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Functional equivalence of topological spaces and topological modules

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Abstract

Let R be a topological ring and E, F be unitary topological R-modules. Denote by $C_p(X, E)$ the class of all continuous mappings of X into E in the topology of pointwise convergence. The spaces X and Y are called $l_p(E, F)$-equivalent if the topological R-modules $C_p(X, E)$ and $C_p(Y, F)$ are topological isomorphic. Some conditions under which the topological property $\mathcal{P}$ is preserved by the $l_p(E, F)$-equivalence (Theorems 6.3, 6.4, 7.3 and 8.1) are given.

Keywords: Function space, topology of pointwise convergence, support, linear homeomorphism, perfect properties, open finite-to-one properties

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1. Preliminaries

Throughout this paper, by a space we will mean a Tychonoff space [9].

A topological semiring is a topological space R equipped with two continuous binary operations $\{+, \cdot\}$, called addition and multiplication, such that (see [10, 11, 15]):

1. $(R, +)$ is a topological commutative monoid with identity element 0 and proprieties: $(a + b) + c = a + (b + c)$, $0 + a = a + 0 = a$, $a + b = b + a$ for all $a, b, c \in R$.

2. $(R, \cdot)$ is a topological monoid with identity element $1 \neq 0$ and proprieties: $(a \cdot b) \cdot c = a \cdot (\cdot c)$, $1 \cdot a = a \cdot 1 = a$, $a \cdot b = b \cdot a$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, $0 \cdot a = 0$ for all $a, b, c \in R$.

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Let G be a topological Abelian group under addition operation and R be a topological semiring. We call G a topological R-module if on it is defined the continuous operation of multiplication $\cdot : R \times G \longrightarrow G$ between an element of R and an element of G, say $ra \in G$, where $r \in R$ and $a \in G$, with the following properties: $1 \cdot a = a$, $0 \cdot a = 0$, $r(a+b) = ra+rb$, $(r+s)a = ra+sa$, $r(sa) = (rs)a$, for any $r, s \in R$ and $a, b \in G$.

Let R be a topological semiring and E, F be topological R-modules. The mapping $\varphi : E \to F$ is a linear mapping if it satisfies the conditions: $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(\alpha x) = \alpha\varphi(x)$ for any $x, y \in E$ and $\alpha \in R$.

Fix a space X, a topological semiring R, and topological R-modules E and F.

By $C(X, E)$ we will denote the family of all E-valued continuous functions with the domain X and by $C_p(X, E)$ we will denote the space $C(X, E)$ endowed with the topology of pointwise convergence. Recall that the family of sets of the form $W(x_1, x_2, ..., x_n, U_1, U_2, ..., U_n) = \{f : C(X, E) : f(x_i) \in U_i \textit{ for any } \ i \leq n\}$, where $x_1, x_2, ..., x_n \in X$, $U_1, U_2, ..., U_n$ are open sets of E and $n \in \mathbb{N}$, is a base of the space $C_p(X, E)$.

By $H_p(E, F)$ we denote the space of all linear mappings of E into F as a subspace of the space $C_p(E, F)$.

The spaces X and Y are called $l_p(E, F)$-equivalent if the spaces $C_p(X, E)$ and $C_p(Y, F)$ are linearly homeomorphic and we denote $X \overset{E,F}{\sim} Y$.

A space X is zero-dimensional if $indX = 0$ (small inductive dimension is zero), i.e., X has a base of clopen (open and closed) subsets.

The following two assertions are evidently.

1.1. Proposition. Fix a topological R-module E. Then $C_p(X, E)$ is a topological R-module and E is embedded in a natural way in $C_p(X, E)$ as a closed submodule of $C_p(X, E)$.

1.2. Proposition. If E is a zero-dimensional topological R-module, then $C_p(X, E)$ is a zero-dimensional topological R-module too.

2. The evaluation mapping

Let X be a space, R be a topological semiring and E be a non-trivial topological R-module. Fix $x \in X$. Then the mapping $\xi_x : C_p(X, E) \to E$ defined by $\xi_x(f) = f(x)$ is called the evaluation mapping at x (see, by instance, [1]).

We now define the canonical evaluation mapping $e_X : X \to C_p(C_p(X, E), E)$, where $e_X(x) = \xi_x$ for any $x \in X$.

The proofs of the following two assertions are standard (see [8]).

2.1. Proposition. The evaluation mapping $\xi_x : C_p(X, E) \to E$ is continuous and linear for every point $x \in X$.

2.2. Proposition. The canonical evaluation mapping $e_X : X \to C_p(C_p(X, E), E)$ is continuous. Moreover, the set $e_X(X)$ is closed in the space $C_p(C_p(X, E), E)$.

Let X and Y be spaces, Φ be a family of functions $f : X \to Y$. We say that Φ *separates points* of X (or simply is *separating* [1]) if for any $x, y \in X$, $x \neq y$, there exists $f \in \Phi$ such that $f(x) \neq f(y)$. We also say that Φ *separates points from closed sets* (or is *regular* [1]) if for any non-empty closed subset B of X, any point

$x \in X \setminus B$ and any two points $a, b \in Y$ there exists $f \in \Phi$ such that $f(x) = a$ and $f(B) = b$.

2.3. Proposition. If $C_p(X, E)$ is a regular family, then the canonical evaluation mapping $e_X : X \to C_p(C_p(X, E), E)$ is a homeomorphism from X to the closed subspace $e_X(X)$ of $C_p(C_p(X, E), E)$.

A space X is called R-Tychonoff if for any closed subset B of X, any point $a \in X \setminus F$ there exists $g \in C(X, R)$ such that $g(a) = 1$ and $B \subseteq g^{-1}(0)$.

The product of R-Tychonoff spaces is an R-Tychonoff space. The subspace of an R-Tychonoff is an R-Tychonoff space.

Remark. Let X be an R-Tychonoff space and E be a non-trivial topological R-module. Then X is a Tychonoff space, and for each closed set B of X, any point $a \in X \setminus B$ and any points $b, c \in E$ there exists $f \in C(X, E)$ such that $f(a) = b$ and $f(B) = c$.

The proofs of the following two assertion is elementary.

2.4. Proposition. If $indX = 0$, then the space X is R-Tychonoff.

Let R be a topological semiring. A topological R-module E is called:

(i) *simple* if it does not contain a non-trivial submodule over R.

(ii) *locally simple* if E is not trivial and there exists an open subset U of E such that $0 \in U$ and U do not contains non-trivial R-submodules of E.

2.5. Example. If R is a field, then R is a simple topological R-module. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{K}$ be the field of complex numbers. Then $\mathbb{K}$ is locally simple and not simple $\mathbb{R}$-module.

We mention the following obvious fact.

2.1. Lemma. *Let R be a topological semiring and E be an R-module. Then Ra is an R-submodule for any $a \in E$.*

Fix a space X and two topological R-modules E and F. We define $M_p(X, E, F) = H_p(C_p(X, E), F)$ the subspace of all linear mappings from $C_p(X, E)$ into F. Let $M_p(X, E) = M_p(X, E, E)$. Now we define $L_p(X, E) \subseteq C_p(C_p(X, E), E)$ as follows $L_p(X, E) = \{\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n : \alpha_i \in R, x_i \in e_X(X), i \leq n \in \mathbb{N}\}$.

By construction, we have $L_p(X, E) \subseteq M_p(X, E)$. As a rule $L_p(X, E) \neq M_p(X, E)$ (see [8]).

2.6. Proposition. Let R be a topological semiring, E be a topological R-module and X be a space. Then for any $g \in C(X, E)$ there exists a unique linear mapping $\overline{g} \in H_p(L_p(X, E), E)$ such that $g = \overline{g} \circ e_X$, where $e_X : X \to L_p(X, E)$ is the evaluation mapping.

Proof. Let $E_f = E$ for any $f \in C_p(X, E)$. By definition, $e_X(X) \subseteq L_p(X, E) \subseteq E^{C(X,E)} = \Pi\{E_f : f \in C(X, E)\}$. We have $e_X(x) = \xi_x$ for any $x \in X$ and $L_p(X, E)$ is the submodule of $E^{C(X,E)}$ generated by the set $e_X(X)$. We consider the projection $\pi_f : E^{C(X,E)} \longrightarrow E_f = E$. Let $\overline{f} = \pi_f | L_p(X, E) : L_p(X, E) \longrightarrow E$ is the desired linear mapping. $\square$

2.7. Theorem. Let R be a semiring, E be a topological R-module and X be a space. Consider the space $e_X(X)$, where $e_X : X \to L_p(X,E)$ is the evaluation mapping. Then the topological R-modules $C_p(X,E)$, $C_p(e_X(X),E)$ and $H_p(L_p(X,E),E)$ are linearly homeomorphic.

Proof. Let E_f for any $f \in C_p(X,E)$. By definition, $e_X(X) \subseteq L_p(X,E) \subseteq E^{C_p(X,E)} = \Pi\{E_f : f \in C(X,E)\}$. We consider the projection $\pi_f : E^{C(X,E)} \longrightarrow E_f = E$. Let $\overline{f} = \pi_f|L_p(X,E) : L_p(X,E) \longrightarrow E$. Then $\overline{f}$ and π_f are continuous linear mappings.

If $g : e_X(X) \to E$ is a continuous mapping, then $g \circ e_X = f$ for a unique $f \in C(X,E)$. Therefore, $g = \pi_f|e_X(X)$ and the correspondence $f \to \pi_f|e_X(X)$ is a linear homeomorphism of $C_p(X,E)$ onto $C_p(e_X(X),E)$.

Hence, without loss of generality, we can assume that $X = e_X(X) \subseteq L_p(X,E)$.

By virtue of Proposition 2.6, the correspondence $\psi : C_p(X,E) \longrightarrow H_p(L_p(X,E),E)$, where $\psi(f) = \overline{f}$, is a one-to-one linear mapping of $C(X,E)$ onto $H_p(L_p(X,E),E)$.

For each $y \in L_p(X,E)$ there exist the minimal $n = n(y) \in \mathbb{N}$, the unique points $x_1(y),...,x_n(y) \in X$ and the unique points $\alpha_1(y),...,\alpha_n(y) \in R$ such that $y = \alpha_1(y)x_1(y) + ... + \alpha_n(y)x_n(y)$. Hence, the correspondence ψ is continuous and linear. Since $\psi(f)|X = f$, the mapping ψ^{-1} is continuous. □

2.8. Corollary. Let X, Y be spaces and R be a locally simple R-module. The spaces $C_p(X,R)$ and $C_p(Y,R)$ are linearly homeomorphic if and only if the spaces $L_p(X,R)$ and $L_p(Y,R)$ are linearly homeomorphic.

2.2. Lemma. *Let X be an R-Tychonoff space, Z be a closed subspace of X, E be a topological R-module and $g : X \longrightarrow E$ be a continuous mapping. For any finite subset B of $X \setminus Z$ and any function $f : B \longrightarrow E$ there exists a continuous function $\varphi : X \longrightarrow E$ such that $f = \varphi|B$ and $\varphi|Z = g|Z$.*

Proof. Fix a family $\{U_x : x \in B\}$ of open subsets of X such that $x \in U_x \subseteq X \setminus Z$ for each $x \in B$ and $U_x \cap U_y = \varnothing$ for each distinct points $x, y \in B$. For each $x \in B$ fix a continuous function $f_x : X \longrightarrow E$ such that $f_x(x) = f(x) - g(x)$ and $f_x(X \setminus U_x) = 0$. Let $f_B(y) = \sum\{f_x(y) : x \in B\}$. By construction, the function f_B is continuous, $f_B(Z) = 0$ and $f_B(x) = f(x) - g(x)$ for each $x \in B$. Obviously, $\varphi = f_B + g$ is the desired function. □

For any subspace Y of a space X we put $C_p(Y|X,E) = \{f|Y : f \in C(X,E)\}$. A subspace Y of X is *E-full* if $C(Y|X,E) = C(Y,E)$.

A space X is called *compactly E-full* if $C(Y|X,E) = C(Y,E)$ for any compact subspace Y of X.

The following assertion is well-known (see [8]).

2.3. Lemma. *Let X be a zero-dimensional space and E be a metrizable space. Then X is a compactly E-full space. Moreover, for any compact subset Y of X and any $f \in C(Y,E)$ there exists $g \in C(X,E)$ such that $g(X) \subseteq f(Y)$ and $f = g|_Y$.*

3. The support mapping

Fix a topological semiring R and non-trivial topological R-modules E and F.

Consider a space X and a functional $\mu \in M_p(X, E, F)$. We put $\mathcal{S}(\mu) = \{B \subseteq X :$ *if* $B \subseteq f^{-1}(0)$, *then* $\mu(f) = 0\}$. Obviously, $X \in \mathcal{S}(\mu)$. Thus the set $\mathcal{S}(\mu)$ is non-empty.

The set $supp_X(\mu)$ is the family of all points $x \in X$ such that for each neighbourhood U of x in X there exists $f \in C_p(X, E)$ such that $f(X \setminus U) = 0$ and $\mu(f) \neq 0$ (see [2, 12], for $E = R = \mathbb{R}$, [3, 14] for $R = \mathbb{R}$, [8] when R is a topological ring).

If $f \in C_p(X, E)$ and U is an open neighbourhood of 0 in E, then we put $A(f, L, U) = \{g \in C_p(X, E) : f(x) - g(x) \in U \text{ for any } x \in L\}$. The family $\{A(f, L, U) : f \in C_p(X, E), L$ *is finite subset of* X, U *is open neighbourhood of* 0 *in* $E\}$ is an open base of the space $C_p(X, E)$.

3.1. Theorem. Let X be a R-Tychonoff space, E and F be non-trivial topological R-modules, $\mu \in M_p(X, E, F)$ and $\mu \neq 0$. If F is a locally simple topological R-module, then:

1. There exists a finite set $K \in \mathcal{S}(\mu)$ such that $supp_X(\mu) \subseteq K$.
2. $supp_X(\mu) \in \mathcal{S}(\mu)$ and $supp_X(\mu)$ is a finite non-empty subset of X.
3. $supp_X(\mu) = \cap\mathcal{S}(\mu)$.

Proof. Fix an open subset U_0 of $C_p(X, E)$ such that $0 \in U_0$ and an open subset W_0 of F such that $0 \in W_0$, W_0 do not contains non-trivial R-submodules of F and $\mu(U_0) \subseteq W_0$.

There exist a finite subset K of X and an open subset V_0 of E such that $0 \in V_0$ and $0 \in A(0, K, V_0) \subseteq U_0$. Hence $\mu(f) \in W_0$ for each $f \in A(0, K, V_0)$.

Let $f \in C_p(X, E)$ and $f(K) = 0$. Then $\alpha f \in A(0, K, V_0)$ for each $\alpha \in R$. Hence $\mu(\alpha f) \in W_0$ for each $\alpha \in R$. Thus $R \cdot \mu(f) \subseteq V_0$ and $R \cdot \mu(f)$ is the trivial R-submodule. Thus $\mu(f) = 0$ and $K \in \mathcal{S}(\mu)$. In this case $supp_X(\mu) \subseteq K$. Hence $supp_X(\mu)$ is a finite set and K is a finite set from $\mathcal{S}(\mu)$.

Let $L \in \mathcal{S}(\mu)$ be a finite set and $x_0 \in L \setminus supp_X(\mu)$. Then $L_1 = L \setminus \{x_0\} \in \mathcal{S}(\mu)$. Really, since $x_0 \notin supp_X(\mu)$, there exists an open subset H of X such that $x_0 \in H$ and $\mu(f) = 0$ provided $f(X \setminus H) = 0$. We can assume that $H \cap L = \{x_0\}$. Let $f \in C_p(X, E)$ and $f(L_1) = 0$. There exists $h \in C(X, E)$ such that $h(x_0) = f(x_0)$ and $h(X \setminus H) = 0$. We put $g(x) = f(x) - h(x)$ for any $x \in X$. Since $h(X \setminus H) = 0$, we have $\mu(h) = 0$. By construction, $g(L) = 0$ and $\mu(g) = 0$. Hence $f = g + h$ and $\mu(f) = \mu(g + h) = \mu(g) + \mu(h) = 0$. Hence $L_1 \in \mathcal{S}(\mu)$. Since $K \in \mathcal{S}(\mu)$ and $K \setminus supp_X(\mu)$ is a finite set, we have $supp_X(\mu) \in \mathcal{S}(\mu)$. In particular, we have $supp_X(\mu) = \cap\mathcal{S}(\mu)$. □

The following assertions are obviously:

3.2. Proposition. Let $n \geq 1$, $x_1, x_2, ..., x_n$ are distinct points of X, $\alpha_1, \alpha_2, ..., \alpha_n \in R$ and $\mu(f) = \Sigma\{\alpha_i f(x_i) : i \leq n\}$ for each for each $f \in C_p(X, E)$, then:

1. $\mu \in L_p(X, E)$ and $supp_X(\mu) \subseteq \{x_1, x_2, ..., x_n\}$.
2. If for each $i \leq n$ the set $\alpha_i E$ is a non-trivial R-submodule of E, then $supp_X(\mu) = \{x_1, x_2, ..., x_n\}$.

4. Topological properties of the mapping $supp_X$

Fix a topological semiring R. Let X be a space, E and F be two non-trivial topological R-modules.

Recall that a set-valued mapping $f : X \to 2^Y$ is lower semicontinuous (l.s.c) if for every open subset U of Y the inverse image of U, $f^{-1}(U) = \{x \in X : f(x) \cap U \neq \varnothing\}$ is open in X.

The correspondence $supp_X$ is a set-valued mapping of the space $M_p(X, E, F)$ into X. For $H \subseteq M_p(X, E, F)$ we put $supp_X(H) = \cup\{supp_X(\mu) : \mu \in H\}$.

4.1. Proposition. If F is a locale simple R-module, then the set-valued mapping $supp_X : M_p(X, E, F) \to X$ is l.s.c.

Proof. We follow very closely the proof of [3, Property 4.2] and [12, Lemma 6.8.2 (4)].

Let U be an open subset of X, and put $V = supp_X^{-1}(U)$, i.e., $V = \{\mu \in M_p(X, E, F) : supp_X(\mu) \cap U \neq \varnothing\}$. Let $\mu \in V$, and take $x_0 \in supp_X(\mu) \cap U$. Fix an open subset W of X such that $x_0 \in W \subseteq cl_X W \subseteq U$. Then there exists $f \in C(X, E)$ such that $f(X \setminus W) = \{0\}$ and $\mu(f) \neq 0$. Let $H = \{\eta \in M_p(X, E, F) : \eta(f) \neq 0\}$. Since the set $\{0\}$ is closed in F, H is the basic open set $W(f, F \setminus \{0\})$ $= \{\eta \in M_p(X, E, F) : \eta(f) \in F \setminus \{0\}\}$ and $\mu \in W(f, F \setminus \{0\})$.

We affirm that $H \subseteq V$. By contradiction, suppose that $\eta \in H \setminus V$, i.e. $\eta(f) \neq 0$ and $supp_X(\eta) \cap U = \varnothing$. Then $X \setminus cl_X W$ is an open neighbourhood of $supp_X(\eta)$ and, since $f(X \setminus cl_X W) = \{0\}$, applying Theorem 3.1, we get that $\eta(f) = 0$. A contradiction, hence V is open in $M_p(X, E, F)$. □

A subset L of a space X is bounded if any continuous real-valued function $f : X \longrightarrow \mathbb{R}$ is bounded on L.

A subset L of a topological R-module E is called:

(i) precompact or totally a-bounded if for any neighbourhood U of 0 in E there exists a finite subset A of E such that $L \subseteq A + U = U + A$;

(ii) a-bounded if for any neighbourhood U of the 0 in E there exists $n \in \mathbb{N}$ such that $L \subseteq nU$.

Any bounded set is precompact. In a topological vector space over field of reals any precompact set is a-bounded.

A topological R-module E is called locally bounded if there exists an a-bounded neighbourhood U of 0 in E such that $E = \cup\{nU : n \in \mathbb{N}\}$ and for any $a \in E$, $a \neq 0$, and any $n \in \mathbb{N}$ there exists $t \in R$ such that $ta \notin nU$. In this case the set U does not contain R-submodules of E and E is a locally simple R-module.

4.2. Example. Let E be a normed vector space over reals $\mathbb{R}$. Then E is a locally bounded $\mathbb{R}$-module.

4.3. Example. Let E be a topological vector space over reals $\mathbb{R}$ and there exists a number $q > 0$ and a functional $||.|| : E \longrightarrow \mathbb{R}$ such that:

1. $0 < q \leq 1$.
2. $||x|| \geq 0$ for any $x \in E$.
3. If $||x|| = 0$, then $x = 0$.
4. $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in E$.
5. $||\lambda x|| \leq |\lambda|^q ||x||$ for all $x \in E$ and $\lambda \in \mathbb{R}$.
6. If $x \neq 0$ then $lim_{\lambda \to +\infty} ||\lambda x|| = +\infty$.

The functional $||.||$ is called a q-norm, if the family $\{V(0, r) = \{x : ||x|| < r\} : r > 0\}$ is a base of E at 0. Any q-normed space is locally bounded.

4.4. Theorem. Let F be a locally bounded topological R-module, B be a submodule of F and X be an R-Tychonoff space with the following properties:

(b) : for any non-bounded subset L of X there exists $f \in C(X, B)$ such that the set $f(L)$ is not a-bounded in F;

(r) : B is topological isomorphic to some R-submodule of E.

Then:

(i) The set $supp_X(H)$ is bounded in X for any a-bounded subset H of $M_p(X, E, F)$.

(ii) The set $supp_X(H)$ is bounded in X for any totally a-bounded subset H of $M_p(X, E, F)$.

(iii) The set $supp_X(H)$ is bounded in X for any bounded subset H of $M_p(X, E, F)$.

Proof. We can assume that $B \subseteq E$ too. Since B is a non a-bounded subset of F there exists an open subset W_0 of F such that $0 \in W_0$ and $B \setminus nW_0 \neq \varnothing$ for each $n \in \mathbb{N}$. Moreover, If $H \subseteq B$ is a non a-bounded of F then H is a non a-bounded of B too.

Since F is locally bounded we can fix an open neighbourhood W_1 of 0 in E such that the set W_1 is a-bounded, $F = \bigcup\{nW_1 : n \in \mathbb{N}\}$ and for any $a \in F$, $a \neq 0$, and for any $n \in \mathbb{N}$ there exists $t \in R$ such that $ta \notin nW_1$.

Now fix two open neighbourhoods W_2 and W_3 of 0 in F such that $W_2 = -W_2 \subset 3W_2 = W_2 + W_2 + W_2 \subseteq W_3 = -W_3 \subseteq W_1 \cap W_0$.

By construction, $W_1 \subseteq kW_2$ for some $k \in \mathbb{N}$.

Hence the sets W_2 and W_3 have the following properties:

- W_2 and W_3 are a-bounded subsets of E;
- $F = \bigcup\{nW_2 : n \in \mathbb{N}\} = \bigcup\{nW_3 : n \in \mathbb{N}\}$;
- if L is a bounded or a precompact subset of F, then $L \subseteq nW_2$ for some $n \in \mathbb{N}$;
- if $a \in F$, $a \neq 0$, then for any $n \in \mathbb{N}$ there exists $t \in R$ such that $ta \notin nW_3$.

Since B is a non a-bounded subset of F and W_3 is an a-bounded of F, we have $B \setminus nW_3 \neq \varnothing$ for each $n \in \mathbb{N}$.

If $\mu \in M_p(X, E, F)$ and $\mu \neq 0$, then $supp_X(\mu)$ is a finite non-empty subset of X.

We can assume that $C(X, B) \subseteq C(X, E)$ and $C(X, B) \subseteq C(X, F)$.

Suppose that the set H is a-bounded or precompact in $M_p(X, E, F)$ and the set $supp_X(H)$ is not bounded in X. Fix $f \in C(X, B)$ such that the set $f(supp_X(H))$ is not a-bounded in F.

By induction, we shell construct a sequence $\{\mu_n : n \in \mathbb{N}\} \subseteq H$, a sequence $\{U_k : k \in \mathbb{N}\}$ of open subsets of X, a sequence $\{x_n \in supp_X(\mu_n) : n \in \mathbb{N}\}$ and a sequence $\{h_k \in C(X, B) : n \in \mathbb{N}\}$ with properties:

1. $x_i \in U_i$, $h_i(X \setminus U_i) = 0$ for any $i \in \mathbb{N}$;
2. $\{U_n : n \in \mathbb{N}\}$ is a discrete family of subsets of X;
3. $\mu_n(h_n) \notin nW$;
4. $supp_X\{\mu_1, \mu_2, ..., \mu_n\} \cap cl_X U_{n+1} = \varnothing$;
5. $f(U_n) \subseteq f(x_n) + W_0$ and $f(x_{n+1}) \notin \bigcup\{f(x_i) + W : i \leq n\}$ for each $n \in \mathbb{N}$;

Fix $\mu_1 \in H$ and $x_1 \in supp_X(\mu_1)$. There exists an open subset U_1 of X and $g_1 \in C(X, B)$ such that $f(U_1) \subseteq W_0 + f(x_1)$, $g_1(X \setminus U_1) = 0$ and $\mu_1(g_1) \neq 0$. There exists $\alpha_1 \in R$ such that $\alpha_1 \mu_1(g) \notin W_3$. We put $h_1 = \alpha_1 g_1$.

Assume that $n \geq 1$ and the objects $\{h_i, x_i, U_i, \mu_i : i \leq n\}$ are constructed. We put $M_n = \bigcup\{supp_X(\mu_i) : i \leq n\}$. The set M_n is finite. Hence the set

$f(supp_X(H)) \setminus f(M_n)$ is not a-bounded in F. For some $m_n \in \mathbb{N}$ we have $f(M_n) \subseteq m_n W_0$.

Fix $\mu_{n+1} \in H$ and $x_{n+1} \in supp_X(H)$ such that $f(x_{n+1}) \in B \setminus m_n W$. There exists an open subset U_{n+1} of X and $g_{n+1} \in C(X, B)$ such that $x_{n+1} \in U_{n+1}$, $f(U_{n+1}) \subseteq f(x_{n+1}) + W_0$, $g_{n+1}(X \setminus U_{n+1}) = 0$, $cl_X U_{n+1} \cap M_n = \varnothing$ and $M_{n+1}(g_{n+1}) \neq 0$. There exists $\alpha_{n+1} \in R$ such that $\alpha_{n+1}\mu_{n+1}(g_{n+1}) \notin (n+1)W$. We put $h_{n+1} = \alpha_{n+1} g_{n+1}$. That completes the inductive construction. The objects $\{x_m, \mu_n, h_n, U_n\}$ are constructed for all $n \in \mathbb{N}$. Let $h = \Sigma\{h_n : n \in \mathbb{N}\}$. Since $\{U_n : n \in \mathbb{N}\}$ is a discrete family and $h_n(X \setminus U_n) = 0$ for any $n \in \mathbb{N}$, we have $h \in C(X, B)$. By construction, $\mu_n(h) = \mu_n(h_n) \notin nW_0$ for any n. Then $\{\mu_n(h) : n \in \mathbb{N}\}$ is a not a-bounded subset of E. Since the set H is a-bounded, the set $\{\mu(h) : \mu \in H\}$ is a-bounded too, a contradiction. The proof is complete. □

Remark. Any normed space is a locally bounded $\mathbb{R}$-module. If E is a non-trivial normed space, then for any non-bounded subset L of the space X there exists $f \in C(X, E)$ such that the set $f(L)$ is not bounded in E. For a normed space E Theorem 4.4 was proved by V. Valov in [14]. For a ring R and $E = F$ Theorem 4.4 was proved in [8].

A space X is μ-complete if any closed bounded subset of X is compact.

A space X is Dieudonné complete if the maximal uniformity on X is complete. Any Dieudonné complete space is μ-complete.

Denote by PX the space X with the G_δ-topology generated by the G_δ-subsets of X. The set $\delta - cl_X H = cl_{PX} H$ is called the G_δ-closure of the set H in X. If $\delta - cl_X H = H$, then we say the set H is G_δ-closed.

If the space X is μ-complete, then any G_δ-closed subspace of X is μ-complete.

A tightness of a space X is the minimal cardinal number τ for which for any subset $L \subseteq X$ and any point $x \in cl_X L$ there exists a subset $L_1 \subseteq L$ such that $|L_1| \leq \tau$ and $x \in cl_X L_1$.

We denote by $t(X)$ and $l(X)$ the tightness and the Lindelöf numbers respectively of a space X.

The following four propositions were proved in [8] (see [1] for $E = \mathbb{R}$).

4.5. Proposition. Assume that E is a metrizable and $l(X^n) \leq \tau$ for any $n \in \mathbb{N}$. Then $t(C_p(X, E)) \leq \tau$.

4.6. Proposition. Let X and E be spaces and $t(X) \leq \aleph_0$. Then $C_p(X, E)$ is a G_δ-closed subspace of the space E^X. Moreover, if E is μ-complete then the space $C_p(X, E)$ is μ-complete too.

4.7. Proposition. Let F and E be topological R-modules and $H_p(F, E)$ be the space of all linear continuous mappings of F into E. Then $H_p(F, E)$ is a closed subspace of the space $C_p(F, E)$.

4.8. Corollary. Let E and F be topological R-modules and $t(F) \leq \aleph_0$. Then $H_p(F, E)$ is a G_δ-closed subset of E^F. In particular, if E is μ-complete, then space $H_p(F, E)$ is μ-complete too.

4.9. Proposition. Let Y be a subspace of the space X, E be a non-trivial topological R-module, X be an R-Tychonoff space and $p_Y(f) = f|_Y$ for each

$f \in C_p(X, E)$. Then the mapping $p_Y : C_p(X, E) \longrightarrow C_p(Y|X, E)$ has the following properties:

(i) p_Y is a continuous mapping.

(ii) If the set Y is closed in X, then the mapping p_Y is open.

(iii) If Y is dense in X, then p_Y is a one-to-one correspondence.

(iv) The subspace $C_p(Y|X, E)$ is dense in the $C_p(Y, E)$.

4.10. Theorem. Let E be a metrizable R-module, F be a locally bounded metrizable R-module, B be a closed submodule of F and X be an R-Tychonoff space with the following properties:

(b) : for any non-bounded subset L of X there exists $f \in C(X, B)$ such that the set $f(L)$ is not a-bounded in F;

(r) : B is topological isomorphic to some R-submodule of E;

(c) : X be an R-Tychonoff compactly E-full space.

Then the space X is μ-complete if and only if the space $M_p(X, E, F)$ is μ-complete.

Proof. By virtue of Proposition 2.3, we can assume that $X = e_X(X)$ is a subspace of the space $M_p(X, E, B)$. From Proposition 2.2 it follows that the subspace X is closed in $M_p(X, E, B)$. Obviously, $M_p(X, E, B)$ is a closed subspace of the space $M_p(X, E, F)$.

Let $M_p(X, E, F)$ be a μ-complete space. Since X is a closed subspaces of $M_p(X, E, B)$ and $M_p(X, E, F)$, the space X is μ-complete too.

Assume that X is a μ-complete space. Let Φ be a closed bounded subset of $M_p(X, E, F)$. Then the closure Y of the set $\cup\{supp_X(\mu) : \mu \in \Phi\}$ is a compact subset of X.

The restriction mapping $p_Y : C_p(X, E) \longrightarrow C_p(Y, E)$ is an open continuous linear mapping of the R-module $C_p(X, E)$ onto the R-module $C_p(Y, E)$.

Claim 1. The dual mapping $\varphi : F^{C(Y,E)} \longrightarrow F^{C(X,E)}$ is a linear embedding and the set $\varphi(F^{C(Y,E)})$ is closed in $F^{C(X,E)}$.

The proof of this fact is similar with the prof of Proposition 0.4.6 from [1].

By construction, we have $\Phi \subseteq \varphi(M_p(Y, E, F)) \subseteq M_p(X, E, F)$.

Claim 2. $\varphi(M_p(Y, E, F))$ is a closed subset of the subspaces $M_p(X, E, F)$ and $C_p(C_p(X, E), E)$ of the space $E^{C(X,E)}$.

Follows from Claim 1 and Proposition 4.7.

Claim 3. $\varphi(C_p(C_p(Y, E), F)) \subseteq C_p(C_p(X, E), F)$.

Follows from the continuity of the mapping p_Y.

Claim 4. The sets $\varphi(M_p(X, E, F))$ and $\varphi(C_p(C_p(Y, E), F))$ are G_δ-closed in $F^{C(X,E)}$.

Since Y is compact, from Proposition 4.5 it follows that $t(C_p(Y, E)) = \aleph_0$. Then, from Proposition 4.6 it follows that $C_p(C_p(Y, E), F)$ is a G_δ-closed subset of the space $F^{C(Y,E)}$. From Claim 1 it follows that $\varphi(C_p(C_p(Y, E), F))$ is G_δ-closed in $F^{C(X,E)}$. Corollary 4.8 completes the proof of the claim.

Let G be the G_δ-closure of the set $C_p(C_p(X, E), E))$ in $E^{C(X,E)}$. We have $M_p(X, E, F) \subseteq G$. Hence Φ is a bounded subset of the space G.

Claim 5. The sets $\varphi(M_p(X, E, F))$ and $\varphi(C_p(C_p(Y, E), F))$ are closed in G.

Follows from Claim 4.

Since F is a metrizable space, F is a μ-complete space. Thus Φ is a closed bounded subset of the μ-complete space G. Therefore the set Φ is compact. The proof is complete. □

5. Relations between linear equivalent spaces

Let R be a topological semiring and E, F be non-trivial locally bounded topological R-modules. The R-module $E \times F$ is locally bounded. We identify E with the R-submodule $E \times \{0\}$ of $E \times F$ and F with the R-submodule $\{0\} \times F$ of $E \times F$.

Fix two non-empty R-Tychonoff spaces X and Y with the properties:

- for any non-bounded subset L of X there exists $f \in C(X, E)$ such that the set $f(L)$ is not a-bounded in E;

- for any non-bounded subset L of Y there exists $f \in C(Y, F)$ such that the set $f(L)$ is not a-bounded in F.

Fix now a continuous linear homeomorphism $u : C_p(X, E) \longrightarrow C_p(Y, F)$. Then the mapping $v : M_p(Y, F, E \times F) \longrightarrow M_p(X, E, E \times F)$, where $v(\eta) = \eta \circ u$ for each $\eta \in M_p(Y, F, E \times F)$, is a linear homeomorphism.

For each $x \in X$ and each $f \in C_p(X, E)$ we put $\varepsilon_x(f) = (\xi_x(f), 0) = (f(x), 0) \in E \subseteq E \times F$. For each $y \in Y$ and each $g \in C_p(Y, F)$ we put $\delta_y(g) = (0, \xi_y(g)) = (0, g(y)) \in F \subseteq E \times F$. Realy, we can assume that $\varepsilon_x = \xi_x$ and $\delta_y = \xi_y$. Obviously, $v^{-1}(\varepsilon_x) = \varepsilon_x \circ u^{-1} \in M_p(Y, F, E \times F) \setminus \{0\}$ and $v(\delta_y) = \delta_y \circ u \in M_p(X, E, E \times F) \setminus \{0\}$. Hence, for each $x \in X$ and each $y \in Y$) we can put $\varphi(x) = supp_Y(v^{-1}(\varepsilon_x))$ and $\psi(y) = supp_X(v(\delta_y))$.

Property 7.1. *$\varphi : X \to Y$ and $\psi : Y \to X$ are l.s.c. set-valued mappings and $\varphi(x)$, $\psi(y)$ are finite non-empty sets for all points $x \in X$ and $y \in Y$.*

Proof. Follows from Proposition 4.1 and Theorem 5.1. □

Property 7.2. *Let $y_0 \in Y$, $f \in C(X, E)$ and $f(\psi(y_0)) = 0$. Then $u(f)(y_0) = 0$.*

Proof. For any $\eta \in M_p(Y, F, E \times F)$ and $g \in C(X, E)$ we have $v(\eta)(g) = \eta(u(g))$ $(\eta \circ u)(g)$. Since $f(supp_X(v(\delta_{y_0}))) = f(\psi(y_0)) = 0$, we have $(\delta_{y_0} \circ u)(f) = 0$ and $u(f)(y_0) = \delta_{y_0}(u(f)) = (\delta_{y_0} \circ u)(f) = 0$. The proof is complete. □

5.1. Corollary. If $f, g \in C(X, E)$ and $f|\psi(y) = g|\psi(y)$, then $u(f)(y) = u(g)(y)$.

Property 7.3. *$x \in \psi(\varphi(x))$ for every point $x \in X$ and $y \in \varphi(\psi(y))$ for every point $y \in Y$.*

Proof. For every $x \in X$ the sets $\varphi(x)$ and $\psi(\varphi(x))$ are finite and closed. Assume that $x_0 \in X$ and $x_0 \notin \psi(\varphi(x_0)) = H$. Fix $f \in C(X, E)$ such that $f(x_0) = b \neq 0$ and $f(H) = f(\psi(\varphi(x_0))) = 0$. Since $\psi(y) \subseteq H$ and $f(H) = 0$ for any $y \in \varphi(x_0)$ by virtue of Property 7.2, we have $u(f)(y) = 0$ for each $y \in \varphi(x_0)$. Since $u(f)(y) = 0$ for each $y \in \varphi(x_0)$, by virtue of Property 7.2, we have $f(x_0) = u^{-1}(u(f))(x_0) = 0$. By construction, we have $f(x_0) \neq 0$, a contradiction. □

Property 7.4. *If H is dense subset of Y, then $\psi(H)$ is a dense subset of X provided u is an injection.*

Proof. Assume that $x_0 \notin cl_X\psi(H)$. Then there exists $f \in C(X,E)$ such that $f(x_0) \neq 0$ and $f(\psi(H)) = 0$. Since $f(\psi(H)) = 0$ for any $y \in Y$, by virtue of Property 7.2, we have $u(f)(y) = 0$ for any $y \in Y$. Thus $u(f) = 0$. Hence $f = 0$, a contradiction. □

From the above properties follows

5.2. Corollary. The space X is separable if and only if the space Y is separable. In general, $d(X) = d(Y)$.

Property 7.5. *$\varphi(H)$ is a bounded set of Y for each bounded set H of X.*

Proof. Let H be a bounded subset of X. Then H is a bounded subset of $M_p(X,E,E\times F)$ and respectively $v^{-1}(H)$ is a bounded subset of $M_p(Y,F,E\times F)$. By Theorem 4.4 the set $supp_Y(v^{-1}(H))$ is a bounded subset of Y. The proof is complete. □

Property 7.6. *Let E and F be metrizable spaces, X be a compactly E-full space and Y be a compactly F-full space. Then the space X is μ-complete if and only if the space Y is μ-complete.*

Proof. Let X be a μ-complete space. Then $M_p(X,E,E\times F)$ and $M_p(Y,F,E\times F)$, by virtue of Theorem 4.10, are μ-complete spaces. By Theorem 4.10 the space Y is μ-complete too. The proof is complete. □

As in [3] we say that the pair of set-valued mappings $\theta : X \longrightarrow Y$ and $\pi : Y \longrightarrow X$ is called lower-reflective if it has the following conditions:
1*l*. θ and π are l.s.c.
2*l*. $\theta(x)$ and $\pi(x)$ are finite sets for all points $x \in X$ and $y \in Y$.
3*l*. $x \in \pi(\theta(x))$ and $y \in \theta(\pi(y))$ for all points $x \in X$ and $y \in Y$.
Also, as in [3] we say that the pair of set-valued mappings $\theta : X \longrightarrow Y$ and $\pi : Y \longrightarrow X$ is called upper-reflective if it has the following conditions:
1*u*. $\theta(F)$ is a bounded subset of Y for each bounded subset F of X.
2*u*. $\pi(\Phi)$ is a bounded subset of X for each bounded subset Φ of Y.
3*u*. $x \in cl_X\pi(\theta(x))$ and $y \in cl_Y\theta(\pi(y))$ for all points $x \in X$ and $y \in Y$.
General conclusion: The set valued mappings $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ forms an equivalence of X and Y in sense of article [3]. Thus the general theorems from [3] can be extended for the mappings in topological R-modules. In the following sections we formulate the general theorems for the R-modules, where R is a topological semiring.

6. Application to perfect properties

We say that the property $\mathcal{P}$ is a perfect property if for any continuous perfect mapping $f : X \longrightarrow Y$ of X onto Y we have $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$. We say that the property $\mathcal{P}$ is a strongly perfect property if it is perfect and any space with property $\mathcal{P}$ is μ-complete.

6.1. Example. From the Example 6.2 [3] the following properties are perfect: to be a compact space; to be a paracompact p-space; to be a paracompact space; to be a metacompact space; to be a k-scattered space; to be a monotonically p-space;

to be a monotonically Čech complete space; to be a Čech complete space; to be a Lindelöf space; to be a Lindelöf Σ-space; to be a subparacompact space; to be a locally compact space.

6.2. Example. The following properties are strongly perfect: to be a compact space; to be a paracompact p-space; to be a paracompact space; to be a μ-complete metacompact space; to be a k-scattered μ-complete space; to be a μ-complete monotonically p-space; to be a μ-complete monotonically Čech complete space; to be a μ-complete Čech complete space; to be a Lindelöf space; to be a Lindelöf Σ-space; to be a μ-complete subparacompact space; to be a μ-complete locally compact space.

A space X is called a wq-space if for any point $x \in X$ there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of X such that $x \in \cap\{U_n : n \in \mathbb{N}\}$ and each set $\{x_n \in U_n : n \in \mathbb{N}\}$ is bounded in X.

A space X is pseudocompact if the set X is bounded in the space X. Any pseudocompact space is a wq-space.

6.3. Theorem. Let R be a topological semiring and E and F be non-trivial locally bounded topological R-modules. Fix two non-empty R-Tychonoff spaces X and Y with the properties:

- for any non-bounded subset L of X there exists $f \in C(X, E)$ such that the set $f(L)$ is not a-bounded in E;

- for any non-bounded subset L of Y there exists $f \in C(Y, F)$ such that the set $f(L)$ is not a-bounded in F.

Assume that $u : C_p(X, E) \longrightarrow C_p(Y, F)$ is a linear homeomorphism. Then:

1. X is a pseudocompact space if and only if Y is a pseudocompact space.
2. If $\mathcal{P}$ is a perfect property and X, Y are μ-complete wq-spaces, then $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$.

Proof. Consider the set-valued mappings $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ constructed in the Section 7.

Let X be a pseudocompact space. Then X is a bounded subset of the space X. Hence $Y = \varphi(X)$ is a bounded subset of Y and Y is a pseudocompact space. Assertion 1 is proved.

Assume that $\mathcal{P}$ is a perfect property and X, Y are μ-complete wq-spaces. Suppose that $X \in \mathcal{P}$. By virtue of Theorem 2.5 from [3], there exist a space Z and two perfect single-valued mappings $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ onto X and Y, respectively. Hence, $Y, Z \in \mathcal{P}$. Assertion 2 is proved. The proof is complete. □

6.4. Theorem. Let R be a topological semiring and E and F be non-trivial metrizable locally bounded topological R-modules. Fix two non-empty spaces X and Y with the properties:

- X is an R-Tychonoff compactly E-full space and for any non-bounded subset L of X there exists $f \in C(X, E)$ such that the set $f(L)$ is not a-bounded in E;

- Y is an R-Tychonoff compactly E-full space and for any non-bounded subset L of Y there exists $f \in C(Y, F)$ such that the set $f(L)$ is not a-bounded in F.

Assume that $u : C_p(X, E) \longrightarrow C_p(Y, F)$ is a linear homeomorphism. Then:

1. The space X is μ-complete if and only if the space Y is μ-complete.

2. X is a compact space if and only if Y is a compact space.

3. If $\mathcal{P}$ is a strongly perfect property and X, Y are wq-spaces, then $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$.

Proof. Consider the set-valued mappings $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ constructed in the Section 7. Assertion 1 follows from Property 7.7.

Assume that $\mathcal{P}$ is a strongly perfect property and X, Y are wq-spaces. Suppose that $X \in \mathcal{P}$. By definition of a strongly perfect property, X is a μ-complete space. From assertion 1 it follows that Y is a μ-complete space too. By virtue of Theorem 2.5 from [3], there exist a space Z and two perfect single-valued mappings $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ onto X and Y, respectively. Hence, $Y, Z \in \mathcal{P}$. Assertion 3 is proved.

Let X be a compact space. By virtue of Theorem 6.3, Y is a pseudocompact space. Hence X and Y are wq-spaces. Assertion 3 completes proof of Assertion 2. The proof is complete. □

7. Application to open properties

We say that the property $\mathcal{P}$ is an of-property (open finite property) if for any continuous open finite-to-one mapping $f : X \longrightarrow Y$ and any subspace Z of X we have $Z \in \mathcal{P}$ if and only if $f(Z) \in \mathcal{P}$ (see [3]).

7.1. Example. From the results from [3] and [5] the following properties are of-properties: to be hereditarily Lindelöf; to be σ-space; to be hereditarily separable; to be σ-metrizable; to be σ-scattered; to be σ-discrete space.

7.2. Example. Let τ be an infinite cardinal. Consider the properties: $X \in e(\tau)$ if and only if $e(X) \leq \tau$; $X \in d(\tau)$ if and only if $d(X) \leq \tau$; $X \in hd(\tau)$ if and only if $hd(X) \leq \tau$; $X \in hl(\tau)$ if and only if $hl(X) \leq \tau$.

Then $e(\tau), d(\tau), hd(\tau), hl(\tau)$ are of-properties.

7.3. Theorem. Let R be a topological semiring and E, F be non-trivial locally bounded topological R-modules. Fix two non-empty R-Tychonoff spaces X and Y with the properties:

- for any non-bounded subset L of X there exists $f \in C(X, E)$ such that the set $f(L)$ is not a-bounded in E;
- for any non-bounded subset L of Y there exists $f \in C(Y, F)$ such that the set $f(L)$ is not a-bounded in F.

Assume that $u : C_p(X, E) \longrightarrow C_p(Y, F)$ is a linear homeomorphism. If $\mathcal{P}$ is an of-property, then $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$.

Proof. Consider the set-valued mappings $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ constructed in the Section 7. As in [3] (see Theorem 2.1 from [3]) we put $Z = \cup\{\{x\} \times \varphi(x) : x \in X\}$ and $S = \cup\{\psi(y) \times \{y\} : y \in Y\}$ as subspaces of the spaces $X \times Y$, $f(x, y) = x$ and $g(x, y) = y$ for any point $(x, y) \in X \times Y$. Then $f : Z \longrightarrow X$ and $g : S \longrightarrow Y$ are continuous open finite-to-one mappings. If $D = Z \cap S$, then from Property 7.4 it follows that $f(D) = X$ and $g(D) = Y$. Hence $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$. The proof is complete. □

8. $l_p(E,F)$-equivalence and metrizability

8.1. Theorem. Let R be a topological semiring and E, F be non-trivial metrizable locally bounded topological R-modules. Fix two non-empty spaces X and Y with the properties:

- X is an R-Tychonoff compactly E-full space and for any non-bounded subset L of X there exists $f \in C(X,E)$ such that the set $f(L)$ is not a-bounded in E;

- Y is an R-Tychonoff compactly E-full space and for any non-bounded subset L of Y there exists $f \in C(Y,F)$ such that the set $f(L)$ is not a-bounded in F.

Let X and Y be $l_p(E)$-equivalent spaces. Then:

1. X is a compact metrizable space if and only if Y is a compact metrizable space.

2. If X is a metrizable space, then the space Y is metrizable if and only if Y is a wq-space.

Proof. Any metrizable space is a wq-space.

Let X be a metrizable space and Y be a wq-space. Since X is metrizable, by virtue of Theorem 6.3, Y is a paracompact p-space. From Theorem 7.3 it follows that Y is a σ-space. If a paracompact space Y is a σ-space and a p-space, then Y is metrizable [13]. Assertion 2 is proved.

Assertion 1 follows from the Assertion 2 and Theorem 6.3. The proof is complete. □

References

[1] A. V. Arhangel'skii, *Topological Function Spaces*, Mathematics and its Applications (Soviet Series), vol. 78, Kluwer Academic Publishers Group, Dordrecht, 1992.

[2] A. V. Arhangel'skii, *On linear homomorphisms of function spaces*, Doklady Acad. Nauk SSSR 264 (1982), no. 6, 1289-1292. English translation: Soviet Math. Dokl. 25 (1982), 852-855.

[3] M.M. Choban, *General theorems on functional equivalence of topological spaces*, Topology Appl. 89 (1998), 223-239.

[4] M.M. Choban, *Algebraical equivalence of topological spaces*, Buletinul Acad. de Ştiinţe a Republicii Moldova, Matematica, 1 (2001), 12-36.

[5] M. Choban, *Open finite-to-one mappings*, Soviet Math. Dokl. 8 (1967), 603-603.

[6] M.M. Choban, *On the theory of topological algebraic systems*, Trudy Moskovskogo Matem. Obshchestva 48 (1985), 106-149. English translation: Trans. Moscow Math. Soc. 48, 1986, 115-159.

[7] M.M. Choban, *Some topics in topological algebra*, Topology Appl. 54 (1993), 183-202.

[8] M.M. Choban, R. N. Dumbrăveanu, *$l_p(R)$-equivalence of topological spaces and topological modules*, Buletinul Academiei de Stiinte a Rep. Moldova, Matematica 1 (2015), 20-47.

[9] R. Engelking, *General Topology,* PWN, Warsawa 1977.

[10] J. S. Golan, *Semirings and their applications,* Springer, 1999.

[11] V. P. Maslov, *Idempotent analysis,* American Mathematical Society, 1992.

[12] J. van Mill, *The infinite-dimensional topology of function spaces*, North-Holland Mathematical Library, Amsterdam, vol. 64, 2001.

[13] A. Okuyama, *A survey of the theory of σ-spaces*, General Topology Appl. 1 (1971), 57-63.

[14] V. Valov, *Function spaces*, Topology Appl. 81 (1997), no. 1, 1-22.

[15] S. Warner, *Topological rings*, North-Holland mathematics studies. Elsevier, v. 178, 1993.