta: \mathcal{F}_{ss}(X, K) \to \mathcal{F}_{ss}(X, K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \ \lambda_A = \bar{0}_K, \\ (\lambda_A)_1 & if\ \ \lambda_A \subseteq (\lambda_A)_1, \\ (\lambda_A)_2 & if (\lambda_A)_1 \subset \lambda_A \subseteq (\lambda_A)_2, \\ (\lambda_A)_3 & if\ \ \lambda_A \subseteq (\lambda_A)_3, \\ (\lambda_A)_4 & if (\lambda_A)_1 \neq \lambda_A \subseteq (\lambda_A)_4, \\ \bar{1}_K & otherwise. \end{cases}$$

To show that (X,θ,K) is semi-regular-$\check{C}\mathcal{F}$-scs, we must find all fuzzy soft points which is disjoint from a closed-fss's in (X,θ,K). Thus we have the following cases:

1- $(\lambda_A)_1$={$(h_1,a_{0.5}),(h_2,a_1\vee b_1)$} is a closed-fss and $\{b_{s_1}^{h_1},s_1>0\}$ be the set of all fuzzy soft points which is disjoint from $(\lambda_A)_1$. For any $s_1>0$, there exists an open-fss $(\lambda_A)_2$={$(h_1,a_1),(h_2,a_1\vee b_1)$} such that $(\lambda_A)_1\subseteq(\lambda_A)_2$ and $b_s^{h_1}\tilde{\notin}\theta((\lambda_A)_2)$.

2- $(\lambda_A)_2$={$(h_1,a_1),(h_2,a_1\vee b_1)$} is a closed-fss and $\{b_{s_1}^{h_1},s_1>0\}$ be the set of all fuzzy soft points which is disjoint from $(\lambda_A)_2$. For any $s_1>0$, there exists an open-fss $(\lambda_A)_2$={$(h_1,a_1),(h_2,a_1\vee b_1)$} such that $(\lambda_A)_2\subseteq(\lambda_A)_2$ and $b_s^{h_1}\tilde{\notin}\theta((\lambda_A)_2)$.

3- $(\lambda_A)_3$={(h_1,b_1)} is a closed-fss and the fuzzy soft points which is disjoint from $(\lambda_A)_3$ are: $\{a_{t_1}^{h_1},t_1>0\}$, $\{a_{t_2}^{h_2},t_2>0\}$ and $\{b_{s_2}^{h_2},s_2>0\}$. For all these fuzzy soft points there exists an open-fss $(\lambda_A)_3$ satisfied the required conditions of semi-regular- $\check{C}\mathcal{F}$-scs. Then (X,θ,K) is semi-regular- $\check{C}\mathcal{F}$-scs.

Theorem 3.8 Every closed $\check{C}\mathcal{F}$-sc subspace (V,θ_V,K) of a semi-regular-$\check{C}\mathcal{F}$-scs (X,θ,K) is a semi-regular-$\check{C}\mathcal{F}$-sc subspace.

Proof. Let x_t^h be a fuzzy soft point in (V,θ_V,K) and ρ_C be a closed-fss in (V,θ_V,K) such that $x_t^h\cap\rho_C=\bar{0}_K$, then $x_t^h\tilde{\notin}\rho_C$. By Proposition 2.15, ρ_C is a closed-fss (X,θ,K) not contains x_t^h. But (X,θ,K) is a semi-regular-$\check{C}\mathcal{F}$-scs. Then there exists an open-fss λ_A such that $\rho_C\subseteq\lambda_A$ and $x_t^h\tilde{\notin}\theta(\lambda_A)$. Now, $\rho_C\subseteq\lambda_A$ and $\rho_C\subseteq\bar{V}_K$, this implies $\rho_c\subseteq\lambda_A\cap\bar{V}_K$ which is an open-fss from Proposition 2.16. Next, we must show $x_t^h\tilde{\notin}\theta_V(\lambda_A\cap\bar{V}_K)$. Suppose, $x_t^h\tilde{\in}\theta_V(\lambda_A\cap\bar{V}_K)=\bar{V}_K\cap\theta(\lambda_A\cap\bar{V}_K)\subseteq\bar{V}_K\cap\theta(\lambda_A)\cap\theta(\bar{V}_K)$, it follows $x_t^h\tilde{\in}\theta(\lambda_A)$ which is a contradiction. Hence (V,θ_V,K) is a semi-regular-$\check{C}\mathcal{F}$-sc subspace. ■

Definition 3.9 An associative fuzzy soft topological space (X,τ_θ,K) of (X,θ,K) is said to be a semi-regular-fsts, if for every fuzzy soft points x_t^h disjoint from a closed-fss ρ_C in (X,τ_θ,K), there exists an open-fss λ_A in(X,τ_θ,K) such that $\rho_C\subseteq\lambda_A$ and $x_t^h\tilde{\notin}\tau_\theta\text{-}cl(\lambda_A)$.

Theorem 3.10 If (X,τ_θ,K) is a semi-regular-fsts, then (X,θ,K) is also a semi-regular-$\check{C}\mathcal{F}$-scs.

Proof. Let x_t^h be a fuzzy soft point disjoint from a closed-fss ρ_C in (X,θ,K). That means $x_t^h\tilde{\notin}\rho_C$. Since ρ_C is a closed-fss in (X,θ,K). It follows that ρ_C is a closed-fss in (X,τ_θ,K). But (X,τ_θ,K) is a semi-regular-fsts. It follows, there exists τ_θ-open-fss λ_A such that $\rho_C\subseteq\lambda_A$ and $x_t^h\tilde{\notin}\tau_\theta\text{-}cl(\lambda_A)$. From Theorem 2.21, we get $x_t^h\tilde{\notin}\theta(\lambda_A)$. Hence, (X,θ,K) is a semi-regular-$\check{C}\mathcal{F}$-scs. ■

Definition 3.11 A $\check{C}\mathcal{F}$-scs (X,θ,K) is said to be pseudo regular-$\check{C}\mathcal{F}$-scs, if it is both quasi regular-$\check{C}\mathcal{F}$-scs and semi-regular-$\check{C}\mathcal{F}$-scs.

Example 3.12 In Example 3.2, (X,θ,K) is pseudo regular-$\check{C}\mathcal{F}$-scs.

The next theorem follows directly from Theorem 3.3 and Theorem 3.8.

Theorem 3.13 Every closed $\check{C}\mathcal{F}$-sc subspace (V, θ_V, K) of pseudo regular-$\check{C}\mathcal{F}$-scs (X, θ, K) is pseudo regular-$\check{C}\mathcal{F}$-sc subspace.

Definition 3.14 An associative fsts (X, τ_θ, K) of (X, θ, K) is said to be pseudo regular-fsts, if it is both quasi regular-fsts and semi-regular-fsts.

Theorem 3.15 If (X, τ_θ, K) is a pseudo regular-fsts, then (X, θ, K) is also a pseudo-regular-$\check{C}\mathcal{F}$-scs.

Proof. The proof follows directly from Theorem 3.5 and Theorem 3.10. ■

Definition 3.16 A $\check{C}\mathcal{F}$-scs (X, θ, K) is said to be regular-$\check{C}\mathcal{F}$-scs, if for every fuzzy soft points x_t^h disjoint from a closed-fss ρ_C, there exist open-fss's λ_A and μ_B such that $x_t^h \tilde{\in} \lambda_A$, $\rho_C \subseteq \mu_B$ and $\lambda_A \cap \mu_B = \bar{0}_K$.

Example 3.17 Let $X=\{a, b\}$, $K=\{h_1, h_2\}$ and $(\lambda_A)_1, (\lambda_A)_2 \in \mathcal{F}_{ss}(X, K)$ such that

$$(\lambda_A)_1=\{(h_1, b_1), (h_2, a_1 \vee b_1)\} \quad and \quad (\lambda_A)_2=\{(h_1, a_1)\}.$$

Define θ: $\mathcal{F}_{ss}(X, K) \to \mathcal{F}_{ss}(X, K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if \ \lambda_A = \bar{0}_K, \\ (\lambda_A)_1 & if \ \lambda_A \subseteq (\lambda_A)_1, \\ (\lambda_A)_2 & if \ \lambda_A \subseteq (\lambda_A)_2, \\ \bar{1}_K & other\ wise. \end{cases}$$

Then (X, θ, K) is regular $\check{C}\mathcal{F}$-scs.

Theorem 3.18 Every closed $\check{C}\mathcal{F}$-sc subspace (V, θ_V, K) of a regular-$\check{C}\mathcal{F}$-scs (X, θ, K) is a regular-$\check{C}\mathcal{F}$-sc subspace.

Proof. Let x_t^h be a fuzzy soft point in (V, θ_V, K) and ρ_C be a closed-fss in (V, θ_V, K) such that $x_t^h \cap \rho_C = \bar{0}_K$, then $x_t^h \tilde{\notin} \rho_C$. By Proposition 2.15, ρ_C is a closed-fss in (X, θ, K) not contains x_t^h. But (X, θ, K) is regular-$\check{C}\mathcal{F}$-scs. Then there exist open-fss's λ_A and μ_B such that $x_t^h \tilde{\in} \lambda_A$, $\rho_C \subseteq \mu_B$ and $\lambda_A \cap \mu_B = \bar{0}_K$. Thus, we have $x_t^h \tilde{\in} \lambda_A \cap \bar{V}_K$ and $\rho_C \subseteq \mu_B \cap \bar{V}_K$ and from Proposition 2.16, $\lambda_A \cap \bar{V}_K$ and $\mu_B \cap \bar{V}_K$ are open-fss's in (V, θ_V, K). Moreover, it is clear that $(\lambda_A \cap \bar{V}_K) \cap (\mu_B \cap \bar{V}_K) = \bar{0}_K$. Hence, (V, θ_V, K) is a regular-$\check{C}\mathcal{F}$-scs. ■

Definition 3.19 An associative fsts (X, τ_θ, K) of (X, θ, K) is said to be regular-fsts, if for every fuzzy soft point x_t^h disjoint from a closed-fss ρ_C, there exist open-fss's λ_A, μ_B such that $x_t^h \tilde{\in} \lambda_A$, $\rho_C \subseteq \mu_B$, and $\lambda_A \cap \mu_B = \bar{0}_K$.

Theorem 3.20 An associative fuzzy soft topological (X, τ_θ, K) is regular-fsts if and only if (X, θ, K) is regular-$\check{C}\mathcal{F}$-scs.

Proof. Let x_t^h be fuzzy soft point disjoint from a closed-fss ρ_C in (X, θ, K), Since (X, τ_θ, K) is regular-fsts, there exist λ_A and μ_B open-fss's in (X, τ_θ, K) such that $x_t^h \tilde{\in} \lambda_A$, $\rho_C \subseteq \mu_B$, and $\lambda_A \cap \mu_B = \bar{0}_K$. From Theorem 2.21, we get λ_A and μ_B are open-fss's in (X, θ, K) such that $x_t^h \tilde{\in} \lambda_A$, $\rho_C \subseteq \mu_B$, and $\lambda_A \cap \mu_B = \bar{0}_K$. Thus, (X, θ, K) is regular-$\check{C}\mathcal{F}$-scs.

Conversely, similar to the first direction. ■

Definition 3.21 Let (X,θ,K) and (Y,θ^*,R) be two ČƑ-*scs's*. A fuzzy soft mapping f_{up}: $(X,\theta,K) \rightarrow (Y,\theta^*,R)$ is said to be Čech fuzzy soft homeomorphism (ČƑs-homeomorphism, for short) mapping, if f_{up} is injective, surjective, ČƑs-continuous and f_{up}^{-1} is ČƑs-continuous mapping.

Proposition 3.22 A fuzzy soft mapping f_{up}: $(X,\theta,K) \rightarrow (Y,\theta^*,R)$ ČƑs-homeomorphism mapping if and only if f_{up} is injective, surjective, ČƑs-continuous and ČƑs-open mapping.

Proof. The proof follows directly from the definition of ČƑs-homeomorphism mapping.

Proposition 3.23 Let $f_{up}: \mathcal{F}_{ss}(X,K) \rightarrow \mathcal{F}_{ss}(Y,R)$ be a fuzzy soft mapping and let x_t^h be a fuzzy soft point in X. Then the image of x_t^h under the fuzzy soft mapping f_{up} is a fuzzy soft point in Y, which is defined as $f_{up}(x_t^h) = u(x)_t^{p(h)}$.

Proof. Let x_t^h be a fuzzy soft point in X. Then from Definition 2.22(1), we have

$$f_{up}(x_t^h)(r)(y) = \begin{cases} \vee_{u(z)=y}\left(\vee_{p(h')=r}\left(x_t^h(h')\right)\right)(z) & \text{if } z \in u^{-1}(y), \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} \vee_{u(z)=y}\left(\begin{cases} x_t & \text{if } h = h' \\ \bar{0}_X & \text{if } h \neq h' \end{cases}\right)(z) & \text{if } z \in u^{-1}(y), \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} \vee_{u(z)=y}(x_t)(z) & \text{if } z \in u^{-1}(y), \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} t & \text{if } x = z, \\ 0 & \text{if } x \neq z. \end{cases}$$
$$= u(x)_t^{p(h)}. \quad ■$$

Proposition 3.24 Let $f_{up}: \mathcal{F}_{ss}(X,K) \rightarrow \mathcal{F}_{ss}(Y,R)$ be a bijective fuzzy soft mapping and let y_s^r be a fuzzy soft point in Y. Then the inverse image of y_s^r under the fuzzy soft mapping f_{up} is a fuzzy soft point in X, which is defined as $f_{up}^{-1}(y_s^r) = x_s^h, p(h) = r$ and $u(x) = y$.

Proof. Let y_s^r be a fuzzy soft point in Y. Then from Definition 2.22(2), we have

$$f_{up}^{-1}(y_s^r)(h)(x) = \begin{cases} y_s^r(p(h))(u(x)) & \text{for } h \in K, \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} y_s(u(x)) & \text{if } p(h) = r, \\ \bar{0}_K & \text{if } p(h) \neq r, \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} s & \text{if } u(x) = y, \\ 0 & \text{otherwise.} \end{cases}$$
$$= x_s^h. \quad ■$$

Theorem 3.25 The property of being regular-ČƑ-scs is topological property.

Proof. Let (X,θ,K) and (Y,θ^*,R) be any two $\check{C}\mathcal{F}$-scs and let $f_{up}:(X,\theta,K)\to(Y,\theta^*,R)$ be a $\check{C}\mathcal{F}s$-homeomorphism mapping and (X,θ,K) is regular-$\check{C}\mathcal{F}$-scs. We want to show (Y,θ^*,R) is also regular-$\check{C}\mathcal{F}$-scs. Let y_s^r be a fuzzy soft point in (Y,θ^*,R) and ρ_C be a closed-fss in (Y,θ^*,R) such that $y_s^r\cap\rho_C=\bar{0}_R$. Since f_{up} is $\check{C}\mathcal{F}s$-homeomorphism mapping, then $f_{up}^{-1}(y_s^r)$ is a fuzzy soft point and $f_{up}^{-1}(\rho_C)$ is a closed-fss in (X,θ,K) such that $f_{up}^{-1}(y_s^r)\cap f_{up}^{-1}(\rho_C)=\bar{0}_K$. But (X,θ,K) is regular-$\check{C}\mathcal{F}$-scs this implies there exist disjoint open-fss's λ_A and μ_B such that $f_{up}^{-1}(y_s^r)\subseteq\lambda_A$ and $f_{up}^{-1}(\rho_C)\subseteq\mu_B$. It follows, $f_{up}(f_{up}^{-1}(y_s^r))\subseteq f_{up}(\lambda_A)$ and $f_{up}(f_{up}^{-1}(\rho_C))\subseteq f_{up}(\mu_B)$. Since f_{up} is $\check{C}\mathcal{F}s$-homeomorphism mapping, then f_{up} is $\check{C}\mathcal{F}s$-open mapping, this yields there exist open-fss's $f_{up}(\lambda_A)$ and $f_{up}(\mu_B)$ in (Y,θ^*,R) such that $y_s^r\subseteq f_{up}(\lambda_A)$ and $\rho_C\subseteq f_{up}(\mu_B)$. Moreover, $f_{up}(\lambda_A)\cap f_{up}(\mu_B)=\bar{0}_R$. Hence, (Y,θ^*,R) is also regular-$\check{C}\mathcal{F}$-scs. ∎

4 Normality in Čech Fuzzy Soft Closure Spaces

In this section, some normality axioms, namely semi-normal, pseudo normal, normal and completely normal in both $\check{C}\mathcal{F}$-scs's and their associative fsts's and study the relationships between them, and study their basic properties as in the previous section.

Definition 4.1 A $\check{C}\mathcal{F}$-scs (X,θ,K) is said to be semi-normal-$\check{C}\mathcal{F}$-scs, if for each pair of disjoint closed-fss's ρ_C and η_D, either there exists an open-fss λ_A such that $\rho_C\subseteq\lambda_A$ and $\theta(\lambda_A)\cap\eta_D=\bar{0}_K$ or there exists an open-fss μ_B such that $\eta_D\subseteq\mu_B$ and $\theta(\mu_B)\cap\rho_C=\bar{0}_K$.

If the both conditions hold, then (X,θ,K) is said to be pseudo normal-$\check{C}\mathcal{F}$-scs.

Example 4.2 Let $X=\{a,b\}$, $K=\{h_1,h_2\}$, and $(\lambda_A)_1,(\lambda_A)_2\in\mathcal{F}_{ss}(X,K)$ such that

$$(\lambda_A)_1=\{(h_1,a_1\vee b_1)\}\quad\text{and}\quad(\lambda_A)_2=\{(h_2,a_1\vee b_1)\}.$$

Define $\theta:\mathcal{F}_{ss}(X,K)\to\mathcal{F}_{ss}(X,K)$ as follows:

$$\theta(\lambda_A)=\begin{cases}\bar{0}_K & if\ \lambda_A=\bar{0}_K,\\ (\lambda_A)_1 & if\ \lambda_A\subseteq(\lambda_A)_1,\\ (\lambda_A)_2 & if\ \lambda_A\subseteq(\lambda_A)_2,\\ \bar{1}_K & otherwise.\end{cases}$$

Then (X,θ,K) is semi-normal-$\check{C}\mathcal{F}$-scs. Since the only disjoint closed-fss's are $(\lambda_A)_1$, $(\lambda_A)_2$ and there exists an open-fss $(\lambda_A)_1$ such that $(\lambda_A)_1\subseteq(\lambda_A)_1$ and $\theta((\lambda_A)_1)\cap(\lambda_A)_2=(\lambda_A)_1\cap(\lambda_A)_2=\bar{0}_K$.

Theorem 4.3 Every closed $\check{C}\mathcal{F}$-sc subspace (V,θ_V,K) of semi-normal-$\check{C}\mathcal{F}$-scs (X,θ,K) is a semi-normal-$\check{C}\mathcal{F}$-sc subspace.

Proof. Let ρ_C and η_D be closed-fss's in (V,θ_V,K) such that $\rho_C\cap\eta_D=\bar{0}_K$. Since $\bar{V}_K$ is closed-fss in (X,θ,K). Then by Proposition 2.15, ρ_C and η_D are disjoint closed-fss's in (X,θ,K). But (X,θ,K) is semi-normal-$\check{C}\mathcal{F}$-scs, it follows there exist an open-fss λ_A such that

$\rho_C \subseteq \lambda_A$ and $\theta(\lambda_A) \cap \eta_D = \bar{0}_K$. Since $\rho_C \subseteq \lambda_A$, then $\rho_C \subseteq \lambda_A \cap \bar{V}_K$ which is open-fss in (V, θ_V, K). And

$$\begin{aligned}\theta_V(\lambda_A \cap \bar{V}_K) \cap \eta_D &= \bar{V}_K \cap \theta(\lambda_A \cap \bar{V}_K) \cap \eta_D \\ &\subseteq \bar{V}_K \cap \theta(\lambda_A) \cap \theta(\bar{V}_K) \cap \eta_D \\ &= \bar{V}_K \cap \theta(\lambda_A) \cap \eta_D \\ &= \bar{0}_K.\end{aligned}$$

Similarly, if there exists an open-fss μ_B such that $\eta_D \subseteq \mu_B$ and $\theta(\mu_B) \cap \rho_C = \bar{0}_K$. We have an open-fss $\mu_B \cap \bar{V}_K$ in (V, θ_V, K) such that $\eta_D \subseteq \mu_B \cap \bar{V}_K$ and $\theta_V(\mu_B \cap \bar{V}_K) \cap \rho_C = \bar{0}_K$. Hence (V, θ_V, K) is a semi-normal- $\check{C}\mathcal{F}$-sc subspace. ■

Definition 4.4 An associative fsts (X, τ_θ, K) of (X, θ, K) is said to be a semi-normal-fsts, if for each pair of disjoint closed-fss's ρ_C and η_D, either there exists an open-fss λ_A such that $\rho_C \subseteq \lambda_A$ and $\tau_\theta\text{-}cl(\lambda_A) \cap \eta_D = \bar{0}_K$, or there exists an open-fss μ_B such that $\eta_D \subseteq \mu_B$ and $\tau_\theta\text{-}cl(\mu_B) \cap \rho_C = \bar{0}_K$.

Theorem 4.5 If (X, τ_θ, K) is a semi-normal-fsts, then (X, θ, K) is also semi- normal-$\check{C}\mathcal{F}$-scs.

Proof. Let ρ_C and η_D be disjoint closed-fss's in (X, θ, K). Then ρ_C and η_D be disjoint closed-fss's in (X, τ_θ, K). By hypothesis, there exists an open-fss λ_A such that $\rho_C \subseteq \lambda_A$ and $\tau_\theta\text{-}cl(\lambda_A) \cap \eta_D = \bar{0}_K$, or there exists an open-fss μ_B such that $\eta_D \subseteq \mu_B$ and $\tau_\theta\text{-}cl(\mu_B) \cap \rho_C = \bar{0}_K$. From Theorem 2.21, we get either there exists an open-fss λ_A in (X, θ, K) such that $\rho_C \subseteq \lambda_A$ and $\theta(\lambda_A) \cap \eta_D = \bar{0}_K$, or there exists an open-fss μ_B in (X, θ, K) such that $\eta_D \subseteq \mu_B$ and $\theta(\mu_B) \cap \rho_C = \bar{0}_K$. Hence, (X, θ, K) is semi-normal-$\check{C}\mathcal{F}$-scs. ■

Definition 4.6 A $\check{C}\mathcal{F}$-scs (X, θ, K) is said to be normal-$\check{C}\mathcal{F}$-scs, if for each pair of disjoint closed-fss's ρ_C and η_D, there exist disjoint open-fss's λ_A and μ_B such that $\rho_C \subseteq \lambda_A$ and $\eta_D \subseteq \mu_B$.

Example 4.7 Let $X = \{a, b\}$, $K = \{h_1, h_2\}$ and let $(\lambda_A)_i \in \mathcal{F}_{ss}(X, K)$, $i = 1,2,3,4$, such that

$$(\lambda_A)_1 = \{(h_1, a_{0.5})\}, \quad (\lambda_A)_2 = \{(h_2, a_{0.5})\},$$

$$(\lambda_A)_3 = \{(h_1, a_{0.5} \vee b_{0.5}), (h_2, a_1 \vee b_1)\} \text{ and } (\lambda_A)_4 = \{(h_1, a_1 \vee b_1), (h_2, a_{0.5} \vee b_{0.5})\}.$$

Define $\theta: \mathcal{F}_{ss}(X, K) \to \mathcal{F}_{ss}(X, K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ (\lambda_A)_1 & if\ \lambda_A \subseteq (\lambda_A)_1, \\ (\lambda_A)_2 & if\ \lambda_A \subseteq (\lambda_A)_2, \\ (\lambda_A)_1 \cup (\lambda_A)_2 & if \lambda_A \subseteq (\lambda_A)_1 \cup (\lambda_A)_2 \\ (\lambda_A)_3 & if \lambda_A \in \left\{ \begin{matrix} (h_1, a_{t_1} \vee b_{s_1}), (h_2, a_{t_2} \vee b_{s_2}); \\ t_1, s_1 \le 0.5, 0.5 < t_2, s_2 \le 1 \end{matrix} \right\}, \\ (\lambda_A)_4 & if \lambda_A \in \left\{ \begin{matrix} (h_1, a_{t_1} \vee b_{s_1}), (h_2, a_{t_2} \vee b_{s_2}); \\ t_2, s_2 \le 0.5, 0.5 < t_1, s_1 \le 1 \end{matrix} \right\}, \\ \bar{1}_K & otherwise. \end{cases}$$

Then (X,θ,K) normal-$\check{C}\mathcal{F}$-scs. Since the only disjoint closed-fss's are $(\lambda_A)_1$, $(\lambda_A)_2$ and there exist disjoint open-fss's $\lambda_A = \{(\mathrm{h}_1, a_{0.5} \vee b_{0.5})\}$ and $\mu_B = \{(\mathrm{h}_2, a_{0.5} \vee b_{0.5})\}$ such that $(\lambda_A)_1 \subseteq \lambda_A$ and $(\lambda_A)_2 \subseteq \mu_B$.

Theorem 4.8 Every closed $\check{C}\mathcal{F}$-sc subspace (V,θ_V,K) of normal-$\check{C}\mathcal{F}$-scs (X,θ,K) is normal-$\check{C}\mathcal{F}$-sc subspace.

Proof. Similar of Theorem 4.3. ■

Definition 4.9 An associative fsts (X,τ_θ,K) of (X,θ,K) is said to be normal-fsts, if for each pair of disjoint closed-fss's ρ_c and η_D in (X,τ_θ,K) there exist disjoint τ_θ-open-fss's λ_A, $\mu_B$in (X,τ_θ,K) such that $\rho_C \subseteq \lambda_A$ and $\eta_D \subseteq \mu_B$.

Theorem 4.10 (X,τ_θ,K) is normal-fsts if and only if (X,θ,K) is normal-$\check{C}\mathcal{F}$-scs.

Proof. The proof follows from the hypothesis and Theorem 2.21. ■

Theorem 4.11 The property of being normal-$\check{C}\mathcal{F}$-scs is topological property.

Proof. Let (X,θ,K) and (Y,θ^*,R) be any two $\check{C}\mathcal{F}$-scs and let $f_{up}\colon (X,\theta,K) \to (Y,\theta^*,R)$ be a $\check{C}\mathcal{F}s$-homeomorphism mapping and (X,θ,K) is normal-$\check{C}\mathcal{F}$-scs. We want to show (Y,θ^*,R) is also normal -$\check{C}\mathcal{F}$-scs. Let ρ_C and η_D be disjoint closed-fss's in (Y,θ^*,R). From hypothesis, f_{up} is is $\check{C}\mathcal{F}s$-continuous mapping and from Theorem 2.25, we get $f_{up}^{-1}(\rho_C)$ and $f_{up}^{-1}(\eta_D)$ are closed-fss's in (X,θ,K) such that $f_{up}^{-1}(\rho_C) \cap f_{up}^{-1}(\eta_D) = \bar{0}_K$. But (X,θ,K) is normal-$\check{C}\mathcal{F}$-scs. This implies, there exist disjoint open-fss's λ_A and μ_B such that $f_{up}^{-1}(\rho_C) \subseteq \lambda_A$ and $f_{up}^{-1}(\eta_D) \subseteq \mu_B$. It follows, $f_{up}(f_{up}^{-1}(\rho_C)) \subseteq f_{up}(\lambda_A)$ and $f_{up}(f_{up}^{-1}(\eta_D)) \subseteq f_{up}(\mu_B)$. Since f_{up} is $\check{C}\mathcal{F}s$-homeomorphism mapping, then f_{up} is $\check{C}\mathcal{F}s$-open mapping, this yields there exist open-fss's $f_{up}(\lambda_A)$ and $f_{up}(\mu_B)$ in (Y,θ^*,R) such that $\rho_C \subseteq f_{up}(\lambda_A)$ and $\eta_D \subseteq f_{up}(\mu_B)$. Moreover, $f_{up}(\lambda_A) \cap f_{up}(\mu_B) = \bar{0}_R$. Hence, (Y,θ^*,R) is also normal-$\check{C}\mathcal{F}$-scs. ■

Definition 4.12 A $\check{C}\mathcal{F}$-scs (X,θ,K) is said to be completely normal-$\check{C}\mathcal{F}$-scs, if for each pair of disjoint closed-fss's ρ_C and η_D there exist disjoint open-fss's λ_A and μ_B such that $\rho_C \subseteq \lambda_A$ and $\eta_D \subseteq \mu_B$ and $\theta(\lambda_A) \cap \theta(\mu_B)=\bar{0}_K$.

Example 4.13 Let $X=\{a,b\}$, $K=\{h_1,h_2\}$ and let $(\lambda_A)_i \in \mathcal{F}_{ss}(\mathrm{X},K)$, $i = 1,2,3,4,5$, such that

$$(\lambda_A)_1=\{(h_1,a_1)\},\quad (\lambda_A)_2=\{(h_1,b_1)\},\quad (\lambda_A)_3=\{(h_1,b_1),(h_2,a_1 \vee b_1)\},$$

$$(\lambda_A)_4=\{(h_1,a_1),(h_2,a_1 \vee b_1)\} \quad \text{and} \quad (\lambda_A)_5=\{(h_1,a_1 \vee b_1)\}.$$

Define $\theta\colon \mathcal{F}_{ss}(\mathrm{X},K) \to \mathcal{F}_{ss}(\mathrm{X},K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ (\lambda_A)_1 & if\ \lambda_A \subseteq (\lambda_A)_1, \\ (\lambda_A)_2 & if\ \lambda_A \subseteq (\lambda_A)_2, \\ (\lambda_A)_3 & if (\lambda_A)_2 \neq \lambda_A \subseteq (\lambda_A)_3, \\ (\lambda_A)_4 & if (\lambda_A)_1 \neq \lambda_A \subseteq (\lambda_A)_4, \\ (\lambda_A)_5 & if (\lambda_A)_1, (\lambda_A)_2 \neq \lambda_A \subseteq (\lambda_A)_5, \\ \bar{1}_K & otherwise. \end{cases}$$

Then (X, θ, K) is completely normal $\check{C}\mathcal{F}$-scs. Since the only disjoint closed-fss's are $(\lambda_A)_1$, $(\lambda_A)_2$ and there exist $(\lambda_A)_1$ and $(\lambda_A)_2$ are disjoint open-fss's such that $(\lambda_A)_1 \subseteq (\lambda_A)_1$, $(\lambda_A)_2 \subseteq (\lambda_A)_2$ and $\theta((\lambda_A)_1) \cap \theta((\lambda_A)_2) = \bar{0}_K$.

Proposition 4.14 Every completely normal-$\check{C}\mathcal{F}$-scs is normal-$\check{C}\mathcal{F}$-scs.

Proof. Suppose (X, θ, K) is completely normal-$\check{C}\mathcal{F}$-scs and let ρ_C, η_D be any disjoint closed-fss's in (X, θ, K). From hypothesis, there exist disjoint open-fss's λ_A and μ_B such that $\rho_C \subseteq \lambda_A$, $\eta_D \subseteq \mu_B$ and $\theta(\lambda_A) \cap \theta(\mu_B) = \bar{0}_K$. By using (C2) of Definition 2.9, we have $\lambda_A \cap \mu_B = \bar{0}_K$. Thus (X, θ, K) is normal- $\check{C}\mathcal{F}$-scs .■

Remark 4.15 The converse of Proposition 4.14 is not true, as Example 4.7.

Theorem 4.16 Every closed $\check{C}\mathcal{F}$-sc subspace (V, θ_V, K) of completely normal- $\check{C}\mathcal{F}$-scs (X, θ, K) is a completely normal-$\check{C}\mathcal{F}$-sc subspace.

Proof. Let ρ_C, η_D be any two disjoint closed-fss's in (V, θ_V, K). Then by Proposition 2.15, ρ_C, η_D are disjoint closed-fss's (X, θ, K). But (X, θ, K) is completely normal-$\check{C}\mathcal{F}$-scs, then there exist λ_A, μ_B disjoint open-fss's such that $\rho_C \subseteq \lambda_A$, $\eta_D \subseteq \mu_B$ and $\theta(\lambda_A) \cap \theta(\mu_B) = \bar{0}_K$. By Proposition 2.16, $\lambda_A \cap \bar{V}_K$ and $\mu_B \cap \bar{V}_K$ are open-fss's in (V, θ_V, K) such that $\rho_C \subseteq \lambda_A \cap \bar{V}_K$ and $\eta_D \subseteq \mu_B \cap \bar{V}_K$. To complete the proof, we must show $\theta_V(\lambda_A \cap \bar{V}_K) \cap \theta_V(\mu_B \cap \bar{V}_K) = \bar{0}_K$. Now,

$$\begin{aligned} \theta_V(\lambda_A \cap \bar{V}_K) \cap \theta_V(\mu_B \cap \bar{V}_K) &= \bar{V}_K \cap \theta(\lambda_A \cap \bar{V}_K) \cap \bar{V}_K \cap \theta(\mu_B \cap \bar{V}_K) \\ &\subseteq \bar{V}_K \cap \theta(\bar{V}_K) \cap \theta(\lambda_A) \cap \theta(\bar{V}_K) \cap \theta(\mu_B) \\ &= \bar{V}_K \cap \theta(\lambda_A) \cap \theta(\mu_B) \\ &= \bar{0}_K. \end{aligned}$$

Hence, (V, θ_V, K) completely normal-$\check{C}\mathcal{F}$-sc subspace. ■

Definition 4.17 An associative fsts space (X, τ_θ, K) of (X, θ, K) is said to be completely normal-fsts, if for each pair of disjoint closed-fss's ρ_C and η_D there exists disjoint open-fss's λ_A and μ_B in (X, τ_θ, K) such that $\rho_C \subseteq \lambda_A$, $\eta_D \subseteq \mu_B$ and τ_θ-*cl*$(\lambda_A) \cap \tau_\theta$-*cl*$(\mu_B) = \bar{0}_K$.

Theorem4.18 If (X, τ_θ, K) is a completely normal-fsts. Then (X, θ, K) is also completely normal-$\check{C}\mathcal{F}$-scs.

Proof. The proof follows from hypothesis and by Theorem 2.21. ■

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