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ENERGY CONDITIONS FOR SOME HAMILTONIAN PROPERTIES OF GRAPHS

Rao Li

The energy of a graph is defined as the sum of the absolute values of the eigenvalues of the graph. In this note, we present energy conditions for some Hamiltonian properties of graphs.

1. INTRODUCTION

All the graphs considered in this note are undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [**1**]. Let G be a graph of order n with e edges. We use $\delta = \delta(G)$ and $\Delta = \Delta(G)$ to denote the minimum and maximum degrees of G, respectively. The 2-degree, denoted $t(v)$, of a vertex v in G is defined as the sum of degrees of vertices adjacent to v. We use $T = T(G)$ to denote the maximum 2-degree of G. Obviously, $T(G) \leq (\Delta(G))^2$. A bipartite graph G is called semiregular if all the vertices in the same vertex part of a bipartition of the vertex set of G have the same degree. The independence number, denoted $\alpha = \alpha(G)$, is defined as the size of the largest independent set in G. The eigenvalues $\mu_1(G) \geq \mu_2(G) \geq ... \geq \mu_n(G)$ of the adjacency matrix $A(G)$ of G are called the eigenvalues of G. The spread, denoted $Spr(G)$, of G is defined as $\mu_1(G) - \mu_n(G)$. The energy, denoted $Eng(G)$, of G is defined as $\sum_{i=1}^{n} |\mu_i(G)|$ (see [**5**]). A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. In this note, we will present energy conditions for Hamiltonicity and traceability of graphs. The main results are as follows.

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Theorem 1. Let G be a k-connected ($k \geq 2$) graph with $n \geq 3$ vertices and e edges. If

$$Eng(G) \geq 2\sqrt{e} + \sqrt{2(n-2)\left(e + \sqrt{T\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor} - \frac{2\delta^2(k+1)}{n-k-1}\right)},$$

then G is Hamiltonian or G is $K_{k,\,k+1}$ with $n = 2k+1$.

Theorem 2. Let G be a k-connected graph with $n \geq 2$ vertices and e edges. If

$$Eng(G) \geq 2\sqrt{e} + \sqrt{2(n-2)\left(e + \sqrt{T\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor} - \frac{2\delta^2(k+2)}{n-k-2}\right)},$$

then G is traceable.

2. LEMMAS

In order to prove Theorems 1 and 2, we need the following lemmas. Lemma 1 below is Theorem 1.5 on Page 26 in [**4**].

Lemma 1. [**4**] *For a graph G with n vertices and e edges,*

$$Spr(G) \leq \mu_1 + \sqrt{2e - \mu_1^2} \leq 2\sqrt{e}.$$

Equality holds throughout if and only if equality holds in the first inequality; equivalently, if and only if $e = 0$ or G is $K_{a,\,b}$ for some a, b with $e = ab$ and $a + b \leq n$.

Lemma 2 below is Corollary 3.4 on Page 2731 in [**7**].

Lemma 2. [**7**] *Let G be a graph. Then $Spr(G) \geq 2\delta\sqrt{\frac{\alpha(G)}{n-\alpha(G)}}$. If equality holds, then G is a semiregular bipartite graph.*

Lemma 3 is Theorem 1 on Page 5 in [**2**].

Lemma 3. [**2**] *Let G be a connected graph. Then $\mu_1 \leq \sqrt{T(G)}$ with equality if and only if G is either a regular graph or a semiregular bipartite graph.*

Lemma 4 follows from Proposition 2 on Page 174 in [**3**].

Lemma 4. [**3**] *Let G be a graph. Then $\mu_n \geq -\sqrt{\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor}$ with equality if and only if G is $K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}$.*

3. PROOFS

Proof of Theorem 1. Let G be a graph satisfying the conditions in Theorem 1. Suppose, to the contrary, that G is not Hamiltonian. Since G is k-connected ($k \geq 2$), G has a cycle. Choose a longest cycle C in G and give an orientation on C. Since G is not Hamiltonian, there exists a vertex $u_0 \in V(G) - V(C)$. By Menger's theorem, we can find s ($s \geq \kappa$) pairwise disjoint (except for u_0) paths P_1, P_2, ..., P_s between u_0 and $V(C)$. Let v_i be the end vertex of P_i on C, where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of v_1, v_2, ..., v_s agrees with the orientation of C. We use v_i^+ to denote the successor of v_i along the orientation of C, where $1 \leq i \leq s$. Since C is a longest cycle in G, we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq s$ and the index $s+1$ is regarded as 1. Moreover, $S := \{u_0, v_1^+, v_2^+, ..., v_s^+\}$ is independent (otherwise G would have cycles which are longer than C). Then $\alpha \geq s + 1 \geq k + 1$.

Some proof techniques in [**6**] will be used in the remainder of the proofs. From Cauchy-Schwarz inequality, we have that

$$Eng(G) = \sum_{i=1}^{n} |\mu_i| = |\mu_1| + |\mu_n| + \sum_{i=2}^{n-1} |\mu_i|$$

$$\leq \mu_1 - \mu_n + \sqrt{(n-2)\sum_{i=2}^{n-1} \mu_i^2}$$

$$= \mu_1 - \mu_n + \sqrt{(n-2)\left(\sum_{i=1}^{n} \mu_i^2 - \mu_1^2 - \mu_n^2\right)}$$

$$= \mu_1 - \mu_n + \sqrt{(n-2)(2e - (\mu_1 - \mu_n)^2 - 2\mu_1\mu_n)}.$$

Then by Lemmas 1, 2, 3, 4, $\alpha \geq k+1$ and assumptions of Theorem 1, we have that

$$2\sqrt{e} + \sqrt{2(n-2)\left(e + \sqrt{T\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2(k+1)}{n-k-1}\right)}$$

$$\leq Eng(G) \leq$$

$$2\sqrt{e} + \sqrt{(n-2)\left(2e + 2\sqrt{T\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} - \frac{4\delta^2\alpha}{n-\alpha}\right)}$$

$$\leq 2\sqrt{e} + \sqrt{2(n-2)\left(e + \sqrt{T\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2(s+1)}{n-s-1}\right)}$$

$$\leq 2\sqrt{e} + \sqrt{2(n-2)\left(e + \sqrt{T\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2(k+1)}{n-k-1}\right)}.$$

Thus

$$Eng(G) = 2\sqrt{e} + \sqrt{2(n-2)\left(e + \sqrt{T\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor} - \frac{2\delta^2(k+1)}{n-k-1}\right)}.$$

Therefore, $\mu_2 = \cdots = \mu_{n-1}$, $Spr(G) = 2\sqrt{e} = 2\delta\sqrt{\frac{\alpha}{n-\alpha}}$, $\alpha = s+1 = k+1$, $\mu_1 = T$, and $\mu_n = -\sqrt{\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor}$. In view of Lemmas 1, 2, 3, 4, we have that S is a largest independent set of size $\alpha = k+1$ and G is $K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}$.

If n is even, then G is $K_{r,r}$ where $n = 2r$ for some integer $r \geq 2$. Thus $r = \alpha = k+1$ and G is Hamiltonian, a contradiction.

If n is odd, then G is $K_{r,r+1}$ where $n = 2r+1$ for some integer $r \geq 2$. Thus $r+1 = \alpha = k+1$ and G $K_{k,k+1}$ with $n = 2k+1$.

This completes the proof of Theorem 1. □

Proof of Theorem 2. Let G be a graph satisfying the conditions in Theorem 2. Suppose, to the contrary, that G is not traceable. Choose a longest path P in G and give an orientation on P. Let x and y be the two end vertices of P. Since G is not traceable, there exists a vertex $u_0 \in V(G) - V(P)$. By Menger's theorem, we can find s ($s \geq k$) pairwise disjoint (except for u_0) paths P_1, P_2, ..., P_s between u_0 and $V(P)$. Let v_i be the end vertex of P_i on P, where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of v_1, v_2, ..., v_s agrees with the orientation of P. Since P is a longest path in G, $x \neq v_i$ and $y \neq v_i$, for each i with $1 \leq i \leq s$, otherwise G would have paths which are longer than P. We use v_i^+ to denote the successor of v_i along the orientation of P, where $1 \leq i \leq s$. Since P is a longest path in G, we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq s-1$. Moreover, $S := \{u_0, v_1^+, v_2^+, ..., v_s^+, x\}$ is independent (otherwise G would have paths which are longer than P). Then $\alpha \geq s+2 \geq k+2$.

Using the proofs which are similar to the ones in Proof of Theorem 1, we have that

$$2\sqrt{e} + \sqrt{2(n-2)\left(e + \sqrt{T\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor} - \frac{2\delta^2(k+2)}{n-k-2}\right)}$$

$$\leq Eng(G) \leq$$

$$2\sqrt{e} + \sqrt{(n-2)\left(2e + 2\sqrt{T\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor} - \frac{4\delta^2\alpha}{n-\alpha}\right)}$$

$$\leq 2\sqrt{e} + \sqrt{2(n-2)\left(e + \sqrt{T\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor} - \frac{2\delta^2(s+2)}{n-s-2}\right)}$$

$$\leq 2\sqrt{e} + \sqrt{2(n-2)\left(e + \sqrt{T\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2(k+2)}{n-k-2}\right)}.$$

Thus

$$Eng(G) = 2\sqrt{e} + \sqrt{2(n-2)\left(e + \sqrt{T\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2(k+2)}{n-k-2}\right)}.$$

Therefore, $\mu_2 = \cdots = \mu_{n-1}$, $Spr(G) = 2\sqrt{e} = 2\delta\sqrt{\frac{\alpha}{n-\alpha}}$, $\alpha = s+2 = k+2$, $\mu_1 = T$, and $\mu_n = -\sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}$. In view of Lemmas 1, 2, 3, 4, we have that S is a largest independent set of size $\alpha = k+2$ and G is $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

If n is even, then G is $K_{r,\, r}$ where $n = 2r$ for some integer r. Thus $r = \alpha = k+2$ and G is traceable, a contradiction.

If n is odd, then G is $K_{r,\, r+1}$ where $n = 2r+1$ for some integer r. Thus $r+1 = \alpha = k+2$ and G is $K_{k+1,\, k+2}$ with $n = 2k+3$ and G is traceable, a contradiction.

This completes the proof of Theorem 2. $\square$

Notice that $\mu_1 \leq \sqrt{T} \leq \Delta$ and G is regular when $\mu_1 = \Delta$. Thus Theorem 1 and Theorem 2 have the following Corollary 1 and Corollary 2, respectively.

Corollary 1. Let G be a k-connected ($k \geq 2$) graph with $n \geq 3$ vertices and e edges. If

$$Eng(G) \geq 2\sqrt{e} + \sqrt{2(n-2)\left(e + \sqrt{\Delta\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2(k+1)}{n-k-1}\right)},$$

then G is Hamiltonian.

Corollary 2. Let G be a k-connected graph with $n \geq 2$ vertices and e edges. If

$$Eng(G) \geq 2\sqrt{e} + \sqrt{2(n-2)\left(e + \sqrt{\Delta\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2(k+2)}{n-k-2}\right)},$$

then G is traceable.

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Dept. of mathematical sciences, University of South Carolina Aiken, Aiken, SC 29801, USA, Email: raol@usca.edu