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Forcing linearity numbers for coatomic modules

Peter R. Fuchs[1]*

Abstract
We show that an integer $n \in \mathbb{N} \cup \{0\}$ is the forcing linearity number of a coatomic module over an arbitrary commutative ring with identity if and only if $n \in \{0, 1, 2, \infty\} \cup \{q+2 \,|\, q$ is a prime power$\}$.

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[1] *Inst. for Algebra, Johannes Kepler University, Austria*
***Corresponding author**: peter.fuchs@jku.at
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1. Introduction

Throughout this paper R shall denote a commutative ring with identity and V a unital right R-module. Consider the set $M_R(V) := \{f : V \to V \,|\, f(vr) = f(v)r$ for all $r \in R, v \in V\}$. Under the operations of pointwise addition and composition of functions, $M_R(V)$ is a near-ring with identity, called the near-ring of homogeneous functions. Note that $M_R(V)$ contains the endomorphism ring $End_R(V)$. The question arises how much linearity is needed on a function $f \in M_R(V)$ to ensure that f is linear on all of V, i.e. $f \in End_R(V)$. More precisely, we say that a collection $\{W_i | i \in I\}$ of proper submodules forces linearity on V, if whenever $f \in M_R(V)$ and f is linear on each $W_i, i \in I$, then $f \in End_R(V)$. Thus $M_R(V) = End_R(V)$ if and only if the empty collection forces linearity on V. The smallest number of modules which force linearity on V gives rise to the forcing linearity number of V.

Definition 1.1. *[3] Let V be an R–module. The forcing linearity number $f\ln(V) \in \mathbb{N} \cup \{0, \infty\}$ of V is defined as follows:*

1. *If $M_R(V) = End_R(V)$, then $f\ln(V) = 0$.*
2. *If $M_R(V) \neq End_R(V)$, and there is some finite collection $\{W_i | 1 \leq i \leq n\}, n \in \mathbb{N}$, of proper submodules of V which forces linearity on V, but no collection of fewer than n proper submodules forces linearity, then we say that $f\ln(V) = n$.*
3. *If neither 1. or 2. holds, then we say that $f\ln(V) = \infty$.*

Forcing linearity numbers have been found for several classes of rings and modules, see for example [3], [4], [5] and their references. In section 2 we determine the forcing linearity number of coatomic modules over an arbitrary commutative ring R with identity. An R–module V is called coatomic, if every proper submodule is contained in a maximal submodule of V. For example a finitely generated module or a semisimple module over any ring is coatomic. For a commutative noetherian local ring, the coatomic modules have been characterized in [7].

2. Forcing linearity numbers of coatomic modules

For an R–module V and subsets S_1, S_2 of V let $(S_1 : S_2) = \{r \in R | S_2 r \subseteq S_1\}$. For $v \in V$ let $Ann(v) = \{r \in R | vr = 0\}$.

Theorem 2.1. *Let V be an R–module and let M,N be maximal submodules of V, $M \neq N$. The following are equivalent:*

1. *The collection $\{M,N\}$ does not force linearity.*
2. $\exists\, w \neq 0 \in V : (M:V) = (N:V) = Ann(w)$.

Proof. $1 \Rightarrow 2$: Since $\{M,N\}$ does not force linearity on V, there exists a function $f \in M_R(V)$ such that f is linear on the submodules M,N, but $f \notin End_R(V)$. Let $u,v \in V$ be such that $w := f(u+v) - f(u) - f(v) \neq 0$. Since $M \neq N$, and M,N are maximal, we have that $M+N = V$. For every $v \in V - M$, $(M:v) = (M:V)$, therefore $(M:V)$ and $(N:V)$ are maximal ideals. If $(M:V) \neq (N:V)$, then $(M:V)+(N:V) = R$, hence $r+s = 1$ for some $r \in (M:V)$, $s \in (N:V)$. Now $wr = f(ur+vr) - f(ur) - f(vr) = f(ur) + f(vr) - f(ur) - f(vr) = 0$, since f is linear on M. Similarly, $ws = 0$, hence $w = w.1 = w(r+s) = 0$, a contradiction. Thus $(M:V) = (N:V)$, and since $(M:V) \subseteq Ann(w)$ and $(M:V)$ is a maximal ideal, it follows that $(M:V) = Ann(w)$.
$2 \Rightarrow 1$: Let $v \in V - M$. Then $(M:v) = (M:V) = Ann(w)$ and $h : V/M \to Rw$, $h(vr/M) := wr$ is an isomorphism. Define a function $f : V \to V$ as follows: For $m \in M, n \in N$ let

$$f(m+n) := \begin{cases} h(n/M) & \text{if } m+n \notin M \cup N \\ 0 & \text{otherwise} \end{cases}$$

Since $M+N = V$, f is defined on V. We show that f is well–defined. Suppose $m_1+n_1 = m_2+n_2$, $m_1,m_2 \in M$, $n_1,n_2 \in N$. If $m_1+n_1 \in M \cup N$, then $f(m_1+n_1) = f(m_2+n_2) = 0$. If $m_1+n_1 \notin M \cup N$, then $n_1/M = n_2/M$, hence $f(m_1+n_1) = h(n_1/M) = h(n_2/M) = f(m_2+n_2)$. Next we show that f is homogeneous. Let $S := V - (M \cup N)$. If $m+n \in S$, then $(N:m) = (N:V)$ and $(M:n) = (M:V)$. By our assumption $(M:V) = (N:V) = Ann(w) \neq R$, hence $(N:m) = (M:n)$. If $r \notin (M:n)$, then $r \notin (N:m)$, which implies that $(m+n)r = mr + nr \in S$, hence $f((m+n)r) = h(nr/M) = h(n/M)r = f(m+n)r$. If $r \in (M:n)$, then $(m+n)r \notin S$, hence $f(m+n)r = h(n/M)r = h(nr/M) = h(0) = 0 = f((m+n)r)$. Now suppose $m+n \notin S$. Then $m+n \in M \cup N$, hence $(m+n)r \in M \cup N$ for all $r \in R$. Thus $f(m+n)r = 0 = f((m+n)r)$. It now follows that $f \in M_R(V)$. Since $f|M = f|N = 0$, f is linear on M and N. However, for $m \in M - N$ and $n \in N - M$, we have that $m+n \in S$, thus $f(m+n) = h(n/M) \neq 0$, since h is an isomorphism, whereas $f(m) + f(n) = 0$, so $f \notin End_R(V)$.Therefore the collection $\{M,N\}$ does not force linearity on V. □

For an $R-$module V let $Rad(V)$ denote the Jacobson radical of V and let $J := Rad(R)$. Recall that an $R-$module V is called local, if V contains a unique maximal submodule.

Theorem 2.2. *For a noncyclic coatomic module V, the following are equivalent:*

1. $f\ln(V) > 2$.
2. *$I := (Rad(V):V)$ is a maximal ideal and $I = Ann(w)$ for some $0 \neq w \in V$.*

Proof. $1 \Rightarrow 2$: Let $\mathbf{M}$ denote the collection of all maximal submodules of V. Since V is coatomic, $\mathbf{M} \neq \emptyset$. If there exist $M_1,M_2 \in \mathbf{M}$ such that $(M_1:V) \neq (M_2:V)$, then by Theorem 2.1 the collection $\{M_1,M_2\}$ forces linearity on V. Thus $(M_1:V) = (M_2:V)$ for all $M_1,M_2 \in \mathbf{M}$ and $I = \bigcap\{(M:V) | M \in \mathbf{M}\} = (M:V)$ for all $M \in \mathbf{M}$, hence $I = (Rad(V):V)$ is a maximal ideal. Like in the proof of Theorem 1, we see that $I = Ann(w)$ for some $w \neq 0$.
$2 \Rightarrow 1$: Suppose that V is a local module with unique maximal submodule M. Let $v \in V - M$. If $vR \neq V$, then vR is contained in a maximal submodule, which implies $vR \subseteq M$, a contradiction. Consequently $vR = V$ for all $v \in V - M$, which contradicts our assumption that V is noncyclic. Therefore there exist at least two maximal submodules. Suppose $f\ln(V) \leq 2$. Then there exists a collection of submodules $\{S_1,S_2\}$ which forces linearity on V. Since V is coatomic, there exist maximal submodules M_1,M_2 such that $S_1 \subseteq M_1$, $S_2 \subseteq M_2$. Without loss of generality we may assume that $M_1 \neq M_2$ (otherwise we can choose another maximal submodule, since V is not local). Then $\{M_1,M_2\}$ also forces linearity on V. We have $(Rad(V):V) \subseteq (M_1:V) \neq R$. By our assumptions $(Rad(V):V)$ is a maximal ideal, hence $(Rad(V):V) = (M_1:V) = (M_2:V)$. Also, $(Rad(V):V) = Ann(w)$ for some $0 \neq w \in V$. Therefore $\{M_1,M_2\}$ does not force linearity by Theorem 1, a contradiction. □

Theorem 2.3. *Let V be coatomic. Suppose $I := (Rad(V):V)$ is a maximal ideal of R and there exists $0 \neq w \in V$ such that $I = Ann(w)$. Then*

$$fln_R(V) = fln_{R/I}(V/Rad(V))$$

Proof. We first show that $f\ln_{R/I}(V/Rad(V)) \leq f\ln_R(V)$. Let $\{W_i | i \in I\}$ be a collection of proper submodules which forces linearity on V. Since V is coatomic, we may assume that each W_i, $i \in I$, is maximal. We show that the collection $\{W_i/Rad(V) | \, i \in I\}$ forces linearity on $V/Rad(V)$. Suppose that this is not the case. Then there exists a homogeneous function $f : V/Rad(V) \to V/Rad(V)$, which is linear on each submodule $W_i/Rad(V)$, $i \in I$, but not linear on $V/Rad(V)$. Let $\pi_M : V/Rad(V) \to V/M$ denote the projection of $V/Rad(V)$ onto V/M for a maximal submodule M. Since f is not linear, there exists a maximal submodule M of V such that $\pi_M f : V/Rad(V) \to V/M$ is not linear. Since I is a maximal ideal, $I = (M : V)$, hence $w(M : V) = 0$, which implies $V/M \simeq wR$. Thus we obtain a homogeneous map $f_1 : V/Rad(V) \to wR$, which is linear on each submodule $W_i/Rad(V)$, $i \in I$. If $g : V \to V$ is defined by $g(v) := f_1(v/Rad(V))$, then $g \in M_R(V)$ and linear on each W_i, $i \in I$, but not linear on V, a contradiction to our assumption that $\{W_i | i \in I\}$ forces linearity on V. For the reverse inequality suppose first that $f\ln_{R/I}(V/Rad(V)) \leq 1$. Since $V/Rad(V)$ is a vector space over the field R/I, it follows from Theorem 3.1 in [3] that $\dim_{R/I}(V/Rad(V)) = 1$. Note that $Rad(V)$ is a superfluous submodule, since V is coatomic. It follows that V is cyclic, hence $f\ln_{R/I}(V/Rad(V)) = 0 = f\ln(V)$. If $dim_{R/I}(V/Rad(V)) = 2$ or $f\ln_{R/I}(V/Rad(V)) \geq 2$ and R/I is infinite, we have that $f\ln_{R/I}(V/Rad(V)) = \infty$ by Theorem 3.1 in [3]. So suppose that $f\ln_{R/I}(V/Rad(V)) \geq 3$ and $|R/I| =: q \in \mathbb{N}$. By [3], 3.8 and 3.10, $f\ln_{R/I}(V/Rad(V)) = q + 2$. Choose $\{r_1, ..., r_q\} \subseteq R$ such that $R/I = \{r_1/I, ..., r_q/I\}$. It suffices to give a collection of $q+2$ proper submodules which forces linearity on V. Let $\{b_i | i \in I\} \subseteq V$ be such that $\{b_i/Rad(V) | i \in I\}$ is a basis of the vector space $V/Rad(V)$. As we have seen above, $|I| \geq 3$, so we can choose pairwise different elements $i_1, i_2, i_3 \in I$. Let $\langle X \rangle$ denote the submodule generated by a subset $X \subseteq V$, and define $S_1 := \langle b_{i_1}, b_{i_2} \rangle + Rad(V)$, $S_2 := \langle b_{i_1} + b_{i_3} \rangle + \langle b_i | i \notin \{i_1, i_3\} \rangle + Rad(V)$, and for $r \in \{r_1, ..., r_q\}$ define $S_r := \langle b_{i_1} + rb_{i_2},\ b_{i_1} + b_{i_3} \rangle + \langle b_i | i \notin \{i_1, i_2, i_3\} \rangle + Rad(V)$. Note that all submodules are proper, since $Rad(V)$ is superfluous. Similarly as in Theorems 3.8, 3.10 in [3], one can prove that the collection $\{S_1, S_2\} \cup \{S_{r_i} | \, i \in \{1, ..., q\}\}$ forces linearity on V. □

For R local and J T-nilpotent, Theorem 2.3 has been proved in [4], Theorem 5.1. The following example shows that Theorem 2.3 is not true in general, if I is not the annihilator of some $0 \neq w \in V$.

Example 2.4. *Let $R := F[[x]]$ denote the ring of formal power series over a field F and let $V := R \times R$. Since R is local with radical $J = (x)$, $Rad(V) = VJ = (x) \times (x)$ and $I = (Rad(V) : V) = (x)$ is maximal. By [3], Corollary 2.4, $f\ln_R(V) = 1$. However, $f\ln_{R/I}(V/Rad(V)) = f\ln_F(F^2) = \infty$, by [3], Theorem 3.1.*

Theorem 2.5. *Let $n \in \mathbb{N} \cup \{0, \infty\}$. Then n is the forcing linearity number of a coatomic module over a commutative ring if and only if $n \in \{0, 1, 2, \infty\} \cup \{q + 2 | \ q \text{ is a prime power}\}$.*

Proof. It is well-known that there exist coatomic modules V over a commutative ring R such that $f\ln_R(V) \in \{0, 1, 2, \infty\}$, see for example [5]. If V is a cyclic module, then $M_R(V) = End_R(V)$, hence $f\ln_R(V) = 0$. Now suppose $f\ln_R(V) > 2$. By Theorem 2.2, $I = (Rad(V) : V)$ is a maximal ideal and $I = Ann(w)$ for some $0 \neq w \in V$. By Theorem 2.3, $f\ln_R(V) = f\ln_{R/I}(V/Rad(V))$ and as we have remarked previously, $f\ln_{R/I}(V/Rad(V)) \in \{\infty\} \cup \{q + 2 | \text{ q is a prime power}\}$. □

It is not known to the author, whether Theorem 2.5 is true for every module over a commutative ring.

There is a class of rings which have the property that every right module is coatomic, or which is easily seen to be equivalent, every nonzero right module has a maximal submodule.

Definition 2.6. *A ring R is called a right max–ring, if every right $R-$module is coatomic. See [6].*

Theorem 2.7. *[2] For a commutative ring R, the following are equivalent:*

1. *R is a max–ring.*
2. *J is T–nilpotent and R/J is von Neumann regular.*

Theorem 2.8. *Let V be a module over a commutative max–ring R. If R is not local, then $f\ln_R(V) \leq 2$.*

Proof. Suppose that R is not local, but $f\ln(V) > 2$. Since R is a max-ring, it follows from Theorem 2.7 and from [1], Proposition 18.3 that $Rad(V) = VJ$. By Theorem 2.2, $(Rad(V) : V) = (VJ : V)$ is a maximal ideal. We have $J \subseteq (VJ : V)$. Suppose that there exists an element $r \in (VJ : V) - J$. Then $r \notin M$ for some maximal ideal M of R. Let R_M, V_M denote the localisations of R, V at M. By [1], Proposition 18.3, $Rad(V_M) = V_M J_M$. Since R is a max-ring J is T-nilpotent, thus J_M is T-nilpotent. It follows from Theorem 2.5 that R_M is a max-ring, hence $Rad(V_M) = V_M J_M \neq V_M$. So let $w/1 \in V_M - Rad(V_M)$. From $r \in (VJ : V)$, $w/1 \cdot r/1 = wr/1 \in V_M J_M$. Since $r \notin M$, $r/1$ is invertible in R_M, hence $w/1 \in V_M J_M = Rad(V_M)$, a contradiction. It now follows that $J = (VJ : V)$ is a maximal ideal of R, which contradicts our assumption that R is not local. □

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