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A study on quasi-pseudometrics

In memory of Professor Lawrence M. Brown

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Abstract

We study some aspects of the space $QPM(X)$ of all quasi-pseudometrics on a set X equipped with the extended T_0-quasi-metric $A_X(f,g) = \sup_{(x,y)\in X\times X}(f(x,y)\dot{-}g(x,y))$ whenever $f,g \in QPM(X)$. We observe that this space is bicomplete and exhibit various closed subspaces of $(QPM(X), \tau((A_X)^s))$.

In the second part of the paper, as a rough way to measure the asymmetry of a quasi-pseudometric f on a set X, we investigate some properties of the value $(A_X)^s(f, f^{-1})$.

Keywords: quasi-pseudometric; T_0-quasi-metric; nonnegatively weightable quasi-pseudometric

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1. Introduction

On the set $QPM(X)$ of all quasi-pseudometrics on the set X we introduce the extended T_0-quasi-metric A_X defined by

$$A_X(f,g) = \sup_{(x,y)\in X\times X}(f(x,y)\dot{-}g(x,y))$$

whenever $f,g \in QPM(X)$.[‡] Let us immediately mention that obviously the specialization order $\leq_{A_X}$ of A_X is the usual order on $QPM(X)$, that is, for $f,g \in$

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[‡]For $a,b \in \mathbb{R}$ we set $a\dot{-}b = \max\{a-b, 0\} = (a-b)\vee 0$.

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$QPM(X)$ we have $f \leq_{A_X} g$ iff $A_X(f,g) = 0$ iff $f(x,y) \leq g(x,y)$ whenever $(x,y) \in X \times X$. [§]

1. Remark. We could also consider the bounded counterpart of A_X defined by $\min\{A_X, 1\}$. In the analogous metric construction this approach was for instance chosen for the studies [23, 24]. Since however we are mainly interested in large distance values as they are investigated for instance in the theory of coarse spaces (e.g. [22]), this is not the approach that we have chosen in this paper.

Below we establish that the space $(QPM(X), A_X)$ is bicomplete. We also show that various natural subspaces of $QPM(X)$ are $\tau((A_X)^s)$-closed and thus bicomplete, for instance the set of all totally bounded quasi-pseudometrics on X, the set of all ultra-quasi-pseudometrics on X and the set of all nonnegatively weightable quasi-pseudometrics on X.

In the second part of the paper we consider for any quasi-pseudometric f on X its *value of asymmetry* defined by $A_f := (A_X)^s(f, f^{-1})$. The definition is obviously motivated by the fact that f is a pseudometric on X if and only if $(A_X)^s(f, f^{-1}) = 0$. [¶]

We discuss some properties of the introduced concept and consider various inequalities that are useful to compute it for suitable quasi-pseudometric spaces (X, f).

2. The space $QPM(X)$ of all quasi-pseudometrics

After recalling the main definitions of the notions used in this paper, we shall establish bicompleteness of the space $(QPM(X), A_X)$ and exhibit various $\tau((A_X)^s)$-closed subspaces of $(QPM(X), A_X)$. For a more detailed discussion of the basic concepts dealt with in this paper the reader may want to consult [7, 13].

1. Definition. Let X be a set and let $d : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then d is called a *quasi-pseudometric* on X if

(a) $d(x,x) = 0$ whenever $x \in X$, and

(b) $d(x,z) \leq d(x,y) + d(y,z)$ whenever $x,y,z \in X$.

We shall say that d is a T_0-*quasi-metric* provided that d also satisfies the following condition (c): For each $x, y \in X$,

$d(x,y) = 0 = d(y,x)$ implies that $x = y$.

The *specialization order* $\leq_d$ of d is defined by $x \leq_d y$ iff $d(x,y) = 0$ whenever $x, y \in X$.

2. Remark. In some cases it is more natural to assume that a quasi-pseudometric d indeed maps into $[0, \infty]$. We shall then speak of an *extended* quasi-pseudometric.[‖] It should also be mentioned that the terminology in the literature is fairly diverse (compare for instance [10, Chapter 6]).

[§]For later use we note that the extended T_0-quasi-metric A_X can indeed be defined for arbitrary functions $f, g : X \times X \to [0, \infty)$. Let us mention that we shall however not define A_X in the case of extended functions f and g in this paper.

[¶]We remark that in the paper [21] a measure of asymmetry is considered that is based on the quotient $\frac{f}{f^{-1}}$ instead of the difference $f \dot{-} f^{-1}$.

[‖]For extended quasi-pseudometrics the triangle inequality is interpreted in the obvious way.

1. Example. (compare for instance [8, Example 2]) On the set $\mathbb{R}$ of the reals set $u(x,y) = x \dot{-} y$ whenever $x, y \in \mathbb{R}$. Then u is the *standard* T_0-quasi-metric on $\mathbb{R}$.

3. Remark. Let d be a quasi-pseudometric on a set X. Then $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x,y) = d(y,x)$ whenever $x, y \in X$ is also a quasi-pseudometric on X, called the *conjugate* or *dual* quasi-pseudometric of d. As usual, a quasi-pseudometric d on X such that $d = d^{-1}$ is called a *pseudometric*. Note that for any (T_0-)quasi-pseudometric d, $d^s = \sup\{d, d^{-1}\} = d \vee d^{-1}$ is a pseudometric (metric).

The following auxiliary result is well known. Its proof is included here for the convenience of the reader.

1. Lemma. (see for instance [14, Lemma 8]) Let (X, d) be a quasi-pseudometric space and $a, b, x, y \in X$. Then $|d(x,y) - d(a,b)| \leq d^s(x,a) + d^s(y,b)$.

Proof. We have that $d(x,y) \leq d(x,a) + d(a,b) + d(b,y)$, and therefore $d(x,y) - d(a,b) \leq d(x,a) + d(b,y)$. Similarly $d(a,b) \leq d(a,x) + d(x,y) + d(y,b)$, and therefore $d(a,b) - d(x,y) \leq d(a,x) + d(y,b)$. Thus $|d(x,y) - d(a,b)| \leq d^s(x,a) + d^s(y,b)$. □

As we have announced above, we equip the set $QPM(X)$ of all quasi-pseudometrics on X with the (extended) function

$$A_X(f,g) = \sup_{(x,y) \in X \times X} (f(x,y) \dot{-} g(x,y))$$

whenever $f, g \in QPM(X)$.

1. Proposition. We have that $(QPM(X), A_X)$ is an extended T_0-quasi-metric space.

Proof. The argument is obvious and left to the reader. □

4. Remark. Note that by definition $A_X(d,e) = A_X(d^{-1}, e^{-1})$ whenever $d, e \in QPM(X)$. In particular for any quasi-pseudometric d on a set X we have that $A_X(d, d^{-1}) = A_X(d^{-1}, d) = (A_X)^s(d, d^{-1})$.

5. Remark. Let X be a set, d a quasi-pseudometric on X and $\underline{0}$ the constant quasi-pseudometric equal to 0. Then $A_X(d, \underline{0})$ is equal to the *diameter* $\delta_d = \sup_{(x,y) \in X \times X} d(x,y)$ of (X, d).

2. Lemma. Let d, e, f, g be quasi-pseudometrics on a set X.

(a) Then $A_X(d+e, f+g) \leq A_X(d,f) + A_X(e,g)$, where $d+e, f+g$ are quasi-pseudometrics on X.

(b) Furthermore $A_X(\alpha d, \alpha f) = \alpha A_X(d,f)$ whenever α is a nonnegative real, where αd and αf are quasi-pseudometrics on X.

(c) If $f \geq g$ and $h \geq e$, then $A_X(f,e) \geq A_X(g,h)$.

Proof. All these computations are straightforward. □

In the following Δ_X will denote the diagonal $\{(x,x) : x \in X\}$ of the set X.

2. Example. Let $\leq$ be a partial order on a set X. Set, for each $x, y \in X$, $d_{\leq}(x,y) = 0$ if $x \leq y$ and $d_{\leq}(x,y) = 1$ otherwise. Then $d_{\leq}$ is a T_0-quasi-metric on X, which is called the *natural* T_0-quasi-metric of $(X, \leq)$ (compare for instance [2, Section 4]). We now consider the following specific example of this construction: Let X be the set of integers $\mathbb{Z}$. Set

$$\leq = \Delta_{\mathbb{Z}} \cup \{(2n, 2n+1) : n \in \mathbb{Z}\} \cup \{(2n, 2n-1) : n \in \mathbb{Z}\}.$$

Then $\leq$ is a partial order on $\mathbb{Z}$. Of course, $\geq = (\leq)^{-1} = \Delta_{\mathbb{Z}} \cup \{(2n+1, 2n) : n \in \mathbb{Z}\} \cup \{(2n-1, 2n) : n \in \mathbb{Z}\}$. We have that $d_{\leq} \wedge (d_{\leq})^{-1} = \underline{0}$, since $\leq \cup (\geq) = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : |x - y| \leq 1\}$. Here we have $(d_{\leq})^{-1} = d_{\geq}$ and $d_{\leq} \wedge d_{\geq}$ is the largest quasi-pseudometric which is $\leq d_{\leq}$ and $\leq d_{\geq}$. **

It follows that $d_{\leq} \wedge (d_{\leq})^{-1} < \min\{d_{\leq}, (d_{\leq})^{-1}\}$. Obviously $\min\{d_{\leq}, (d_{\leq})^{-1}\}$ does not satisfy the triangle inequality.

3. Lemma. Let X be a set and functions $d_1, d_2 : X \times X \to [0, \infty)$ be given. Set $b := \min\{d_1, d_2\}$ and $s := d_1 \vee d_2 = \max\{d_1, d_2\}$. ††

Then $(A_X)^s(d_1, d_2) = (A_X)^s(s, b)$. (Of course, $A_X(b, s) = 0$.)

Proof. By Lemma 2(c) we have that $A_X(s,b) \geq A_X(d_1, d_2)$ and analogously $A_X(s,b) \geq A_X(d_2, d_1)$. Therefore $A_X(s,b) \geq (A_X)^s(d_1, d_2)$.

Let $x, y \in X$. By considering the various possibilities in any case we have that $s(x,y) - b(x,y) \leq (d_1(x,y) - d_2(x,y)) \vee (d_2(x,y) - d_1(x,y)) \leq A_X(d_1, d_2) \vee A_X(d_2, d_1) = (A_X)^s(d_1, d_2)$. Hence $A_X(s,b) \leq (A_X)^s(d_1, d_2)$. We conclude that $A_X(s,b) = (A_X)^s(d_1, d_2)$. □

1. Corollary. Let X be a set and functions $d_1, d_2 : X \times X \to [0, \infty)$ be given, and s and b as defined in Lemma 3.

Then $A_X(s, d_2) = A_X(d_1, d_2)$ and $A_X(d_1, b) = A_X(d_1, d_2)$.

Proof. By Lemma 2(c) we have that $A_X(s, d_2) \geq A_X(d_1, d_2)$.

Let $x, y \in X$. By considering the various possibilities, in any case we have $s(x,y) - d_2(x,y) \leq d_1(x,y) - d_2(x,y) \leq A_X(d_1, d_2)$ and thus $A_X(s, d_2) \leq A_X(d_1, d_2)$.

The second part of the proof is similar: $A_X(d_1, b) \geq A_X(d_1, d_2)$ by Lemma 2(c). Let $x, y \in X$. Then by considering the various possibilities, in any case we have $d_1(x,y) - b(x,y) \leq d_1(x,y) - d_2(x,y) \leq A_X(d_1, d_2)$. Therefore $A_X(d_1, b) \leq A_X(d_1, d_2)$. □

2. Proposition. Let X be a set and functions $d, e, f, g : X \times X \to [0, \infty)$ be given. Then $A_X(d \vee e, f \vee g) \leq A_X(d, f) \vee A_X(e, g)$.

Proof. Let $x, y \in X$. Then we consider the four cases:

Case 1: $(d \vee e)(x,y) = d(x,y)$ and $(f \vee g)(x,y) = f(x,y)$. Then $(d \vee e)(x,y) - (f \vee g)(x,y) \leq A_X(d, f)$.

Case 2: $(d \vee e)(x,y) = d(x,y)$ and $(f \vee g)(x,y) = g(x,y)$. Then $(d \vee e)(x,y) - (f \vee g)(x,y) \leq d(x,y) - f(x,y) \leq A_X(d, f)$, because $f(x,y) \leq g(x,y)$.

**The general construction of the infimum of two quasi-pseudometrics will be discussed briefly below in the last section of this paper.

††Note that if d_1, d_2 are quasi-pseudometrics, then s is a quasi-pseudometric, while b need not satisfy the triangle inequality, as Example 2 shows.

Case 3: $(d \vee e)(x,y) = e(x,y)$ and $(f \vee g)(x,y) = f(x,y)$. Then $(d \vee e)(x,y) - (f \vee g)(x,y) \le e(x,y) - g(x,y) \le A_X(e,g)$, because $g(x,y) \le f(x,y)$.

Case 4: $(d \vee e)(x,y) = e(x,y)$ and $(f \vee g)(x,y) = g(x,y)$. Then $(d \vee e)(x,y) - (f \vee g)(x,y) \le A_X(e,g)$.

The assertion follows. □

2. Corollary. Let X be a set and functions $d,e,f,g : X \times X \to [0,\infty)$ be given. Then $A_X(\min\{d,e\}, \min\{f,g\}) \le A_X(d,f) \vee A_X(e,g)$.

Proof. Let $x,y \in X$. Then we consider the four cases:

Case 1: $(\min\{d,e\})(x,y) = d(x,y)$ and $(\min\{f,g\})(x,y) = f(x,y)$.
Then $(\min\{d,e\})(x,y) - (\min\{f,g\})(x,y) \le A_X(d,f)$.

Case 2: $(\min\{d,e\})(x,y) = d(x,y)$ and $(\min\{f,g\})(x,y) = g(x,y)$.
Then $(\min\{d,e\})(x,y) - (\min\{f,g\})(x,y) = d(x,y) - g(x,y) \le A_X(e,g)$, because $e(x,y) \ge d(x,y)$.

Case 3: $(\min\{d,e\})(x,y) = e(x,y)$ and $(\min\{f,g\})(x,y) = f(x,y)$.
Then $(\min\{d,e\})(x,y) - (\min\{f,g\})(x,y) = e(x,y) - f(x,y) \le A_X(d,f)$, because $d(x,y) \ge e(x,y)$.

Case 4: $(\min\{d,e\})(x,y) = e(x,y)$ and $(\min\{f,g\})(x,y) = g(x,y)$.
Then $(\min\{d,e\})(x,y) - (\min\{f,g\})(x,y) \le A_X(e,g)$.

The assertion follows. □

4. Lemma. Let $d_n\,(n \in \mathbb{N})$ and d be quasi-pseudometrics on a set X such that $\lim_{n\to\infty} A_X(d,d_n) = 0$. Then $\lim_{n\to\infty} A_X(d^{-1},(d_n)^{-1}) = 0$ and

$$\lim_{n\to\infty} A_X(d^s,(d_n)^s) = 0.$$

Proof. The first statement follows from Remark 4. The second statement is a consequence of Proposition 2: Indeed we conclude that $A_X(d^s,(d_n)^s) \le A_X(d,d_n) \vee A_X(d^{-1},(d_n)^{-1})$ whenever $n \in \mathbb{N}$. The assertion now is a consequence of the first statement. □

3. Example. Let X be a set and for each $\lambda \in [0,1]$ set $K(f,g,\lambda) = \lambda f + (1-\lambda)g$ where $f,g \in QPM(X)$ (compare [19]).

Note that $K(f,g,\lambda) = K(g,f,1-\lambda)$ whenever $f,g \in QPM(X)$ and $\lambda \in [0,1]$.

Furthermore, obviously, each $K(f,g,\lambda)$ is a quasi-pseudometric on X, $K(f,g,0) = g$ and $K(f,g,1) = f$.

Let $\lambda,\lambda' \in [0,1]$. Suppose that $\lambda' \le \lambda$.

Then by a straightforward computation we see that

$$A_X(K(f,g,\lambda),K(f,g,\lambda')) = (\lambda - \lambda')A_X(f,g)$$

and

$$A_X(K(f,g,\lambda'),K(f,g,\lambda)) = (\lambda - \lambda')A_X(g,f).$$

In particular, since for any quasi-pseudometric d on a set X we have that $A_X(d,d^{-1}) = A_X(d^{-1},d)$ by Remark 4, for any $\lambda,\lambda' \in [0,1]$ we get that

$$A_X(K(d,d^{-1},\lambda),K(d,d^{-1},\lambda')) = A_X(K(d,d^{-1},\lambda'),K(d,d^{-1},\lambda)) =$$

$$|\lambda - \lambda'|A_X(d,d^{-1}).$$

3. Corollary. Let X be a set and let d be a quasi-pseudometric on X. Set $d^+ = d + d^{-1}$. Then d^+ is a quasi-pseudometric on X.

We have $A_X(d, \frac{d^+}{2}) = A_X(K(d, d^{-1}, 1), K(d, d^{-1}, \frac{1}{2})) = \frac{1}{2}A_X(d, d^{-1})$ and similarly $A_X(\frac{d^+}{2}, d^{-1}) = A_X(K(d, d^{-1}, \frac{1}{2}), K(d, d^{-1}, 0)) = \frac{1}{2}A_X(d, d^{-1})$.

Indeed

$$A_X(d, \frac{d^+}{2}) = A_X(\frac{d^+}{2}, d^{-1}) =$$

$$\frac{1}{2}A_X(d, d^{-1}) = \frac{1}{2}A_X(d^{-1}, d) = A_X(d^{-1}, \frac{d^+}{2}) = A_X(\frac{d^+}{2}, d).$$

Proof. The assertion follows from Remark 4 and Example 3. □

3. The d_{ab}-construction

In the following we recall a modification of a T_0-quasi-metric d studied in [8, Section 5]. Below we give some of the details of the proofs that were omitted in [8, 9].

3. Proposition. (compare [8, Lemma 2]) Given a T_0-quasi-metric d on X and $a, b \in X$ be such that $d(a, b) > 0$ and $d(b, a) > 0$, we define $d_{ab}(x, y) = \min\{d(x, a) + d(b, y), d(x, y)\}$ whenever $x, y \in X$. Then d_{ab} is the largest T_0-quasi-metric satisfying $e \leq d$ on X such that $e(a, b) = 0$.

Proof. The statement that $d_{ab} \leq d$ is obvious by definition of d_{ab}. Furthermore $d_{ab}(a, b) = 0$, hence $d_{ab} < d$. It is easy to see that d_{ab} is a quasi-pseudometric: We only have to show that $d_{ab}(x, z) \leq d_{ab}(x, y) + d_{ab}(y, z)$ whenever $x, y, z \in X$.

We consider the four cases:
(1) $d_{ab}(x, y) = d(x, y)$ and $d_{ab}(y, z) = d(y, z)$.
(2) $d_{ab}(x, y) = d(x, a) + d(b, y)$ and $d_{ab}(y, z) = d(y, z)$.
(3) $d_{ab}(x, y) = d(x, y)$ and $d_{ab}(y, z) = d(y, a) + d(b, z)$.
(4) $d_{ab}(x, y) = d(x, a) + d(b, y)$ and $d_{ab}(y, z) = d(y, a) + d(b, z)$.

In Case (1) we obtain $d_{ab}(x, z) \leq d(x, z) \leq d(x, y) + d(y, z)$.

In Case (2) we obtain $d_{ab}(x, z) \leq d(x, a) + d(b, z) \leq d(x, a) + d(b, y) + d(y, z)$.

In Case (3) we obtain $d_{ab}(x, z) \leq d(x, a) + d(b, z) \leq d(x, y) + d(y, a) + d(b, z)$.

In Case (4) we obtain $d_{ab}(x, z) \leq d(x, a) + d(b, z) \leq d(x, a) + d(b, y) + d(y, a) + d(b, z)$.

Hence we are done. In the proof of [8, Lemma 2] it is argued that d_{ab} satisfies the T_0-condition (c), because d does so and because $d(b, a) > 0$.

Let us now note that if $e \leq d$ is a quasi-pseudometric on X such that $e(a, b) = 0$, then we have that for any $x, y \in X$, $e(x, y) \leq e(x, a) + e(a, b) + e(b, y) \leq d(x, a) + d(b, y)$ and $e(x, y) \leq d(x, y)$. Therefore $e \leq d_{ab}$. □

6. Remark. Let (X, d) be a T_0-quasi-metric space and let $a, b \in X$ be $\leq_d$-incomparable. Then $(d_{ab})^{-1} = (d^{-1})_{ba}$ according to [9, Remark 1]: Indeed let $x, y \in X$. Then $(d_{ab})^{-1}(x, y) = \min\{d(y, a) + d(b, x), d(y, x)\} = \min\{d^{-1}(x, b) + d^{-1}(a, y), d^{-1}(x, y)\} = (d^{-1})_{ba}(x, y)$.

4. Proposition. Let d be a T_0-quasi-metric on a set X and let $a, b \in X$ be incomparable with respect to the specialization order of d, that is, $d(a, b) > 0$ and $d(b, a) > 0$.

(a) We have that $A_X(d_{ab}, d) = 0$.
(b) Moreover the equation $A_X(d, d_{ab}) = d(a, b)$ holds.

Proof. (a) The statement immediately follows from $d_{ab} \leq d$.

(b) By definition $A_X(d, d_{ab}) = \sup_{(x,y)\in X\times X}(d(x, y)\dot{-}d_{ab}(x, y))$. We need to consider two possible differences in the latter expression: $d(x, y)\dot{-}d(x, y) = 0$ or $d(x, y)\dot{-}(d(x, a) + d(b, y))$. But $d(x, y) - d(x, a) - d(b, y) \leq d(a, b)$ by the triangle inequality. Note that equality in the latter inequality holds for $(x, y) = (a, b)$. Indeed $d(a, b)\dot{-}d_{ab}(a, b) = d(a, b) - 0$. We conclude that $A_X(d, d_{ab}) = d(a, b)$. □

5. Proposition. Let (X, d) be a T_0-quasi-metric space and let $a, b \in X$ be $\leq_d$-incomparable. Then $d(b, a) \leq A_X(d_{ab}, (d_{ab})^{-1}) \leq d(a, b) + A_X(d, d^{-1})$.

Proof. The first inequality follows from the fact that $d_{ab}(b, a) - (d_{ab})^{-1}(b, a) = d(b, a) - 0 = d(b, a)$.

We then have the following chain of inequalities: By the triangle inequality, Remark 6 and Proposition 4 we see that $A_X(d_{ab}, (d_{ab})^{-1}) \leq A_X(d_{ab}, d) + A_X(d, d^{-1})+A_X(d^{-1}, (d_{ab})^{-1}) = 0+A_X(d, d^{-1})+A_X(d^{-1}, (d^{-1})_{ba}) = A_X(d, d^{-1})+ d^{-1}(b, a)$. □

4. Corollary. Let (X, m) be a metric space and let $a, b \in X$ be two distinct points in X. Then $A_X(m_{ab}, (m_{ab})^{-1}) = m(a, b)$.

Proof. The result follows from Proposition 5, since m is a metric and $A_X(m, m^{-1}) = 0$. □

4. Some bicomplete subspaces of the space of all quasi-pseudometrics

An (extended) quasi-pseudometric space (X, d) is called *bicomplete* if the (extended) pseudometric space (X, d^s) is complete, that is, each d^s-Cauchy sequence in X converges with respect to the pseudometric topology $\tau(d^s)$.

5. Lemma. The extended metric space $(QPM(X), (A_X)^s)$ is complete, hence $(QPM(X), A_X)$ is bicomplete.

Proof. The standard proof that the set of real-valued functions on a set X with the uniform sup-metric is complete shows that each Cauchy sequence $(d_n)_{n\in\mathbb{N}}$ of quasi-pseudometrics in $(QPM(X), (A_X)^s)$ has a $[0, \infty)$-valued limit function a on $X \times X$ to which it converges uniformly. Therefore we only need to show that a is a quasi-pseudometric on X. But this follows from the observation that the pointwise limit of a sequence of quasi-pseudometrics is a quasi-pseudometric: Indeed for each $x \in X$ we have $d(x, x) = \lim_{n\to\infty} d_n(x, x) = \lim_{n\to\infty} 0 = 0$. Furthermore we see that for any $x, y, z \in X$ we have that $d_n(x, z) \leq d_n(x, y) + d_n(y, z)$. Therefore taking limits in the reals equipped with the usual topology, we get that $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$. □

A quasi-pseudometric d on a set X is called *bounded* if there is $b \in [0, \infty)$ such that $d(x, y) \leq b$ whenever $x, y \in X$, that is, its diameter $\delta_d < \infty$. By $BQPM(X)$ we shall denote the set of bounded quasi-pseudometrics on X.

6. Proposition. The set $BQPM(X)$ of bounded quasi-pseudometrics is closed in $(QPM(X), \tau((A_X)^s))$.

Proof. Suppose that $(d_n)_{n\in\mathbb{N}}$ is a sequence of bounded quasi-pseudometrics on X such that $(A_X)^s(d, d_n) \to 0$ where $d \in QPM(X)$. There is $n \in \mathbb{N}$ such that $(A_X)^s(d_n, d) < 1$. By assumption there is $a \in [0, \infty)$ such that $\delta_{d_n} \leq a$. Then for any $(x, y) \in X \times X$ we have that $d(x, y) \leq (d(x, y) - d_n(x, y)) + d_n(x, y) \leq 1 + a$. Therefore the quasi-pseudometric d is bounded, too. □

6. Lemma. Given a set X with at least 2 points, the set of all T_0-quasi-metrics is not closed in $(QPM(X), \tau((A_X)^s))$.

Proof. For any fixed T_0-quasi-metric d on X, the indiscrete quasi-pseudometric $i(x, y) = 0$ whenever $(x, y) \in X \times X$ is obviously the uniform limit of the sequence $(\frac{1}{n}d)_{n\in\mathbb{N}}$ in $(QPM(X), \tau((A_X)^s))$, but i is not a T_0-quasi-metric in case that X contains at least two points. □

7. Proposition. Let X be a set and $PM(X)$ the set of all pseudometrics belonging to $QPM(X)$. Then $PM(X)$ is closed in $(QPM(X), \tau((A_X)^s))$.

Proof. Suppose that the sequence $(m_n)_{n\in\mathbb{N}}$ of pseudometrics on X converges to the quasi-pseudometric d on X in the sense that $(A_X)^s(m_n, d) \to 0$. Therefore $d(x, y) = \lim_{n\to\infty} m_n(x, y) = \lim_{n\to\infty} m_n(y, x) = d(y, x)$ whenever $x, y \in X$. The statement follows. □

Recall that a quasi-pseudometric d on a set X is called *totally bounded* provided that given any $\epsilon > 0$, there is a finite subset F_ϵ of X such that for each $x \in X$ there is $f \in F_\epsilon$ such that $d^s(x, f) < \epsilon$.

Of course, the standard proof shows that each totally bounded quasi-pseudometric is bounded: Indeed given a totally bounded quasi-pseudometric d on X choose a finite subset F_1 of X as given by the definition. Then for any $x, y \in X$ we have that $d(x, y) \leq 1 + \max_{f,f'\in F_1} d(f, f') + 1$ by an obvious application of the triangle inequality.

8. Proposition. Let X be a set and let $TQPM(X)$ be the set of all totally bounded quasi-pseudometrics on X.

Then $TQPM(X)$ is closed in $(QPM(X), \tau((A_X)^s))$.

Proof. Let $(d_n)_{n\in\mathbb{N}}$ be a sequence of totally bounded quasi-pseudometrics on X converging to a quasi-pseudometric d in $(QPM(X), \tau((A_X)^s))$.

Let $\epsilon > 0$. There is $m \in \mathbb{N}$ such that $(A_X)^s(d, d_m) < \epsilon$. Furthermore there is a finite subset F of X such that for any $x \in X$ there is an $f \in F$ such that $(d_m)^s(x, f) < \epsilon$. Thus for any $x \in X$ there is $f \in F$ such that $d(x, f) \leq (d(x, f) - d_m(x, f)) + d_m(x, f) \leq (A_X)^s(d, d_m) + \epsilon = 2\epsilon$ and similarly, $d(f, x) \leq (d(f, x) - d_m(f, x)) + d_m(f, x) \leq (A_X)^s(d, d_m) + \epsilon = 2\epsilon$. We conclude that d is totally bounded. □

Recall that a quasi-pseudometric d on a set X is called an *ultra-quasi-pseudometric* provided that $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ whenever $x, y, z \in X$. The latter inequality is called the *strong triangle inequality* for d.

9. Proposition. The set of all ultra-quasi-pseudometrics on a set X is $\tau((A_X)^s)$-closed in $QPM(X)$.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of ultra-quasi-pseudometrics on X converging to the quasi-pseudometric d with respect to the topology $\tau((A_X)^s)$.

Using (uniform) convergence, the existence of $x, y, z \in X$ such that $d(x, z) > \max\{d(x, y), d(y, z)\}$ would imply the existence of an $n \in \mathbb{N}$ such that $d_n(x, z) > \max\{d_n(x, y), d_n(y, z)\}$ —a contradiction. The assertion follows. □

7. Lemma. Each quasi-pseudometric space (X, d) with d having a finite range is bicomplete.

Proof. The statement obviously holds for the indiscrete quasi-pseudometric on X. So we can assume that d is not indiscrete. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a d^s-Cauchy sequence in X. Then there is $\epsilon > 0$ such that $\epsilon \leq \min(d(X \times X) \setminus \{0\})$. Hence we have that there is $N_\epsilon \in \mathbb{N}$ such that $0 = d(x_n, x_m) < \epsilon$ whenever $n, m \in \mathbb{N}$ with $n, m \geq N_\epsilon$. We conclude that $(x_n)_{n\in\mathbb{N}}$ converges to x_{N_ϵ} in (X, d^s) and thus d is bicomplete. □

Our next example shows that the subset of complete pseudometrics need not be closed in $(QPM(X), \tau((A_X)^s))$, which also shows that the subset of bicomplete quasi-pseudometrics need not be closed in $(QPM(X), \tau((A_X)^s))$.

4. Example. Let $X = [0, 1) \subseteq \mathbb{R}$ and let $d(x, y) = |x - y|$ whenever $x, y \in X$ be the usual metric on X.

Furthermore for any $x \in X$ suppose that $p(x) = 0.e_1e_2e_3 \ldots e_n \ldots$ is a fixed decimal representation of x with infinitely many digits. Of course, $d(x, y) = |p(x) - p(y)|$ whenever $x, y \in X$.

For each $n \in \mathbb{N}$ let $p_n(x) = 0.e_1e_2 \ldots e_n$. Of course, for each $n \in \mathbb{N}$, $d_n(x, y) = |p_n(x) - p_n(y)|$ whenever $x, y \in X$ is a pseudometric. Note that each d_n has a finite range.

Obviously $\lim_{n\to\infty}(A_X)^s(d_n, d) = 0$, since by Lemma 1

$$(A_X)^s(d_n, d) = \sup_{(x,y)\in X\times X} |d_n(x, y) - d(x, y)|$$

$$= \sup_{(x,y)\in X\times X} ||p_n(x) - p_n(y)| - |p(x) - p(y)||$$

$$\leq \sup_{x\in X} |p(x) - p_n(x)| + \sup_{y\in X} |p(y) - p_n(y)| \leq \frac{2}{10^n}.$$

Furthermore $(1 - \frac{1}{n})_{n\in\mathbb{N}}$ is a d-Cauchy sequence that is not convergent in $(X, \tau(d))$ and thus d not complete. However by Lemma 7 each pseudometric d_n is complete and $(A_X)^s(d_n, d) \to 0$.

The following concept was introduced by Steve Matthews.

2. Definition. (see for instance [5, 18, 15]) Let (X, f) be a quasi-pseudometric space. If there exists a function $w : X \to [0, \infty)$ such that $f(x, y) + w(x) = f(y, x) + w(y)$ whenever $x, y \in X$, then f is called *nonnegatively weightable* and w is said to be a *nonnegative weight* for (X, f).

7. Remark. Note that the weight of a nonnegatively weightable quasi-pseudometric is not unique; for instance for a given metric space (X, m) any nonnonegative real constant function yields a nonnegative weight function.

That is why in the proof given below, if $n \in \mathbb{N}$ and w_n is a weight function for a nonnegatively weightable quasi-pseudometric space (X, d_n), we cannot expect that the sequence $(w_n)_{n\in\mathbb{N}}$ converges to some nonnegative weight function of $\lim_{n\to\infty} d_n$, even if the latter limit exists. □

10. Proposition. The set $WQPM(X)$ of all nonnegatively weightable quasi-pseudometrics on X is $\tau((A_X)^s)$-closed in $QPM(X)$.

Proof. Suppose that $(d_n)_{n\in\mathbb{N}}$ is a sequence of nonnegatively weightable quasi-pseudometrics on X and $(A_X)^s(d, d_n) \to 0$ where $d \in QPM(X)$. For each $n \in \mathbb{N}$ and $x, y \in X$ set $F_n(x, y) := d_n(x, y) - d_n(y, x)$, that is, F_n is the *disymmetry* function of d_n in the sense of [5].

Then $|F_n(x, y) - F_m(x, y)| \leq |d_n(x, y) - d_m(x, y)| + |d_n(y, x) - d_m(y, x)|$ whenever $x, y \in X$ and $n, m \in \mathbb{N}$.

Since $(d_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(QPM(X), (A_X)^s)$, we conclude that for each $(x, y) \in X \times X$, $(F_n(x, y))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{R}, u^s)$.

For each $(x, y) \in X \times X$ set $F(x, y) = \lim_{n\to\infty} F_n(x, y)$. By the previous argument we see that indeed $\lim_{n\to\infty}(A_X)^s(F_n, F) = 0$.

It is known by [5, Theorem 3.5] and readily checked that, by the weightability of d_n, $F_n(x, z) = F_n(x, y) + F_n(y, z)$ whenever $n \in \mathbb{N}$ and $x, y, z \in X$. By taking limits we have therefore $F(x, z) = F(x, y) + F(y, z)$ whenever $x, y, z \in X$. We deduce that $F(x, y) = d(x, y) - d(y, x) = \phi(y) - \phi(x)$ for some function $\phi : X \to \mathbb{R}$ by Sincov's functional equation [11].

It remains to be seen that we can choose the function ϕ in such a way that $\phi(y) \geq 0$ whenever $y \in X$.

By the argument above we can find $n \in \mathbb{N}$ such that $|F_n(x, y) - F(x, y)| < 1$ whenever $(x, y) \in X \times X$.

Fix $x \in X$. Since F_n stems from a nonnegatively weightable quasi-pseudometric d_n with a nonnegative weight $\phi_n : X \to [0, \infty)$, we have $F_n(x, y) = d_n(x, y) - d_n(y, x) = \phi_n(y) - \phi_n(x) \geq -\phi_n(x)$ whenever $y \in X$.

Hence $-\phi_n(x) \leq F_n(x, y)$ whenever $y \in X$ and therefore $-\phi_n(x) - F(x, y) \leq F_n(x, y) - F(x, y) < 1$. Thus $-\phi_n(x) - 1 \leq F(x, y) = \phi(y) - \phi(x)$ whenever $y \in X$. We conclude that $-\phi_n(x) + \phi(x) - 1 \leq \phi(y)$ whenever $y \in X$. Therefore $w(y) := \phi(y) + \phi_n(x) - \phi(x) + 1$ whenever $y \in X$ is a nonnegative weight for d. □

5. The difference approach to the skewness of a quasi-pseudometric

In this section we are interested in measuring the asymmetry or skewness of a T_0-quasi-metric f on a set X. Several methods suggest themselves.

For instance we could compare the specialization orders $\leq_f$ and $\leq_{f^{-1}}$, or we could compare the topologies $\tau(f)$ and $\tau(f^{-1})$. Observe that $\leq_f = \leq_{f^{-1}}$ iff the specialization order $\leq_f$ is equality, that is, f is a T_1-quasi-metric. (A quasi-pseudometric d on X satisfying the condition that $d(x, y) \neq 0$ whenever $x, y \in X$ with $x \neq y$ is called a T_1*-quasi-metric*.) Of course, $\tau(f) = \tau(f^{-1})$ if and only if for any $x \in X$ and sequence $(x_n)_{n\in\mathbb{N}}$ in X, $\lim_{n\to\infty} f(x, x_n) = 0$ iff $\lim_{n\to\infty} f^{-1}(x, x_n) = 0$.

We could also study relationships between the induced quasi-uniformities $\mathcal{U}_f$ and $\mathcal{U}_{f^{-1}}$, or the induced totally bounded quasi-uniformities $(\mathcal{U}_f)_\omega$ and $(\mathcal{U}_{f^{-1}})_\omega$.‡‡ Observe that $\mathcal{U}_f = \mathcal{U}_{f^{-1}}$ iff $\mathcal{U}_f$ is a uniformity. Similarly $(\mathcal{U}_f)_\omega = (\mathcal{U}_{f^{-1}})_\omega$ iff $(\mathcal{U}_f)_\omega$ is a uniformity (compare [7, Corollary 1.40]).

In the following we shall consider a metric approach to asymmetry that is more in the spirit of paper [5] where the function $F(x,y) = d(x,y) - d(y,x)$ (whenever $x, y \in X$) of disymmetry is considered. The following sets might be of special interest for a more detailed study on asymmetry, which will be conducted elsewhere.

5. Example. Let (X,d) be a T_0-quasi-metric space and let $k, r \in [0,\infty)$.

(a) Let $S_{d,k} = \{(x,y) \in X \times X : |d(x,y) - d(y,x)| \leq k\}$. Then $S_{d,k}$ is a $\tau(d^s) \times \tau(d^s)$-closed symmetric reflexive relation. We can call it the *set of k-symmetric pairs*.

(b) $A_{d,k} = \{(x,y) \in X \times X : |d(x,y) - d(y,x)| \geq k\}$ is a $\tau(d^s) \times \tau(d^s)$-closed symmetric relation. We can call it the *set of k-asymmetric pairs*.

(c) Further interesting tools to measure asymmetry could be the sets of reals $\sigma_{d,k;r} = \{d(x,y) : (x,y) \in X \times X \text{ and } |d(y,x) - r| \leq k\}$ and $\alpha_{d,k;r} = \{d(x,y) : (x,y) \in X \times X \text{ and } |d(y,x) - r| \geq k\}$.

In particular we can speak of a *symmetric pair* $(x,y) \in X \times X$ if $d(x,y) = d(y,x)$ and call $x \in X$ a *symmetric point* of (X,d) provided that $d(x,y) = d(y,x)$ whenever $y \in X$.

In the present paper we shall concentrate on investigating the following much simpler concept.

3. Definition. Let (X,d) be a quasi-pseudometric space. We define $A_d := A_X(d,d^{-1}) = \sup_{(x,y)\in X\times X}(d(x,y) \dot{-} d(y,x)) = \sup_{(x,y)\in X\times X}|d(x,y) - d(y,x)|$.

8. Remark. Of course if X is finite, it may be more reasonable to consider the T_0-quasi-metric $S_X(d,e) := \sum_{(x,y)\in X\times X}(d(x,y) \dot{-} e(x,y))$ for $d, e \in QPM(X)$ and then for instance to investigate the value

$$S_X^{\oplus}(d,d^{-1}) = \frac{1}{2} \sum_{(x,y)\in X\times X} |d(x,y) - d(y,x)|$$

in order to make sure that all the relevant differences can contribute to the value of asymmetry.

But we shall restrict our study in the following to the value $A_X(d,d^{-1})$, which is much easier to handle.

Let us consider some examples.

6. Example. Let $X = [a,b]$ be the closed interval with endpoints a and b of the set $\mathbb{R}$. Then $A_u = A_X(u,u^{-1}) \geq u(b,a) - u^{-1}(b,a) = b - a$, where u denotes also the restriction of u to $[a,b]$.

The following observation was already stated in the introduction.

‡‡Here as usual, for any quasi-uniformity $\mathcal{U}$ on a set X, $\mathcal{U}_\omega$ will denote the finest totally bounded quasi-uniformity coarser than $\mathcal{U}$ on X.

9. Remark. Let f be a quasi-pseudometric on a set X. Then $A_f = 0$ if and only if f is a pseudometric on X.

7. Example. Let (X, d, w) be a nonnegatively weighted quasi-pseudometric space, that is, $d(x, y) + w(x) = d(y, x) + w(y)$ whenever $x, y \in X$ where $w : X \to [0, \infty)$ is the weight function. Therefore $A_d = \sup_{(x,y)\in X\times X} |w(y) - w(x)|$.

8. Example. Let $X = [0, \infty)$ and for all $x, y \in X$ set $d(x, y) = 0$ if $x \leq y$ and $d(x, y) = x$ if $x \not\leq y$, where $\leq$ is the usual order on X. We first note that d is a T_0-ultra-quasi-metric on X : Observe that if $x, y \in X$ such that $x < y$, then $d^s(x, y) \geq y$, which shows that the T_0-condition (c) is satisfied by d.

We next verify that d satisfies the strong triangle inequality: Otherwise there are $x, y, z \in X$ such that $d(x, z) \not\leq \max\{d(x, y), d(y, z)\}$. Then $x \not\leq z$ and thus $d(x, z) = x$. Note that the case that $x \leq y$ and $y \leq z$ is impossible, since $x \not\leq z$.

If $x \not\leq y$, then $d(x, y) = x$ and the strong triangle inequality for d is satisfied.

On the other hand, if $x \leq y$ and $y \not\leq z$, then $d(y, z) = y$ and the strong triangle inequality is satisfied for d, because $d(x, z) \leq d(y, z)$. Hence d is a T_0-ultra-quasi-metric.

We now conclude the following: Let $x, y \in [0, \infty)$. If $y < x$, then $d(x, y)\dot{-}d(y, x) = x\dot{-}0 = x$. If $y = x$, then $d(x, y)\dot{-}d(y, x) = 0\dot{-}0 = 0$. If $y > x$, then $d(x, y)\dot{-}d(y, x) = 0\dot{-}y = 0$.

Therefore for each $x \in X$, $\sup_{y\in X}(d(x, y)\dot{-}d(y, x)) = x$ and for each $y \in X$, $\sup_{x\in X}(d(x, y)\dot{-}d(y, x)) = \infty$. In particular $A_d = \infty$. □

8. Lemma. Let (X, d) be a quasi-pseudometric space. Then $A_d \leq \delta_d$ where δ_d denotes the diameter of (X, d).

Proof. For any $(x, y) \in X \times X$ we have that $d(x, y) - d(y, x) \leq d(x, y)$. □

9. Lemma. Let d, d' be quasi-pseudometrics on a set X and $\lambda \in [0, \infty)$. Then the following inequalities hold:

(a) $A_{\lambda d} = \lambda A_d$.

(b) $A_{d+d'} \leq A_d + A_{d'}$.

(c) $A_{d\vee d'} \leq A_d \vee A_{d'}$. Furthermore $A_{\min\{d,d'\}} \leq A_d \vee A_{d'}$ (where $\min\{d, d'\}$ in general is not a quasi-pseudometric on X).

(d) $A_d = A_{d^{-1}}$.

Proof. The statements follow from Lemma 2(b), Lemma 2(a), Proposition 2, Corollary 2 and Remark 4. □

10. Remark. Given a quasi-pseudometric d on a set X, we cannot establish any nontrivial lower bounds for $A_{d+d^{-1}}$ and $A_{d\vee d^{-1}}$ in (b) and (c) above: Note that for any quasi-pseudometric d on X we have that $A_{d+d^{-1}} = 0 = A_{d\vee d^{-1}}$. Considering the space $(\mathbb{R}, u)$, we observe that $u \wedge u^{-1} = \min\{u, u^{-1}\} = \underline{0}$ is the constant indiscrete quasi-pseudometric equal to 0 on $\mathbb{R} \times \mathbb{R}$. Since $A_{\underline{0}} = 0$, we deduce that there is also no nontrivial lower bound for $A_{d\wedge d^{-1}}$.

The following result shows that quasi-pseudometrics that are close to each other have asymmetry values that are close to each other, too.

10. Lemma. For any quasi-pseudometrics p and q on a set X such that $(A_X)^s(p,q) < \infty$ we have that either $(A_X)^s(p,p^{-1}) = (A_X)^s(q,q^{-1}) = \infty$ or $|(A_X)^s(p,p^{-1}) - (A_X)^s(q,q^{-1})| \leq 2(A_X)^s(p,q)$.

Proof. Suppose that $(A_X)^s(p,p^{-1}) = \infty$. Then by the triangle inequality we have that $(A_X)^s(p,p^{-1}) \leq (A_X)^s(p,q) + (A_X)^s(q,q^{-1}) + (A_X)^s(q^{-1},p^{-1})$. By Remark 4 and our assumption we see that $(A_X)^s(q,q^{-1}) = \infty$, too. The case that $(A_X)^s(q,q^{-1}) = \infty$ implies similarly that $(A_X)^s(p,p^{-1}) = \infty$.

So assume that both $(A_X)^s(p,p^{-1})$ and $(A_X)^s(q,q^{-1})$ are $< \infty$. By Remark 4 we conclude analogously as in Lemma 1 that $|(A_X)^s(p,p^{-1}) - (A_X)^s(q,q^{-1})| \leq (A_X)^s(p,q) + (A_X)^s(p^{-1},q^{-1}) = 2(A_X)^s(p,q)$. □

According to [21, p. 131] a *costfunction* is an arbitrary function $g : [0,\rightarrow) \longrightarrow [0,\rightarrow)$ with $g(0) = 0$ that is concave (so $g((1-\lambda)s + \lambda t) \geq (1-\lambda)g(s) + \lambda g(t)$ whenever $s,t \in [0,\infty)$ and $\lambda \in [0,1]$).* For instance $g(x) = \sqrt{x}$ whenever $x \in [0,\infty)$ defines such a costfunction.

11. Proposition. Let d be a quasi-pseudometric on a set X and let g be a costfunction on $[0,\infty)$. Then $A_{g\circ d} \leq g(A_d)$.

Proof. We first note that $g \circ d$ is a quasi-pseudometric on X (compare [21, Theorem 5, Lemma 3 (2) and (3)]). Now we are going to establish the stated inequality.

Case 1: Let $x,y \in X$. If $g(d(x,y)) - g(d(y,x)) \leq 0$, then obviously $g(d(x,y)) - g(d(y,x)) \leq 0 = g(0) \leq g(A_d)$, because g is nondecreasing [21, Lemma 3 (3)] and $0 \leq A_d$.

Case 2: Suppose now that $g(d(x,y)) - g(d(y,x)) > 0$. Thus $g(d(x,y)) > g(d(y,x))$. Then $d(x,y) \leq d(y,x)$ is impossible, since g is nondecreasing [21, Lemma 3 (3)]. Thus necessarily $d(x,y) > d(y,x)$. Therefore $g(d(x,y)) = g(d(x,y) - d(y,x) + d(y,x)) \leq g(d(x,y) - d(y,x)) + g(d(y,x))$ using [21, Lemma 3 (2)]. It follows that $g(d(x,y)) - g(d(y,x)) \leq g(d(x,y) - d(y,x)) \leq g(A_d)$, since g is nondecreasing and $d(x,y) - d(y,x) \leq A_d$. We conclude that $A_{g\circ d} \leq g(A_d)$. □

9. Example. Let (X,d) be a quasi-pseudometric space. It is well known that $\frac{d}{1+d}$ is a bounded quasi-pseudometric on X. See for instance [21, Example 1]: Indeed it suffices to note that $s \mapsto \frac{s}{1+s}$ is a costfunction. By Proposition 11 we then have that $A_{\frac{d}{1+d}} \leq \frac{A_d}{A_d+1}$ if $A_d < \infty$, and $A_{\frac{d}{1+d}} \leq 1$ if $A_d = \infty$.

6. Asymmetrically normed real vector spaces

We next recall the concept of an asymmetric norm (see for instance [6]; compare [21, Section 2.5] or [20, p. 183]), which leads to many interesting examples of quasi-pseudometrics.

4. Definition. Let X be a real vector space and let $\|\cdot| \to [0,\infty)$ be a map such that

(1) $\|0| = 0$.

(2) $\|x+y| \leq \|x| + \|y|$ whenever $x,y \in X$.

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* For possible use in our two next results we also set $g(\infty) := \sup_{x\in[0,\infty)} g(x)$.

(3) $\|\alpha x| = \alpha \|x|$ whenever $x \in X$ and $\alpha \geq 0$. Furthermore suppose that $\|x| = \|-x| = 0$ implies that $x = 0$.

The function $\|\cdot|$ is called an *asymmetric norm* on X. It is known that each asymmetrically normed vector space X induces a T_0-quasi-metric d on X by setting $d(x,y) = \|x-y|$ whenever $x, y \in X$.

To motivate the preceding definition we recall the concept of the asymmetric segment.

10. Example. [1, Remark 2] Let $X = [0,1]$. Find $a, b \in [0,\infty)$ such that $a+b \neq 0$. Set $d_{[ab]}(x,y) = (x-y)a$ if $x > y$ and $d_{[a,b]}(x,y) = (y-x)b$ if $y \geq x$. Then $([0,1], d_{[ab]})$ is a T_0-quasi-metric space induced by the asymmetric norm $n_{[ab]}$ on $\mathbb{R}$ defined by $n_{[ab]}(x) = xa$ if $x > 0$ and $n_{[ab]}(x) = -xb$ if $x \leq 0$.

The following related example then yields another illustration of Proposition 11.

11. Example. Let $X = [-1,1]$ be the real interval and set for $x, y \in X$ $d(x,y) = |x-y|$ if $x \geq y$ and $d(x,y) = 2|x-y|$ if $x < y$. Then by Example 10 d is a T_0-quasi-metric on X.

Using the costfunction $g(x) = \sqrt{x}$ $(x \in [0,\infty))$ we compute that

$A_d = \sup_{(x,y) \in X \times X} |x-y| = 2$ and hence $\sqrt{A_d} = \sqrt{2}$,

while $A_{\sqrt{d}} = \sup_{(x,y) \in X \times X} (\sqrt{2}-1)\sqrt{|x-y|} = 2 - \sqrt{2}$, which is indeed $< \sqrt{2}$.

11. Remark. Given a set X, it is often useful to abuse the notation and write $A_X(f,g) = \|f-g|$ where $f, g \in QPM(X)$, although in this case obviously not all conditions of Definition 4 are satisfied, since the vector space structure is missing.

12. Proposition. Let X be a non-trivial real vector space, let $\|\cdot|$ be an asymmetric norm on X and let d be the induced T_0-quasi-metric as defined above. Then $A_d = \sup_{x \in X} |\|-x| - \|x||$. Hence $A_d = \infty$ if $\|\cdot|$ is not a norm. □

Proof. The first statement is obvious. For the second statement, without loss of generality there is $x_0 \in X$ such that $\|-x_0| > \|x_0|$. Let $\alpha > 0$. Then $d(0,\alpha x_0) - d(\alpha x_0, 0) = \|0-\alpha x_0| - \|\alpha x_0 - 0| = \alpha(\|-x_0| - \|x_0|)$, which can be made arbitrarily large by choosing α appropriately. □

12. Remark. In [21] a multiplicative approach to an asymmetry measure σ_d of a T_0-quasi-metric d on a set X (with at least two elements) is chosen: σ_d is computed as

$$\sup_{(x,y) \in (X \times X) \setminus \Delta_X} \frac{d(x,y)}{d(y,x)},$$

where the latter expression is defined to be infinite in case that $d(y,x) = 0$ for some $(x,y) \in (X \times X) \setminus \Delta_X$. Hence this definition is mainly suitable for a T_1-quasi-metric. We also note that this approach is very useful in an asymmetrically normed space $(X, \|\cdot|)$, since in this case for an induced T_1-quasi-metric d the value σ_d does not depend on the length $\|z|$ of the vector $z \in X$ and thus can be determined on the unit sphere $\{z \in X : \|z| = 1\}$ (see Proposition 12 and compare [21, Lemma 10]).

We refer the reader to [4, Section 4] for a short discussion of connections between additive and multiplicative approaches to distance functions.

7. Some properties of A_d where d is a quasi-pseudometric

Given a quasi-pseudometric d on a set X, in this section we prove two simple facts about the asymmetry value A_d of d.

13. Proposition. Let (X,d) be a quasi-pseudometric space such that the topology $\tau(d^s)$ is compact. Then there is $(a,b) \in X \times X$ such that $A_d = d(a,b) - d(b,a)$, that is, the supremum A_d is attained.

Proof. We sketch the standard argument. By compactness of the pseudometric topology $\tau(d^s)$, we see that d is bounded. Hence $A_d < \infty$ by Lemma 8. Therefore there is a sequence $(x_n, y_n)_{n\in\mathbb{N}}$ in $X \times X$ such that the real sequence $(F(x_n,y_n))_{n\in\mathbb{N}}$, where for each $n \in \mathbb{N}$ $F(x_n,y_n) = d(x_n,y_n) - d(y_n,x_n)$, converges to the value A_d. By compactness of $\tau(d^s)$ there is a subsequence $(n_k)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ and $x, y \in X$ such that $(x_{n_k})_{k\in\mathbb{N}}$ resp. $(y_{n_k})_{k\in\mathbb{N}}$ $\tau(d^s)$-converges to x resp. y in X. Since $\lim_{n\to\infty} F(x_n,y_n) = A_d$, we conclude that $F(x,y) = A_d$ by continuity of d on $(X \times X, \tau(d^s) \times \tau(d^s))$. $\square$

11. Lemma. Let (X,d) be a quasi-pseudometric space and $Y \subseteq X$. Then

$$\sup_{(x,y)\in Y\times Y} |d(x,y) - d(y,x)| \leq \sup_{(x,y)\in X\times X} |d(x,y) - d(y,x)|.$$

Proof. The argument is obvious. $\square$

Our next result considers a density condition under which the inverse inequality also holds.

14. Proposition. Let Y be a subspace of a quasi-pseudometric space (X,d) such that $\mathrm{cl}_{\tau(d^s)} Y = X$. Then $A_Y(d|_{Y\times Y}, d^{-1}|_{Y\times Y}) = A_X(d, d^{-1})$.

Proof. Let $x, y \in \mathrm{cl}_{\tau(d^s)} Y$. Then there are sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X such that $d^s(x,x_n) \to 0$ and $d^s(y_n,y) \to 0$. Fix $n \in \mathbb{N}$. Then $|d(x,y) - d(y,x)| \leq |d(x,y) - d(x_n,y_n)| + |d(x_n,y_n) - d(y_n,x_n)| + |d(y_n,x_n) - d(y,x)| \leq d^s(x,x_n) + d^s(y,y_n) + |d(x_n,y_n) - d(y_n,x_n)| + d^s(y_n,y) + d^s(x_n,x) \leq 2d^s(x_n,x) + 2d^s(y_n,y) + \sup_{(x,y)\in Y\times Y} |d(x,y) - d(y,x)|$ by Lemma 1. Therefore

$$\sup_{(x,y)\in X\times X} |d(x,y) - d(y,x)| \leq \sup_{(x,y)\in Y\times Y} |d(x,y) - d(y,x)|.$$

Hence the stated equality is established. $\square$

5. Corollary. Let (X,d) be a T_0-quasi-metric with bicompletion $(\widetilde{X}, \widetilde{d})$ (see [13, Example 2.7.1]). Then $A_X(d, d^{-1}) = A_{\widetilde{X}}(\widetilde{d}, (\widetilde{d})^{-1})$.

Proof. It is known that X is $\tau((\widetilde{d})^s)$-dense in $\widetilde{X}$. $\square$

8. The q-hyperconvex hull of a T_0-quasi-metric space

We first recall some basic facts about the q-hyperconvex hull of a T_0-quasi-metric space. For additional information we refer the reader to [12, 17] and the literature cited in these papers.

Let (X,d) be a T_0-quasi-metric space. We consider the set Q_X of all function pairs $f = (f_1, f_2)$ on (X,d), where $f_i : X \to [0,\infty)$ $(i = 1,2)$, satisfying

$f_1(x) = \sup\{d(y,x)\dot{-}f_2(y) : y \in X\}$ and $f_2(x) = \sup\{d(x,y)\dot{-}f_1(y) : y \in X\}$ whenever $x \in X$.

We equip Q_X with the T_0-quasi-metric D defined by

$$D(f,g) = \sup_{x\in X}(f_1(x)\dot{-}g_1(x)) = \sup_{x\in X}(g_2(x)\dot{-}f_2(x))$$

whenever $f, g \in Q_X$.

Then the map e defined for each $x \in X$ by $x \mapsto e(x) = f_x$ where $(f_x)_1(y) := d(x,y)$ and $(f_x)_2(y) := d(y,x)$ whenever $y \in X$ yields an isometric embedding of (X,d) into (Q_X, D). The T_0-quasi-metric space (Q_X, D) is called the *q-hyperconvex hull of* (X,d).

Let us mention that for each $f, g \in Q_X$, we have

$$D(f,g) = \sup\{(D(f_{x_1}, f_{x_2}) - D(f_{x_1}, f) - D(g, f_{x_2})) \vee 0 : x_1, x_2 \in X\} \quad (*)$$

according to [12, Remark 7].

15. Proposition. Let (X,d) be a T_0-quasi-metric space and let (Q_X, D) be its q-hyperconvex hull. Then $\delta_d = A_D = \delta_D$.

Proof. We first show that the diameter δ_D of the q-hyperconvex hull (Q_X, D) of a T_0-quasi-metric space (X,d) is equal to the diameter δ_d of (X,d).

Obviously $\delta_D \geq \delta_d$, since (X,d) embeds as an isometric subspace into (Q_X, D). Note that for any $f, g \in Q_X$ we have that by the result $(*)$ stated above,

$$D(f,g) = \sup_{(x,y)\in X\times X}\{D(x,y) - D(x,f) - D(g,y), 0\} = \sup_{(x,y)\in X\times X} D(x,y) \leq \delta_d.$$

Thus $\delta_D \leq \delta_d$. Hence the equality of the two diameters δ_D and δ_d is established.

We next consider now the case that the diameter $\delta_d < \infty$. Define a function pair $\bot$ by setting $\bot_1(x) = 0$ and $\bot_2(x) = \sup_{a\in X} d(x,a)$ whenever $x \in X$. Furthermore define a function pair $\top$ by setting $\top_1(x) = \sup_{a\in X} d(a,x)$ and $\top_2(x) = 0$ whenever $x \in X$.

One verifies that $\bot, \top \in Q_X$ by checking the defining equations: Indeed for each $x \in X$,

$$\bot_1(x) = 0 = \sup_{y\in X}(d(y,x)\dot{-}\bot_2(y)) = \sup_{y\in X}(d(y,x)\dot{-}\sup_{a\in X} d(y,a))$$

and similarly

$$\bot_2(x) = \sup_{y\in X}(d(x,y)\dot{-}\bot_1(y)) = \sup_{y\in X}(d(x,y)\dot{-}0).$$

Analogously for each $x \in X$,

$$\top_1(x) = \sup_{y\in X} d(y,x) = \sup_{y\in X}(d(y,x)\dot{-}\top_2(y)) = \sup_{y\in X}(d(y,x)\dot{-}0)$$

and

$$\top_2(x) = 0 = \sup_{y\in X}(d(x,y)\dot{-}\top_1(y)) = \sup_{y\in X}(d(x,y)\dot{-}\sup_{a\in X} d(a,y)).$$

Hence $\bot, \top \in Q_X$, as asserted.

Furthermore one computes

$$D(\bot, f) = \sup_{x\in X}(\bot_1(x)\dot{-}f_1(x)) = \sup_{x\in X}(0\dot{-}f_1(x)) = 0$$

and similarly $D(f,\top) = \sup_{x\in X}(\top_2(x)\dot{-}f_2(x)) = \sup_{x\in X}(0\dot{-}f_2(x)) = 0$ whenever $f \in Q_X$. Hence $\perp$ is the bottom and $\top$ the top of Q_X with respect to the specialization order $\leq_D$ of D on Q_X.

Thus $D(\top,\perp) - D(\perp,\top) = D(\top,\perp) - 0 = \sup_{x\in X}(\top_1(x)\dot{-}\perp_1(x))$ $= \sup_{x\in X}(\sup_{a\in X} d(a,x)\dot{-}0) = \delta_d$. We conclude that $A_D \geq \delta_d$.

Hence we know by Lemma 8 that $A_d \leq \delta_d \leq A_D \leq \delta_D \leq \delta_d$ and conclude that $\delta_d = A_D = \delta_D$.

Suppose now that (X,d) is an unbounded T_0-quasi-metric space and let (Q_X, D) be the q-hyperconvex hull of (X,d).

Choose $x_0 \in X$. For each $n \in \mathbb{N}$ set $X_n = \{x \in X : d^s(x_0,x) \leq n\}$ and denote the restriction of d to $X_n \times X_n$ by d_n.

Note that for each $n \in \mathbb{N}$ we have that $\delta_{d_n} \leq 2n$, thus (X_n, d_n) is bounded. We also observe that $\bigcup_{n\in\mathbb{N}} X_n = X$ where the sequence $(X_n)_{n\in\mathbb{N}}$ of subspaces of X is increasing.

Let (Q_{X_n}, D_n) denote the q-hyperconvex hull of the subspace (X_n, d_n) of (X,d). Denote by $\top_n$ resp. $\perp_n$ the top resp. bottom element of (Q_{X_n}, D_n), as constructed in the first part of the present proof.

For each $n \in \mathbb{N}$ consider an isometry $\tau_n : Q_{X_n} \to Q_X$ as given in [1, Proposition 4].*

For each $n \in \mathbb{N}$ set $f_n := \tau_n(\top_n)$ and $g_n := \tau_n(\perp_n)$. We have that

$$\delta_{d_n} = \sup_{x\in X_n}(\sup_{a\in X_n} d_n(a,x)) = D_n(\top_n,\perp_n) = D(\tau_n(\top_n),\tau_n(\perp_n)) = D(f_n,g_n)$$

and $0 = D_n(\perp_n,\top_n) = D(\tau_n(\perp_n),\tau_n(\top_n)) = D(g_n,f_n)$ whenever $n \in \mathbb{N}$, as we have noted above.

Thus $A_D \geq D(f_n,g_n) - D(g_n,f_n) = D(f_n,g_n) - 0 = \delta_{d_n}$ whenever $n \in \mathbb{N}$ and therefore $A_D \geq \sup_{n\in\mathbb{N}} \delta_{d_n} = \delta_d$. Consequently in the unbounded case $A_d \leq \delta_d \leq A_D \leq \delta_D \leq \delta_d$, too. Hence the stated equality is also established in the case that $\delta_d = \infty$. □

12. Example. Let (X,m) be a metric space and let (Q_X, D) be its q-hyperconvex hull. Then $A_m = 0$, but $A_D = \delta_m$.

Proof. The assertion follows from the previous result and the trivial fact that $A_m = 0$. □

9. The Hausdorff quasi-pseudometric

In this section we consider a T_0-quasi-metric space (X,d) with associated Hausdorff quasi-pseudometric space $(\mathcal{B}_0(X), d_H)$ where $\mathcal{B}_0(X)$ denotes the set of all *bounded* nonempty subsets of (X,d).

Recall that for any $A, B \in \mathcal{B}_0(X)$ we define $d_{H^-}(A,B) = \sup_{a\in A} d(a,B)$ and $d_{H^+}(A,B) = \sup_{b\in B} d(A,b)$. It is known that d_{H^-} and d_{H^+} are both quasi-pseudometrics on $\mathcal{B}_0(X)$. Finally we set $d_H = d_{H^+} \vee d_{H^-}$. Then d_H is the *Hausdorff quasi-pseudometric* on $\mathcal{B}_0(X)$ (compare for instance [3, 16]).

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* The latter result states that if (Z,d) is a T_0-quasi-metric space and S is a nonempty subspace of (Z,d), then there exists an isometric embedding $\tau : Q_S \to Q_Z$ such that $\tau(f)|_S = f$ whenever $f \in Q_S$.

Below we shall make use of the fact that $(d_{H^+})^{-1} = (d^{-1})_{H^-}$, which can be verified by a straightforward computation with the help of the definitions of d_{H^+} and $d_{H^{-1}}$.

16. Proposition. Let (X, d) be a T_0-quasi-metric space. Then $A_{d_{H^+}} = \delta_d$.

Proof. By Lemma 8 we have $A_{d_{H^+}} \leq \delta_{d_{H^+}}$. Furthermore the inequality $\delta_{d_{H^+}} \leq \delta_d$ holds by the definition of d_{H^+} : Indeed in order to reach a contradiction suppose that for some $A, B \in \mathcal{B}_0(X)$ we have $d_{H^+}(A, B) > \delta_d$. Then there must be $b \in B$ such that $d(A, b) > \delta_d$ and so for each $a \in A$ we have that $d(a, b) > \delta_d$ —a contradiction. Hence $\delta_{d_{H^+}} \leq \delta_d$.

Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence in $X \times X$ such that $(d(x_n, y_n))_{n\in\mathbb{N}}$ converges to δ_d, where δ_d could possibly be infinite.

Set for each $n \in \mathbb{N}$, $A_n = \{x_n, y_n\}$ and $B_n = \{x_n\}$. Obviously all these sets belong to $\mathcal{B}_0(X)$. Then $d_{H^+}(B_n, A_n) - d_{H^+}(A_n, B_n) = d(x_n, y_n) - 0$ whenever $n \in \mathbb{N}$. We conclude that $A_{d_{H^+}} \geq \delta_d$.

Hence the stated equality $A_{d_{H^+}} = \delta_d$ is established. □

6. Corollary. Let (X, d) be a T_0-quasi-metric space. Then $A_{d_{H^-}} = \delta_d$.

Proof. We conclude by Proposition 16 and Lemma 9(d) that
$A_{d_{H^-}} = A_{((d^{-1})_{H^+})^{-1}} = A_{(d^{-1})_{H^+}} = \delta_{d^{-1}} = \delta_d$. □

7. Corollary. Let (X, d) be a T_0-quasi-metric space. Then $A_{d_H} \leq A_{d_{H^+}} \vee A_{d_{H^-}} = \delta_d$.

Proof. The statement follows from the definition $d_H = d_{H^+} \vee d_{H^-}$ and Lemma 9(c), Corollary 6 and Proposition 16. □

13. Remark. Let (X, m) be a metric space. Then m_H is a pseudometric, since $(m_{H^+})^{-1} = (m^{-1})_{H^-} = m_{H^-}$. Thus $A_{m_H} = 0$.

10. The infimum-problem

We finish this paper by stating a problem. Given two quasi-pseudometrics f and g on a set X, $f \wedge g$ denotes the largest quasi-pseudometric which is $\leq f$ and $\leq g$.

Indeed the following explicit form of $f \wedge g$ is well known (compare [21, Lemma 6]).

12. Lemma. Let X be a set and let f, g be quasi-pseudometrics on X. For any $x, y \in X$ set $(f \wedge g)(x, y) = \inf\{\sum_{i=0}^{n-1} h(x_i, x_{i+1}) : x_0 = x, x_n = y; x_1, \ldots, x_{n-1} \in X; n \in \mathbb{N}; h \in \{f, g\}\}$. Then $f \wedge g$ is the largest quasi-pseudometric which is $\leq f$ and $\leq g$.

Proof. The standard proof is left to the reader. □

14. Remark. Note that for any $d \in QPM(X)$, $d \wedge d^{-1}$ is indeed a pseudometric.

Proof. For any $x, y \in X$, by definition we clearly have that $(d \wedge d^{-1})(x, y) = (d \wedge d^{-1})(y, x)$. □

Of course, $d_1 \wedge d_2 \leq \min\{d_1, d_2\}$ and the two functions can be distinct, as Example 2 above shows. The authors have only been able to establish the upper bound for $A_{d_1 \wedge d_2}$ given in Lemma 13 below. It should be mentioned that on the other hand Plastria obtained an interesting upper bound for $\sigma_{d_1 \wedge d_2}$, the corresponding multiplicative counterpart of $A_{d_1 \wedge d_2}$: He namely proved that $\sigma_{d_1 \wedge d_2} \leq \sigma_{d_1} \vee \sigma_{d_2}$ [21, Lemma 14.6].

13. Lemma. Let d_1, d_2 be quasi-pseudometrics on a set X. Then $A_{d_1 \wedge d_2} \leq \delta_{d_1} \wedge \delta_{d_2}$.

Proof. We have that $A_{d_1 \wedge d_2} \leq \delta_{d_1 \wedge d_2} \leq \delta_{d_i}$ whenever $i \in \{1, 2\}$ by Lemma 8. □

1. Problem. Let d_1 and d_2 be quasi-pseudometrics on a set X. Is it possible that $A_{d_1 \wedge d_2} > A_{d_1} \vee A_{d_2}$?

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