Hacettepe Journal of Mathematics and Statistics
Volume 46 (1) (2017), 15 – 32

L^M-valued equalities, L^M-rough approximation operators and ML-graded ditopologies

Alexander Šostak * and Aleksandrs Eļkins†

Abstract

We introduce a certain many-valued generalization of the concept of an L-valued equality called an L^M-valued equality. Properties of L^M-valued equalities are studied and a construction of an L^M-valued equality from a pseudo-metric is presented. L^M-valued equalities are applied to introduce upper and lower L^M-rough approximation operators, which are essentially many-valued generalizations of Z. Pawlak's rough approximation operators and of their fuzzy counterparts. We study properties of these operators and their mutual interrelations. In its turn, L^M-rough approximation operators are used to induce topological-type structures, called here ML-graded ditopologies.

Keywords: L^M-valued equalities, L^M-rough approximation operators, ML-graded ditopologies.

2000 AMS Classification: Primary 54A40, 03E75, 18B99; Secondary 46A08

1. Introduction

After the inseption of the concepts of an L-valued equality and an L-valued set by U. Höhle [19], the study of the category of L-valued sets itself, as well as of different mathematical structures, specifically topological and algebraic, on L-valued sets attracted interest of many researchers, see e.g. [20], [21], [22], [24], [45] just to mention a few of them. In Section 3 of this paper we introduce the concept of an L^M-valued set (Definition 3.1), where L is an iccl-monoid (Subsection 2.1.1) and M is an arbitrary infinitely distributive lattice. An L^M-valued set is, in a certain sense, a many-valued version of the concept of an L-valued set. We

*Department of Mathematics, University of Latvia, Riga, LV-1002 and Institute of Mathematics and CS, Riga, LV-1459, LATVIA
Email : aleksandrs.sostaks@lu.lv, aleksandrs.sostaks@lumii.lv
†Department of Mathematics, University of Latvia, Riga, LV-1002 LATVIA
Email : aleksandrs.elkins@gmail.com

Doi : 10.15672/HJMS.2016.400

consider different special kinds of L^M-valued equalities, and study their properties in Section 3 of this paper. Further, in Section 6, we construct an L^M-valued equality E_ρ from an ordinary pseudo-metric ρ on a set X and investigate properties of the obtained L^M-valued set (X, E_ρ). We consider this construction to be an important source for creation many examples of L^M-valued sets.

Aiming to define a precise mathematical tool which would be appropriate and effective to deal with big data, Z. Pawlak [32] introduced in 1983 the concept of a rough set. Pawlak's work was followed by many other researches. In particular, in 1991 D. Dubois and H. Prade [12] published a paper in which fuzzy rough sets were defined. In this way Pawlak's ideas, aimed specifically to deal with the analysis of big data, were alloyed with L. Zadeh' s vision [49] to develop a precise mathematical tool, which would be appropriate to deal with unprecise and vague objects. This combination gave rise to a new field of mathematical reserach, the field interesting and important both from theoretical and practical points of view. Namely, we mean the theory of upper and lower fuzzy rough approximation operators. In this paper, basing on the concept of an L^M-valued set, we introduce a certain many-valued generalization of this theory. It is done in Section 4 consisting of three subsection: Subsection 4.1 where we define and study upper L^M-rough approximation operators induced by L^M-valued equalities, Subsection 4.2 dealing with lower L^M-rough approximation operators induced by L^M-valued equalities, and Subsection 4.3 where some additional properties of these operators, in particular their mutual interrelations, are considered.

Topological properties of upper and lower Pawlak's rough approximation operators where first noticed in 1988 by A.Skowron [39] and A. Wiweger [47]. J. Kortelainen [26] was probably the first one to discover deep connections between fuzzy upper and lower fuzzy rough approximation operators on one side and (Alexandroff) fuzzy topologies on the other. Later the link between fuzzy rough approximation operators and topological L-fuzzy closure and L-fuzzy interior operators was in the center of interest of different authors, see e.g. , [13], [18], [23], [30], [33], [34], [44], [48].[‡] In our paper, we use upper and lower L^M-approximation operators in order to define M-graded L-fuzzy topologies, or ML-graded topologies for short [6], on L^M-valued sets. This is done in Section 5 under an additional assumption that the lattice M is completely distributive.

2. Prerequisites: The context of the work

2.1. Lattices, iccl-monoids and residuated lattices. In this work the two objects, lattices L and M, will play the fundamental role.

2.1.1. *Lattices.* By $L{=}(L, \leq_L, \wedge_L, \vee_L)$ we denote a complete lattice, that is a lattice in which arbitrary suprema (joins) and infima (meets) exist. In particular, the top 1_L and the bottom 0_L elements in L exist and $0_L \neq 1_L$. A lattice $(L, \leq_L$

[‡]Although the authors of these papers speak about *fuzzy topologies*, in fact they are dealing with *fuzzy ditopologies* [4], [5] since the families of fuzzy open and fuzzy closed sets obtained in this way remain unrelated unless some additional assumptions are made, for example under the assumption that the range L of fuzzy sets is an MV-algebra

$, \wedge_L, \vee_L)$ is called infinitely distributive or a frame if $\wedge$ distributes over arbitrary joins:

$$\alpha \wedge_L \left(\bigvee_i \beta_i\right) = \bigvee_i (\alpha \wedge_L \beta_i) \quad \forall \alpha \in L,\ \forall \{\beta_i : i \in I\} \subseteq L.$$

In the sequel we usually omit the subscript L in notation of $\leq, \wedge, \vee$ when this could not lead to misunderstanding.

2.1.2. *iccl-monoids.* Following e.g. [19, 20] by an integral commutative cl-monoid (iccl-monoid for short) we call a tuple $(L, \leq, \wedge, \vee, *)$ where $(L, \leq, \wedge, \vee)$ is a complete lattice and $(L, *, 1_L)$ is a monoid such that:

(1cl) $*$ is commutative: $\alpha * \beta = \beta * \alpha$ for all $\alpha, \beta \in L$;
(2cl) $*$ is associative: $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$ for all $\alpha, \beta, \gamma \in L$;
(3cl) $*$ distributes over arbitrary joins: $\alpha * \left(\bigvee_{i \in I} \beta_i\right) = \bigvee_{i \in I} (\alpha * \beta_i)$ for all $\alpha \in L$, for all $\{\beta_i \mid i \in I\} \subseteq L$,
(4cl) $\alpha * 1_L = \alpha$ for all $\alpha \in L$.

It is known and easy to prove that $\alpha * 0_L = 0_L$ for every $\alpha \in L$ and that $*$ is monotone:

$$\alpha \leq \beta \Longrightarrow \alpha * \gamma \leq \beta * \gamma$$

Note that an iccl-monoid can be characterized also as an integral commutative quantale in the sense of K.I. Rosenthal [37].

2.1. Example. Among the most important examples of iccl-monois are the following three.

- Let $L = [0, 1]$ and $* = \wedge$. In this case iccl-monoid $(L, \leq, \wedge, \vee, *)$ just reduces to the underlying lattice $(L, \leq, \wedge, \vee, \wedge)$.
- Let $L = [0, 1]$ and let $\alpha * \beta := \alpha \cdot \beta$ be the product. Then we come to the so called *product* t-norm.
- Let $L = [0, 1]$ and $\alpha * \beta = \max(\alpha + \beta - 1, 0)$. Then $*$ is the well-known Łukasiewicz t-norm.

The monoidal operation $* : [0, 1] \times [0, 1] \to [0, 1]$ in these cases is usually referred to as a left semi-continuous t-norm, the term originating from the classic paper by [29]. These and other t-norms were studied and used by many authors, see e.g. fundamental monographs [38] and [25].

2.1.3. *Residuated lattices.* In an iccl-monoid a further binary operation $\mapsto$, residuation, is defined:

$$\alpha \mapsto \beta = \bigvee \{\lambda \in L \mid \lambda * \alpha \leq \beta\}\ \forall \alpha, \beta \in L.$$

Residuation is connected with operation $*$ by Galois connection, see [15]:

$$\alpha * \beta \leq \gamma \Longleftrightarrow \alpha \leq (\beta \mapsto \gamma).$$

An iccl-monoid $(L, \leq, \wedge, \vee, *)$ extended by $\mapsto$, that is the tuple $(L, \leq, \wedge, \vee, *, \mapsto)$, is known also as a residuated lattice [31].

In the following proposition we collect well-known properties of the residium which will be used in the main text:

2.2. Proposition. see e.g. [19], [20]

(1) $\left(\bigvee_i \alpha_i\right) \mapsto \beta = \bigwedge_i (\alpha_i \mapsto \beta)$ *for all* $\{\alpha_i \mid i \in I\} \subseteq L$, *for all* $\beta \in L$;

(2) $\alpha \mapsto (\bigwedge_i \beta_i) = \bigwedge_i (\alpha \mapsto \beta_i)$ *for all* $\alpha \in L$, *for all* $\{\beta_i \mid i \in I\} \subseteq L,$;
(3) $1_L \mapsto \alpha = \alpha$ *for all* $\alpha \in L$;
(4) $\alpha \mapsto \beta = 1_L$ *whenever* $\alpha \leq \beta$
(5) $\alpha * (\alpha \mapsto \beta) \leq \beta$ *for all* $\alpha, \beta \in L$;
(6) $(\alpha \mapsto \beta) * (\beta \mapsto \gamma) \leq \alpha \mapsto \gamma$ *for all* $\alpha, \beta, \gamma \in L$;
(7) $\alpha \mapsto \beta \leq (\alpha * \gamma \mapsto \beta * \gamma)$ *for all* $\alpha, \beta, \gamma \in L$.
(8) $\alpha * \beta \leq \alpha \wedge \beta$ *for any* $\alpha, \beta \in L$.
(10) $(\alpha * \beta) \mapsto \gamma = \alpha \mapsto (\beta \mapsto \gamma)$ *for any* $\alpha, \beta, \gamma \in L$.

2.1.4. *Lattice M.* By M we denote a complete infinitely distributive lattice $(M, \leq_M, \wedge_M, \vee_M)$, whose bottom and top elements are denoted by 0_M and 1_M respectively. As different from the lattice L, we do not exclude here the trivial case, that is M can be the one-element lattice $\bullet$ and hence in this case $0_M = 1_M$. Although in the larger part of this work M can be an arbitrary infinitely distributive lattice, when applying our results for constructing M-graded L-fuzzy ditopologies in Section 5, we additionally assume that M is completely distributive. Actually we will use not the original definition of complete distributivity, see e.g [15, Definition I-2-8], but its characterization given by G.N. Raney [36]. Namely, given a complete lattice M and $\beta, \alpha \in M$ following [36], see also [15, Excercise IV-3-31], we introduce the so called *"wedge below"* relation $\lhd$ on M as follows:

$$\beta \lhd \alpha \iff \Big(\text{ if } K \subseteq M \text{ and } \alpha \leq \bigvee K \text{ then } \exists \gamma \in K,\ \alpha \leq \gamma \Big).$$

As shown by G.N. Raney [36], a lattice M is completely distributive if and only if relation $\lhd$ has the *approximation property*, that is

$$\alpha = \bigvee \{\beta \in M \mid \beta \lhd \alpha\} \text{ for every } \alpha \in M.$$

Moreover, relation $\lhd$ has the following nice properties (see [15, 36]) used in the sequel:

($\lhd$ 1) $\beta \lhd \alpha$ implies $\beta \leq \alpha$;
($\lhd$ 2) $\gamma \leq \beta \lhd \alpha \leq \delta$ implies $\gamma \lhd \delta$;
($\lhd$ 3) $\beta \lhd \alpha$ implies that there exists $\gamma \in L$ such that $\beta \lhd \gamma \lhd \alpha$.

2.2. Fuzzy sets. [49], [17] Recall that an L-fuzzy subset of a set X, where L is a complete lattice, is a mapping $A : X \to L$. Given a family $\{A_i \mid i \in I\}$ its union $\bigvee_i A_i : X \to L$ and intersection $\bigwedge_i A_i : X \to L$ are defined respectively by

$$\Big(\bigvee_i A_i\Big)(x) = \sup_{i \in I} A_i(x), \ \Big(\bigwedge_i A_i\Big)(x) = \inf_{i \in I} A_i(x).$$

2.3. L-relations, L-valued equalities and L-valued sets. [§]

Given sets X, Y and an iccl-monoid L, by an L-relation between X and Y we call a mapping $R : X \times Y \to L$. In case $X = Y$, an L-relation $E : X \times X \to L$ is called *an L-valued equality* if it is

(1) reflexive, that is $E(x,x) = 1_L$ for every $x \in X$;
(2) symmetric, that is $E(x,y) = E(y,x)$ for all $x, y \in X$;
(3) transitive, that is $E(x,y) * E(y,z) \leq E(x,z)$ for all $x, y, z \in X$.

[§] The concepts called here an L-relation and L-valued equivalence under different names and with different degrees of generality appear in many papers, see e.g. [46], [50], [1], [2], etc.

A pair (X, E), where $E : X \times X \to L$ is an L-valued equality on X, is called an L-valued, or a many-valued, set.

When dealing with fuzzy subsets of L-valued sets, the property of extensionality plays an important role. This property was considered by many authors, see e.g. U. Höhle [19], [20], F. Klawon [24], etc:

A fuzzy set A in an L-valued set (X, E) is called *extensional* if

$$A(x) * E(x, x') \leq A(x') \; \forall x, x' \in X.$$

The smallest extensional fuzzy set $\tilde{A}$ in (X, E) that is larger than or equal to A ($A \leq \tilde{A}$) is called the extensional hull of A. Explicitly the extensional hull of A can be defined by

$$\tilde{A}(x) = \bigvee_{x' \in X} \left(E(x, x') * A(x') \right),$$

see e.g. [19], [20], [24].

In particular, identifying an element x_0 with the characteristic function $\chi_{\{x_0\}}$ of the one-element set $\{x_0\}$, we get the extensional hull of the point x_0 called a fuzzy singleton:

$$\tilde{\chi}_{x_0} = E(x_0, x).$$

3. L^M-valued equalities and L^M-valued sets

3.1. L^M-fuzzy sets. Let, as it was assumed, $L = (L, \leq_L, \wedge_L, \vee_L, *)$ be an iccl-monoid and $M = (M, \leq_M, \wedge_M, \vee_M)$ be a complete infinitely distributive lattice. Then the powerset $L^M = \{\varphi \mid \varphi : M \to L\}$ becomes an iccl-monoid by point-wise extension of operations $\leq_L, \wedge_L, \vee_L, *$ from L to L^M:

$$(\varphi \wedge \psi)(\alpha) = \varphi(\alpha) \wedge \psi(\alpha); (\varphi \vee \psi)(\alpha) = \varphi(\alpha) \vee \psi(\alpha); (\varphi * \psi)(\alpha) = \varphi(\alpha) * \psi(\alpha)$$

for all $\varphi, \psi \in L^M$ and every $\alpha \in M$.

Applying the standard definition of a fuzzy set to this situation, we say that an L^M-fuzzy subset A of a set X is just a mapping $A : X \to L^M$. However, the special form of the range set L^M allows to interpret A either as a mapping assigning to each $x \in X$ the mapping $A(x) = \varphi_x : M \to L$, or as an L-fuzzy subset $\tilde{A} \in L^{X \times M}$ of $X \times M$, that is as a mapping $\tilde{A} : X \times M \to L$ assigning to a pair $(x, \alpha) \in X \times M$ the element $A(x, \alpha) = A(x)(\alpha) \in L$. This interpretation of an L^M-fuzzy set A allows to represent it as the family $\{A^\alpha : \alpha \in M\}$ of L-fuzzy subsets $A^\alpha \in L^X$ of X ordered by the elements of M, where the L-fuzzy sets A^α are defined by $A^\alpha(x) = A(x, \alpha)$.

3.2. L^M-valued equalities: Definitions and basic properties. Adjusting the defintion of an L-valued relation (see Definition 2.3) to our situation we get the following:

3.1. Definition. Given a set X, *an L^M-valued equality* on it is a mapping $E : X \times X \to L^M$ such that

$(1EL^M)$ $E(x, x)(\alpha) = 1_L$ for every $x \in X$ and every $\alpha \in M$;
$(2EL^M)$ $E(x, y)(\alpha) = E(y, x)(\alpha)$ for all $x, y \in X$ and every $\alpha \in M$;
$(3EL^M)$ $E(x, y)(\alpha) * E(y, z)(\alpha) \leq E(x, z)(\alpha)$ for all $x, y, z \in X$, $\alpha \in M$.

($4EL^M$) $E(x,y)(\cdot)$ is not-increasing, that is
$\alpha < \beta \implies E(x,y)(\alpha) \geq E(x,y)(\beta)$ for all $x, y \in X$, $\alpha, \beta \in M$.

Sometimes we will speak about some special properties of an L^M-valued relations collected in the next definition:

3.2. Definition. An L^M-valued equality E will be called *upper semi-continuous* if

($5EL^M$) $E(x,y)\left(\bigvee_{i\in I}\alpha_i\right) = \bigwedge_{i\in I} E(x,y)(\alpha_i)$ for all $x, y \in X$, $\{\alpha_i \mid i \in I\} \subseteq M$.

An L^M-valued equality E will be called *lower semi-continuous* if

($6EL^M$) $E(x,y)\left(\bigwedge_{i\in I}\alpha_i\right) = \bigvee_{i\in I} E(x,y)(\alpha_i)$ for all $x, y \in X$, $\{\alpha_i \mid i \in I\} \subseteq M$.

An L^M-valued equality satisfying both properties ($5EL^M$) and ($6EL^M$) is called *continuous*.

An L^M-valued equality E will be called *global* if it satisfies properties ($7EL^M$) and ($8EL^M$) below:

($7EL^M$) $E(x,y)(0_M) = 1_L$ for all $x, y \in X$,

($8EL^M$)

$$E(x,y)(1_M) = \begin{cases} 1_L & \text{if } x = y \\ 0_L & \text{otherwise.} \end{cases}$$

Note that each one of the properties ($5EL^M$) and ($6EL^M$) implies the property ($4EL^M$).

3.3. Remark. Sometimes we interpret an L^M-equality $E : X \times X \to L^M$ as a mapping $\tilde{E} : X \times X \times M \to L$ defined by $\tilde{E}(x,y,\alpha) = E(x,y)(\alpha)$ satisfying corresponding analogues of conditions ($1EL^M$) - ($8EL^M$) reformulated in an obvious way. In what follows we will use both entries $E(x,y)(\alpha)$ and $\tilde{E}(x,y,\alpha)$ and interpret E as a mapping $E : X \times X \to L^M$ and as a mapping $\tilde{E} : X \times X \times M \to L$, when it is more convenient. Besides we usually write just E instead of $\tilde{E}$ when it cannot lead to misunderstanding.

The proof of the following proposition is straightforward:

3.4. Proposition. *A mapping $E : X \times X \times M \to L$ is an L^M-valued equality on a set X if and only if for every $\alpha \in M$ the restriction E^α of E to $X \times X \times \{\alpha\}$ is an L-valued equality on X and $\alpha \leq \beta \Longrightarrow E^\alpha \geq E^\beta$. Thus an L^M- valued equality on a set X can be represented as a non-increasing family of L-valued equalities on this set ordered by the elements of the lattice M.*

3.5. Example. Let (X, E) be an L-valued set and M be an arbitrary complete lattice. Then setting $\tilde{E}(x,y,\alpha) = E(x,y)$ for every $\alpha \in M$ we obtain a continuous L^M-valued equality $\tilde{E} : X \times X \times M \to L$. In this way the L-valued set (X, E) can be identified with the L^M-valued set $(X, \tilde{E})$. In particular, in the role of M, one can take here the one-element lattice $M = \bullet$.

3.6. Definition. An L^M-fuzzy set B is called extensional, if $B(x,\alpha) * E(x, x'\alpha) \leq B(x',\alpha)$ for every $x, x' \in X$ and for every $\alpha \in M$. By the L^M-etensioanal hull of an L-fuzzy set $A \in L^X$ we call the smallest extensional L^M-fuzzy set $B \in (L^M)^X$ which is larger than or equal to A, that is $A(x) \leq B(x,\alpha)$ for all $x \in X$ and for all $\alpha \in M$.

From the definitions one can straightforward get the following

3.7. Proposition. *An L^M-fuzzy set B is extensional if and only for each $\alpha \in M$ the L-fuzzy set B^α is extensional. Specifically, an L^M-fuzzy set B is the extensional hull of the L^M-fuzzy set A if an only for each $\alpha \in M$ B^α is the extensional hull of A^α.*

4. L^M-rough approximation operators on an L^M-valued set

4.1. Upper L^M-rough approximation operator on an L^M-valued set. Let $E : X \times X \to L^M$ be an L^M-valued equality on a set X. Given an L-fuzzy set $A \in L^X$ we define an L^M-fuzzy set $u_E(A) \in (L^M)^X$ as follows:

$$u_E(A)(x)(\alpha) = \bigvee_{x' \in X} (E(x, x')(\alpha) * A(x')).$$

In such a way we obtaine an operator $u_E : L^X \to= (L^M)^X$ that, in an obvious way, can be interpreted also as an operator $u_E : L^X \to L^{M \times X}$

4.1. Definition. Let (X, E) be an L^M-valued set We call operator $u_E : L^X \to (L^M)^X$ *the upper L^M-fuzzy rough approximation operator* induced on the L^M-valued set (X, E).

Such operator can be represented as a family of L-fuzzy rough approximation operators $\{u_E^\alpha : L^X \to L^X : \alpha \in M\}$ defined by

$$u^\alpha(A)(x) = u(A)(x)(\alpha)\ \forall A \in L^X,\ \forall x \in X.$$

This family is ordered by elements of the lattice M in such a way that

$$\alpha \leq \beta \Longrightarrow u_E^\alpha(A) \geq u_E^\beta(A)\ \forall A \in L^X,$$

see Proposition 4.2 ($5u$).

We define the *reduced composition* $u_E \odot u_E : L^X \to (L^M)^X$ for operator u_E by setting

$$(u_E \odot u_E)(A)(x)(\alpha) = u_E(u_E(A)(x)(\alpha))(x)(\alpha)\ \forall A \in L^X,\ \forall x \in X.$$

The most important properties of operator u_E are collected in the following proposition:

4.2. Proposition. *Let (X, E) be an L^M-valued set and $u_E : L^X \to (L^M)^X$ be the induced upper L^M-fuzzy rough approximation operator. Then $u_E : L^X \to (L^M)^X$ has the following properties:*

- ($1u$) $u_E(0_X)(x, 0_M) = 0_L$ *for all* $x \in X$*;*
- ($2u$) $u_E(A)(x, \alpha) \geq A(x)$ *for every* $x \in X, \alpha \in M$.
- ($3u$) $u_E(\bigvee_i A_i) = \bigvee_i u_E(A_i)\ \forall \{A_i \mid i \in I\} \subseteq L^X$ *in particular*
- ($3'u$) $u_E(A_1 \vee A_2) = u_E(A_1) \vee u(A_2) \forall A_1, A_2 \in L^X$;
- ($4u$) $(u_E \odot u_E)(A) = u_E(A)\ \forall A \in L^X$;
- ($5u$) $\alpha \leq \beta \Rightarrow u_E(A)(x, \alpha) \geq u_E(A)(x, \beta)\ \forall x \in X$;
- ($6u$) *If E is upper semicontinuous, then* $u_E(A)(x, \bigwedge_i \alpha_i) = \bigvee_i u_E(A)(x, \alpha_i)$ *for every set* $\{\alpha_i \mid i \in I\} \subseteq M$;
- ($7u$) *If E is global, then* $u_E(A)(x, 0_M) = \bigvee_{x' \in X} A(x')$ *and* $u_E(A)(x, 1_M) = A(x)$.

Proof Statement $(1u)$ is obvious. Statement $(2u)$ follows easily taking into account reflexivity of the L^M-relation E.

We prove property $(3u)$ as follows:

$$u_E(\bigvee_i A_i)(x)(\alpha) = \bigvee_{x'}(E(x,x,\alpha')*(\bigvee_i A_i(x'))) = \bigvee_{x'}(\bigvee_i E(x,x',\alpha)*A_i(x')) =$$

$$\bigvee_i(\bigvee_{x'}(E(x,x',\alpha)*A_i(x'))) = \bigvee_i(u_E(A_i)(x,\alpha)) = \left(\bigvee_i(u_E(A_i)\right)(x,\alpha).$$

To prove property $(4u)$ we fix $\alpha \in M$ and $x \in X$ and taking into account transitivity of the L^M-relation we have:

$$(u_E \odot u_E)(A)(x)(\alpha) = u_E^\alpha(u_E^\alpha(A))(x) = \bigvee_{x'}(u_E^\alpha(A)(x') * E^\alpha(x,x')) =$$

$$\bigvee_{x''}\bigvee_{x'}(A(x'') * E^\alpha(x,x') * E^\alpha(x',x'')) \leq \bigvee_{x''} A(x'') * E^\alpha(x,x'') =$$

$$u_E^\alpha(A)(x) = u_E(A)(x)(\alpha)$$

Since the converse inequality follows from $(2u)$, we get property $(4u)$.

Property $(5u)$ is clear from the definitions taking into account that the L^M-valued equality E is non-increasing.

We prove property $(6u)$ as follows. Let $\{\alpha_i \mid i \in I\} \subseteq M$ and let $\alpha = \bigwedge_{i\in I} \alpha_i$. Then for every $x \in X$ we have:

$$u_E(A)(x,\alpha) = u_E(A)\left(x, \bigwedge_i \alpha_i\right) = \bigvee_{x'}\left(\bigvee_{i\in I}(E(x,x',\alpha_i) * A(x'))\right) =$$

$$\bigvee_{i\in I}\bigvee_{x'}(E(x,x',\alpha_i) * A(x')) = \bigvee_{i\in I} u_E(A)(x,\alpha_i).$$

In case E is global, we prove property $(7u)$ as follows:

$$u_E(A)(x,0_M) = \bigvee_{x'}(E(x,x',0_M) * A(x')) = \bigvee_{x'}(1_L * A(x')) = \bigvee_{x'} A(x') \quad \text{and}$$

$$u_E(A)(x,1_M) = \bigvee_{x'} E(x,x',1_M) * A(x) = A(x).$$

□

4.3. Corollary. *L-fuzzy set $u_E(A) \in (L^M)^X$ is the L^M-extensional hull of the L-fuzzy set $A \in L^X$.*

The proof is straightforward from the definitions and taking into account property $(2u)$ in Proposition 4.2.

4.2. Lower L^M-rough approximation operator on an L^M-valued set. Let $E : X \times X \to L^M$ be an L^M-valued equality on a set X. Given an L-fuzzy set $A \in L^X$ we define the L^M-fuzzy set $l_E(A) \in (L^M)^X$ as follows:

$$l_E(A)(x)(\alpha) = \bigwedge_{x'\in X}(E(x,x')(\alpha) \mapsto A(x')).$$

In such a way we obtain an operator $l_E : L^X \to (L^M)^X$. In an obvious way it can be interpreted also as an operator $l_E : L^X \to L^{M\times X}$

4.4. Definition. Let (X,E) be an L^M-valued set. We call $l_E : L^X \to (L^M)^X$ by the *lower L^M-fuzzy rough approximation operator* induced by the L^M-valued equality E.

Such operator can be represented as a family of lower L-fuzzy rough approximation operators $\{l_E^\alpha : L^X \to L^X : \alpha \in M\}$ defined by

$$l^\alpha(A)(x) = l(A)(x)\ \forall A \in L^X,\ \forall x \in X.$$

This family is ordered by elements of the lattice M in such a way that

$$\alpha \le \beta \Longrightarrow l_E^\alpha(A) \le l_E^\beta(A)\ \forall A \in L^X,$$

see Proposition 4.5 (5l).

We define the *reduced composition* $l_E \odot l_E : L^X \to (L^M)^X$ for operator l_E by setting

$$(l_E \odot l_E)(A)(x)(\alpha) = l_E(l_E(A)(x)(\alpha))(x)(\alpha)\ \forall A \in L^X,\ \forall x \in X.$$

The most important properties of this operator are collected in the following proposition:

4.5. Proposition. *Let (X,E) be an L^M-valued set.*

(1l) $l_E(1_X)(x,\alpha) = 1_L\ \forall \alpha \in M,\ \forall x \in X$;
(2l) $A(x) \ge l_E(A)(x,\alpha)\ \forall A \in L^X,\ \forall \alpha \in M$;
(3l) $l_E(\bigwedge_i A_i) = \bigwedge_i l_E(A_i)\ \forall \{A_i \mid i \in I\} \subseteq L^X$ *in particular*
(3′l) $l_E(A_1 \wedge A_2) = l_E(A_1) \wedge u(A_2) \forall A_1, A_2 \in L^X$;
(4l) $(l_E \odot l_E)(A)(x)(\alpha) = l_E(A)(x)(\alpha)$;
(5l) *If E is non-increasing, then $\alpha \le \beta \Longrightarrow l_E(A)(x,\alpha) \le l_E(A)(x,\beta)$;*
(6l) *If E is upper semicontinuous, then $l_E(A)(x, \bigvee_i \alpha_i) = \bigwedge_i l_E(A)(x,\alpha_i)$;*
(7l) *If E is global, then $l_E(A)(x,0_M) = \bigwedge_{x'} A(x')$ and $l_E(A)(x,1_M) = A(x)$.*

Proof Statement (1l) is obvious. Statement (2l) follows easily taking into account reflexivity of the L^M-equivalence E. We prove property (3l) as follows:

$$l_E(\bigwedge_i A_i)(x,\alpha) = \bigwedge_{x'} \left(E(x,x',\alpha) \mapsto \bigwedge_i A_i(x')\right) = \bigwedge_{x'} \bigwedge_i (E(x,x',\alpha) \mapsto A_i(x')) =$$
$$\bigwedge_i \bigwedge_{x'} (E(x,x',\alpha) \mapsto A_i(x')) = \bigwedge_i l_E(A_i).$$

To prove property (4l) we take into account transitivity of the L-valued equality E^α and are reasoning as follows:

$$(l_E \odot l_E)(A)(x)(\alpha) = l_E^\alpha(l_E^\alpha(A))(x) = \bigwedge_{x'} (E^\alpha(x,x') \mapsto l_E^\alpha(A)(x')) =$$
$$\bigwedge_{x'} (E^\alpha(x,x') \mapsto \bigwedge_{x'} (E^\alpha(x',x'') \mapsto A(x''))) =$$
$$\bigwedge_{x'} (\bigwedge_{x''} (E^\alpha(x,x') * E^\alpha(x',x'') \mapsto A(x''))) \ge$$
$$\bigwedge_{x''} (E^\alpha(x,x'') \mapsto A(x'')) = l_E^\alpha(A)(x) = l_E(A)(x)(\alpha).$$

Since the converse inequality follows from (2l), we get property (4l).

Property (5l) is clear from the definitions taking into account that the L^M-valued equality E is non-increasing.

We prove property (6l) as follows. Let $x \in X$ and $\{\alpha_i : i \in I\} \subseteq M$. Then

$$l_E A(x, \bigvee_i \alpha_i) = \bigwedge_{x'} \left(E(x,x',\bigvee_i \alpha_i) \mapsto A(x')\right) =$$
$$\bigwedge_{x'} \left(\bigwedge_i E(x,x',\alpha_i) \mapsto A(x')\right) = \bigwedge_{x'} \bigwedge_i (E(x,x',\alpha_i) \mapsto A(x')) =$$

$$\bigwedge_i \bigwedge_{x'} (E(x,x',\alpha_i) \mapsto A(x')) = \bigwedge_i l_E(A)(x,\alpha_i).$$

To prove property $(7l)$ we notice that in case of the global L^M-valued equality we have

$$l_E(A)(x,0_M) = \bigwedge_{x'} (E(x,x',0_M) \mapsto A(x')) =$$

$$\bigwedge_{x'} (1_L \mapsto A(x')) = 1 \mapsto \bigwedge_{x'} A(x') = \bigwedge_{x'} A(x');$$

$$l_E(A)(x,1_M) = \bigwedge_{x'} (E(x,x',1_M) \mapsto A(x')) = E(x,x,1_M) \mapsto A(x)) = A(x).$$

4.3. Additional properties of the L^M-rough approximation operators. In this section first of all, we are interested in the interchange properties of the upper and lower rough approximation operators $u_E : L^X \to (L^M)^X$ and $l_E : L^X \to (L^M)^X$. Since we need to deal with combination of operators u_E and l_E, we have to specify how to "compose" them. We define the operation of reduced composition $u_E \odot l_E{}^X \to (L^M)^X$ and $l_E \odot u_E{}^X \to (L^M)^X$ in the same manner as it was done in the previous two subsections:

$$(u_E \odot l_E)(A)(x)(\alpha) = u_E(l_E(A)(x)(\alpha))(x)(\alpha) \ \forall A \in L^X, \ \forall x \in X;$$

$$(l_E \odot u_E)(A)(x)(\alpha) = l_E(u_E(A)(x)(\alpha))(x)(\alpha) \ \forall A \in L^X, \ \forall x \in X.$$

4.6. Proposition. *Given an L^M-valued set (X,E) we have $u_E \odot l_E = l_E$, or, explicitely,*

$$u_E(l_E(A)(x)(\alpha))(x,\alpha) = l_E(A)(x,\alpha) \text{ for any } x \in X \text{ and any } \alpha \in M.$$

Proof From the definition of the operators $u_E, l_E : L^X \to (L^M)^X$ and operation $\odot$ we have:

$$(u_E \odot l_E)(A)(x)(\alpha) =$$

$$\bigvee_{y\in X} \Big(E(x,y,\alpha) * \bigwedge_{z\in X} (E(z,y,\alpha) \mapsto A(z))\Big) \leq$$

$$\bigvee_{y\in X} \bigwedge_{z\in X} (E(x,y,\alpha) * (E(z,y,\alpha) \mapsto A(z))) \leq$$

$$\bigwedge_{z\in X} \bigvee_{y\in X} E(x,y,\alpha) * (E(z,y,\alpha) \mapsto A(z)) \leq$$

$$\bigwedge_{z\in X} \bigvee_{y\in X} ((E(x,y,\alpha) \mapsto E(y,z,\alpha)) \mapsto A(z)) \leq$$

$$\bigwedge_{z\in X} (\bigwedge_{y\in Y} (E(x,y,\alpha) \mapsto E(y,z,\alpha)) \mapsto A(z)) \leq$$

$$\bigwedge_{z\in X} (E(x,z,\alpha) \mapsto A(z)) = l_E(A)(x)(\alpha).$$

The first two inequalities in the above series are obvious; The third and the fourth inequalities in the above series are ensured by the easily established inequalities $a * (b \mapsto c) \leq (a * b \mapsto c)$ and $\bigvee_i (a_i \mapsto b) \leq (\bigwedge_i a_i \mapsto b)$ which hold in every iccl-monoid; the last inequality follows from the definition of an L-valued equality: the condition $E(x,y,\alpha) \leq E(x,z,\alpha) * E(z,y,\alpha)$ implies that $E(x,z,\alpha) \leq E(z,y,\alpha) \mapsto E(y,x,\alpha), \forall y \in X$.

Thus we have $(u_E \odot l_E)(A)(x)(\alpha) \leq l_E(A)(x)(\alpha)$. We complete the proof noticing that the inequality $l_E(A)(x)(\alpha) \leq (u_E \odot l_E)(A)(x)(\alpha)$ is obvious. □

4.7. Remark. In case M is the one-element lattice, the corresponding result is contained in [14] In particular, in the special case when $L = [0,1]$ is viewed as a Gödel algebra, that is $* = \wedge$ is the minimum t-norm and M is the one point lattice, the statement of the above theorem is contained in Proposition 9 in [35].

4.8. Proposition. *For every L-fuzzy set A in an L^M-valued set (X,E) its lower L^M-rough approximation $l_E(A)$ is an extensional fuzzy set.*

Proof From Proposition 4.6 we know that $u_E \odot l_E = l_E$, that is for every $\alpha \in M$ and for every $A \in L^X$ the equality $u_E^\alpha(l_E^\alpha(A)) = l_E^\alpha(A)$ holds. Now from Proposition 3.7 it follows that $l_E^\alpha(A)$ is extensional for every $\alpha \in M$. Finally, applying Proposition 4.3 we conclude that $l_E(A)$ is extensional. □

4.9. Definition. Let (X,E) be an L^M-valued set (X,E) and $A \in L^X$ be its L-fuzzy subset. By the extensional kernel of A in (X,E) we call the smallest extensional L^M-fuzzy set $A^0 \in (L^M)^X$ which is smaller than or equal to A.

From the definitions one can easily prove

4.10. Proposition. *$A^0 \in (L^M)^X$ is the extensional kernel of $A \in L^X$ if and only if for each $\alpha \in L$ the L-fuzzy set $(A^0)^\alpha$ is the extensional kernel of A in the L-valued set (X, E^α).*

4.11. Proposition. *Let A be an L-fuzzy subset of an L^M-valued set (X,E) and let A^0 be its kernel. Then $A^0 \leq l_E(A)$*

Proof Referring to Proposition 3.7 we conclude that for every $\alpha \in M$ L-fuzzy set $A^{0,\alpha}$ is extensional in (X,E). Therefore we have

$$(A^0)^\alpha(x) * E^\alpha(x,x') \leq (A^0)^\alpha(x') \text{ for every } x, x' \in X,$$

and hence

$$(A^0)^\alpha(x) \leq E^\alpha(x,x') \mapsto (A^0)^\alpha(x') \leq E^\alpha(x,x') \mapsto A(x'),\ \forall x, x' \in X.$$

It follows from here that

$$(A^0)^\alpha(x) \leq \bigwedge\nolimits_{x' \in X} (E^\alpha(x,x') \mapsto A(x')) = l_E^\alpha(A)(x),\ \forall x \in X,$$

that is $(A^0)^\alpha \leq l^\alpha(A)$.
Referring to Proposition 3.7 again we conclude that $A^0 \leq l_E(A)$ □

From Propositions 4.6 and 4.11 we get the following important result:

4.12. Theorem. *For every L-fuzzy set A in an L^M-valued set (X,E) the lower fuzzy rough approximation operator l_E assigns to A its kernel A^0: That is $l_E(A) = A^0$.*

From this theorem we get

4.13. Corollary. *The equality $l_E \odot u_E = u_E$ holds. Explicetely*

$$(l_E \odot u_E)(A)(x)(\alpha) = u_E(A)(x)(\alpha)$$

for every L-fuzzy set A in an L^M-valued set (X,E).

4.14. Remark. Let $L = \{0, 1\} =: 2$, $M = \bullet$ be the one-element lattice and let $E : X \times X \to \{0, 1\}$ be an equivalence relation. Obviously, in this case E is actually the crisp equivalence relation on X. Then the images of a set $A \in 2^X$ under operators $u_E : 2^X \to 2$ and $l_E : 2^X \to 2$ make the pair $(u_E(A), l_E(A))$ which is actually Pawlak's originally defined rough set $(A^\blacktriangledown, A^\blacktriangle)$ determined by the set A. Indeed, notice first that $u_E(A)$ in this case is just the set of all elements $x \in A$ whose classes $[x]_E$ of E-equivalence have non-empty intersections with A: $[x]_E \cap A \neq \emptyset$, and hence $u_E(A) = A^\blacktriangledown$. On the other hand, $l_E(A)$ is the set of all elements $x \in A$, whose classes of equivalence $[x]_E$ are contained in A: $[x]_E \subseteq A$, and hence $l_E(A) = A^\blacktriangle$.

5. ML-graded ditopology induced by an L^M-valued equality

In this section we apply upper and lower L^M-rough approximation operators induced by an L^M-valued equality on a set X in order to present a construction of an ML-graded ditopology on this set. However first we have to make comments on the terminology used here.

Generalizing the concept of an L-fuzzy topology in the sense of Chang-Goguen (see [7], [17] [16]), T. Kubiak [27] and A.Šostak [40] independently introduced an alternative, in a certain sense more consequent, concept of a fuzzy topology. According to this definition the topology itself is an L-fuzzy subset (and not a crisp one as it is in the case of Chang-Goguen's definition) of the family of L-fuzzy subsets of the ground set X, see Subsection 5.1 To distinguish such approach from the one in the sense of Chang-Goguen, we call it here a *graded topology.*¶ In order to specify the role of the iccl-monoid L and the lattice M in this case, we use a more precise term *an M-graded L-fuzzy topology* or an *ML-graded topology* for short.

In classical topology, as well as, to a large extent, in fuzzy topology, the notion of an open set is usually taken as the primitive and that of a closed set being an auxiliary one, since closed sets are easily obtained from open by taking complements. However in some cases it is reasonable to consider open and closed sets as independent notions. This is especially crucial when dealing with L-fuzzy topologies in case when the lattice L is not equipped with an order reversing involution. To handle with such and analogous more general problems, L.M. Brown with coaurthors has developed the theory of a dichotomous topology, or just ditopology in short [3], [4], [5], etc. Developing the idea of a ditopology, we have introduced and studied the graded version of a ditopology in [6]. In the context of this work the term ML-ditopology on a set means just a pair of mutually independent mappings $\mathcal{T} : L^X \to (L^M)^X$ and $\mathcal{K} : L^X \to (L^M)^X$ satisfying certain topological axioms, see Subsections 5.1, 5.2 for the precise definitions. It is the aim of this section to elaborate a construction of ML-ditopologies induced on L^M-valued sets by L^M-valued equalities.

5.1. ML-graded topology on an L^M-valued set. Let (X, E) be an L^M-valued set and let $l_E : L^X \to (L^M)^X$ be the lower L^M-rough approximation operator induced on this set. Further, let as before, its α-levels $l_E^\alpha : L^X \to L^X$ be defined

¶This term was already used by some authors, [8], [9].

by $l_E^\alpha(A)(x) = l_E(A)(x,\alpha)$. Then the properties $(1l) - (4l)$ of l_E related to l_E^α can be reformulated as follows:

($1l^\alpha$) $l^\alpha(1_X) = 1_L$;
($2l^\alpha$) $A \geq l_E^\alpha(A)\ \forall A \in L^X$;
($3l^\alpha$) $l_E^\alpha(\bigwedge_i A_i) = \bigwedge_i l_E(A_i)\ \forall \{A_i \mid i \in I\} \subseteq L^X$ in particular
(3'l^α) $l_E^\alpha(A_1 \wedge A_2) = l_E^\alpha(A_1) \wedge u(A_2) \forall A_1, A_2 \in L^X$.
($4l^\alpha$) $l_E^\alpha(l_E^\alpha(A)) = l_E^\alpha(A)\ \forall A \in L^X$;

However, this means that $l_E^\alpha : L^X \to L^X$ can be interpreted as an L-fuzzy interior operator on the set X. (This fact is well-known, see, e.g. [28], [41], [42]). Hence by setting $T_\alpha = \{A \in L^X : l_E^\alpha(A) = A\}$, we obtain the L-fuzzy topology corresponding to this L-fuzzy interior operator. Moreover, the property $(3l)$ allows to conclude that it is actually an Alexandroff L-fuzzy topology (see e.g. [26], [10]), that is the intersection axiom holds also for infinite families. Thus for each α the family T_α satisfies the following axioms of an Alexandroff L-fuzzy topology:

(1) $1_X \in T_\alpha$;
(2) $\{A_i : i \in I\} \subseteq T_\alpha \Longrightarrow \bigwedge_i A_i \in T_\alpha$;
(3) $\{A_i : i \in I\} \subseteq T_\alpha \Longrightarrow \bigvee_i A_i \in T_\alpha$

Taking such L-fuzzy topologies for all $\alpha \in M$, we obtain the family $\{T_\alpha : \alpha \in M\}$. Besides, since $l_E^\alpha \leq l_E^\beta$ whenever $\alpha \leq \beta$, we conclude that

$$\alpha \leq \beta \implies T_\alpha \supset T_\beta,$$

that is the family $\{T_\alpha : \alpha \in M\}$ is non-increasing. We use this family of L-fuzzy topologies to define an (Alexandroff) ML-graded topology $\mathcal{T}$ on the set X, by setting

$$\mathcal{T}(A) = \bigvee\{\alpha \in M : A \in T_\alpha.\}$$

5.1. Theorem. *If M is completely distributive, then $\mathcal{T}$ is an M-graded L-fuzzy topology on the L^M-valued set (X,E), that is $\mathcal{T} : L^X \to M$ satisfies the following axioms:*

(1) $\mathcal{T}(1_X) = 1_M$;
(2) $\mathcal{T}(\bigwedge_i A_i) \geq \bigwedge_i \mathcal{T}(A_i)$ *for every family* $\{A_i : i \in I\} \subseteq L^X$;
(3) $\mathcal{T}(\bigvee_i A_i) \geq \bigwedge_i \mathcal{T}(A_i)$ *for every family* $\{A_i : i \in I\} \subseteq L^X$;

Proof The first property is obvious, since $1_X \in T_\alpha$ for all $\alpha \in M$.

To prove the second property, take any family $\{A_i : i \in I\} \subseteq L^X$ and assume that $\bigwedge_i \mathcal{T}(A_i) = \alpha$. In case $\alpha = 0_M$ the inequality is obvious, therefore we assume that $\alpha > 0_M$. Take any $\beta \lhd \alpha$ where $\lhd$ is the way below relation on the completely distributive lattice M. From the definition of $\mathcal{T}$ it is clear that $A_i \in T_\beta$ for every $i \in I$ and hence, recalling that T_β is an Alexandroff L-fuzzy topology, we conclude that also $\bigwedge_i A_i \in T_\beta$. Therefore $\mathcal{T}(\bigwedge_i A_i) \geq \beta$. Since this is true for any $\beta \lhd \alpha$ and lattice M is completely distributive, we conclude that $\mathcal{T}(\bigwedge_i A_i) \geq \alpha = \bigwedge_i \mathcal{T}(A_i)$.

The proof of the third property is similar and we omit it.

5.2. Graded co-topology of an L^M-valued set. Let (X,E) be an L^M-valued set and let $u_E : L^X \to (L^M)^X$ be the upper rough approximation operator induced by the L^M-valued equality E on the set X. Further, as before, let its α-levels $u_E^\alpha : L^X \to L^X$ be defined by $u_E^\alpha(A)(x) = u_E(A)(x,\alpha)$. Then properties $(1u) - (4u)$ of the upper L^M-rough approximation operator u_E related to u^α_E can be reformulated as follows:

(1u^α) $u^\alpha(1_X) = 1_L$;
(2u^α) $A \leq u^\alpha_E(A)\ \forall A \in L^X$;
(3u^α) $u^\alpha_E(\bigvee_i A_i) = \bigvee_i l_E(A_i)\ \forall \{A_i \mid i \in I\} \subseteq L^X$ in particular
(3'u^α) $u^\alpha_E(A_1 \wedge A_2) = u^\alpha_E(A_1) \wedge u(A_2) \forall A_1, A_2 \in L^X$.
(4u^α) $u^\alpha_E(u^\alpha_E(A)) = u^\alpha_E(A)\ \forall A \in L^X$;

However, this means that $u^\alpha_E : L^X \times L^X$ can be interpreted as an L-fuzzy closure operator on the set X (This fact is well-known, see, e.g. [28], [41], [42]). Hence by setting $K_\alpha = \{A \in L^X : u^\alpha_E(A) = A\}$, we obtain the L-fuzzy co-topology corresponding to this L-fuzzy closure operator. Moreover, the property (3u) allows to conclude that it is actually an Alexandroff L-fuzzy co-topology [10]: this means that the union axiom holds also for infinite families. Thus, for each α the family K_α satisfies the following axioms of an Alexandtoff L-fuzzy co-topology:

(1) $1_X \in K_\alpha$;
(2) $\{A_i : i \in I\} \subseteq K_\alpha \Longrightarrow \bigvee_i A_i \in K_\alpha$;
(3) $\{A_i : i \in I\} \subseteq K_\alpha \Longrightarrow \bigwedge_i A_i \in K_\alpha$

Taking such L-fuzzy co-topologies for all $\alpha \in M$, we obtain the family $\{K_\alpha : \alpha \in M\}$. Besides, since $u^\alpha_E \geq u^\beta_E$ whenever $\alpha \leq \beta$, we conclude that

$$\alpha \leq \beta \implies K_\alpha \supset K_\beta,$$

that is the family $\{K_\alpha : \alpha \in M\}$ is non-increasing. We use this family of L-fuzzy co-topologies to define an (Alexandroff) L-fuzzy co-topology $\mathcal{K}$ on the set X, by setting

$$\mathcal{K}(A) = \bigvee\{\alpha \in M : A \in K_\alpha.\}$$

5.2. Theorem. *If M is completely distributive, then $\mathcal{K}$ is an M-graded L-fuzzy co-topology on the L^M-valued set (X, E). This means that the mapping $\mathcal{K} : L^X \to M$ satisfies the following axioms:*

(1) $\mathcal{K}(1_X) = 1_M$;
(2) $\mathcal{K}(\bigvee_i A_i) \geq \bigwedge_i \mathcal{K}(A_i)$ *for every family* $\{A_i : i \in I\} \subseteq L^X$;
(3) $\mathcal{K}(\bigwedge_i A_i) \geq \bigwedge_i \mathcal{K}(A_i)$ *for every family* $\{A_i : i \in I\} \subseteq L^X$;

Proof The first property is obvious, since $1_X \in K_\alpha$ for all $\alpha \in M$.

To prove the second property, take any family $\{A_i : i \in I\} \subseteq L^X$ and assume that $\bigwedge_i \mathcal{K}(A_i) = \alpha$. In case $\alpha = 0_M$ the inequality is obvious, therefore we assume that $\alpha > 0_M$. Take any $\beta \lhd \alpha$ where $\lhd$ is the wedge-below relation in the completely distributive lattice. Then from the definition of $\mathcal{K}$ it is clear that $A_i \in K_\beta$ for every $i \in I$, and hence, recalling that K_β is an Alexandroff L-fuzzy co-topology, we conclude that also $\bigvee_i A_i \in K_\beta$. Therefore $\mathcal{K}(\bigvee_i A_i) \geq \beta$. Since this is true for any $\beta \lhd \alpha$ and lattice M is completely distributive, we conclude that $\mathcal{K}(\bigvee_i A_i) \geq \alpha = \bigwedge_i \mathcal{K}(A_i)$.

The proof of the third property is similar and we omit it.

6. Construction of an L^M-valued equality from a pseudo-metric

In this section we construct an L^M-valued equality E_ρ from an ordinary pseudo-metric ρ on a set X. We think that this construction presents an important source for creation of many examples of L^M-valued sets with prescribed properties.

Let $L = M = [0,1]$ be the unit intervals viewed as lattices and let $* : L \times L \to L$ be a continuous t-norm. Further, let X be a set and $\rho : X \times X \to [0,1]$ be a pseudo-metric on this set. We define a mapping $E_\rho : X \times X \times [0,1] \to [0,1]$ by setting

$$E_\rho(x,y)(\alpha) = \begin{cases} \frac{1-\alpha}{1-\alpha+\alpha\rho(x,y)} & \text{if } \alpha \neq 1 \text{ or } \rho(x,y) \neq 0 \\ 1 & \text{if } \alpha = 1 \text{ and } \rho(x,y) = 0. \end{cases}$$

It is easy to see that the mapping $E(x,y)(\cdot) : [0,1] \to [0,1]$ is continuous for any $x, y \in [0,1]$. Indeed, the statement is obvious if $\rho(x,y) \neq 0$ or $\alpha \neq 1$, otherwise $\lim_{\alpha \to 1} E(x,y)(\alpha) = \lim_{\alpha \to 1} \frac{1-\alpha}{1-\alpha+\alpha\rho(x,y)} = 1$.

6.1. Proposition. *For every pseudo-metric $\rho : X \times X \to [0,1]$ the mapping $E_\rho : X \times X \times [0,1] \to [0,1]$ satisfies conditions $(1EL^M)$, $(2EL^M)$, $(4EL^M)$, $(5EL^M)$, $(6EL^M)$, $(7EL^M)$ and $(8EL^M)$. The mapping $E_\rho : X \times X \times [0,1] \to [0,1]$ satisfies condition $(3EL^M)$ in cases of the product t-norm $* = \cdot$ and of the Łukasiewicz t-norm $* = *_L$. If ρ is an ultra pseudo-metric, then mapping $E_\rho : X \times X \times [0,1] \to [0,1]$ satisfies condition $(3EL^M)$ in case of the minimum t-norm $* = \wedge$.*

The validity of conditions $(1EL^M)$ and $(2EL^M)$ follows directly from the definition of the mapping $E_\rho : X \times X \times [0,1] \to [0,1]$.

To prove $(3EL^M)$ consider separately the cases of the three t-norms:

$* = \wedge$ Since in this case ρ is assumed to be an ultra pseudo-metric, we have $\rho(x,y) \leq \max\{\rho(x,z), \rho(z,y)\}$ for all x, y, z. It is straightforward to conclude from here that

$$\frac{1-\alpha}{1-\alpha+\alpha\rho(x,y)} \geq \frac{1-\alpha}{1-\alpha+\alpha\rho(x,z)} \bigwedge \frac{1-\alpha}{1-\alpha+\alpha\rho(z,y)}.$$

$* = \cdot$ The inequality

$$\frac{1-\alpha}{1-\alpha+\alpha\rho(x,y)} \geq \frac{1-\alpha}{1-\alpha+\alpha\rho(x,z)} \cdot \frac{1-\alpha}{1-\alpha+\alpha\rho(z,y)}$$

can be easily established taking into account the triangular property $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ of the pseudo-metric ρ.

$* = *_L$ It is well known that $*_L \leq \cdot$ and hence this property follows from the analogous property of the product t-norm establish above.

Property $(4EL^M)$ follows directly from the definition of the L^M valued equality E_ρ.

To prove Property $(5EL^M)$ let $\alpha = \bigvee_{n \in \mathbb{N}} \alpha_n$ for some $\alpha \in [0,1]$ and $\{\alpha_n : n \in \mathbb{N}\} \subset [0,1]$. Without loss of generality we may assume that

$$n \leq n+1 \Rightarrow \alpha_n \leq \alpha_{n+1} \text{ for every } n \in \mathbb{N}.$$

Then, referring to the continuity and already the established non-increaseness of the mapping $E_\rho(x,y) : [0,1] \to [0,1]$ we have

$$E_\rho(x,y,\alpha) = E_\rho(x,y, \lim_{n\to\infty} \alpha_n) = \lim_{n\to\infty} E_\rho(x,y,\alpha_n) = \bigwedge\nolimits_{n \in \mathbb{N}} E_\rho(x,y,\alpha_n).$$

To prove Property ($6EL^M$), let $\alpha = \bigwedge_{n\in\mathbb{N}} \alpha_n$ for some $\alpha_n \in [0,1]$. Without loss of generality we may assume that $n \le n+1 \Rightarrow \alpha_n \ge \alpha_{n+1}$ for every $n \in \mathbb{N}$. Then, referring to the continuity and already established non-increaseness of the mapping $E_\rho(x,y) : [0,1] \to [0,1]$ we have

$$E_\rho(x,y,\alpha) = E_\rho(x,y,\lim_{n\to\infty} \alpha_n) = \lim_{n\to\infty} E_\rho(x,y,\alpha_n) = \bigvee\nolimits_{n\in\mathbb{N}} E_\rho(x,y,\alpha_n).$$

From the definition of E_ρ it is clear that $E_\rho(x,y,0) = 1$ for every $x, y \in X$ and

$$E_\rho(x,y)(1_M) = \begin{cases} 1_L & \text{if } x = y \\ 0_L & \text{otherwise,} \end{cases}$$

and hence ($7EL^M$) and ($8EL^M$) hold. □

6.2. Corollary. *In case $* = \cdot$ and $* = *_L$ the mapping $E_\rho : X \times X \to [0,1] \to [0,1]$ is a global continuous L^M-equality for any pseudo-metric $\rho : X \times X \to [0,1]$. If ρ is an ultra pseudo-metric, then E_ρ is a global continuous L^M-valued equality in case $* = \wedge$.*

6.3. Remark. It is well known that for every pseudo-metric $d : X \times X \to (0,\infty)$ there exist fuzzy metrics $\rho : X \times X \to [0,1]$ equivalent to the given pseudo-metric d. By saying *equivalent* we mean that d and ρ induce the same topology on the set X. Therefore, if we start with an arbitrary pseudo-metric $d : X \times X \to (0,\infty)$, then we take the equivalent pseudo-metric $\rho : X \times X \to [0,1]$ defined by $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$ as its counterpart. In this case L^M-valued equality E_ρ can be rewritten as

$$E_d(x,y,\alpha) = \frac{(1-\alpha)(1+d(x,y)}{1-\alpha+d(x,y}.$$

7. Conclusion

We have introduced the notions of an L^M-valued equality and an L^M-valued set, which conceptionally generalize the concepts of an L-valued equality and an L-valued set, well-known to people working in this field. We have studied the basic properties of these concepts. An example of an L^M-valued equality induced by a bounded pseudometric was presented. We showed that L^M-equalities induce in a natural way a certain kind of many-valued rough approximation operators; we call them an upper and a lower L^M-rough approximation operators. Finally we apply these operators to construct an ML-graded ditopology on the L^M-valued set.

We view this work as the first part of the reserach in this direction. Among important, in our opinion, issues, which remained beyond the scope of this work, we mention here the following:

In this work we did not touch the question how special properties of L^M-equalities (upper and lower semicontinuity, etc.,) are reflected in the structure of the constructed ML-graded topologies? Can we characterize the class of ditopologies which are induced by an L^M-equality with a certain property? In particular, how do the levels $\mathcal{T}_\alpha$ and $\mathcal{K}_\alpha$ of the ML-graded topology $\mathcal{T}$ and $\mathcal{K}$ are related to the L-fuzzy topology T_α and co-topology $\mathcal{K}_\alpha$ depending on the properties of the L^M-valued equality E ?

Having L^M-valued sets on one side and ML-graded ditopogical spaces on the other it seems important to study their relations on the categorical level, that is when certain ordinary functions or fuzzy functions [11], [43], [24] are taken as morphisms in the corresponding category. A similar question was studied for ordinary L-valued sets in our paper [14].

Related to the previous question: what are the connections between the operations in the (prospective!) category of L^M-valued sets (products, coproducts, etc) and the corresponding operations in the category of LM-graded ditopologies?

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