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A study of the quasi covering dimension for finite spaces through the matrix theory

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Abstract

We use matrices to study the dimension function $\dim_q$, calling quasi covering dimension, for finite topological spaces, which is always greater than or equal to the classical covering dimension dim. In particular, we present algorithms in order to compute the $\dim_q(X)$ of an arbitrary finite topological space X.

Keywords: Covering dimension, quasi covering dimension, quasi cover, dense set.

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1. Preliminaries and notations

In this section we recall the notion of the topological covering dimension. We refer to [3, 6] for more details.

A *cover* of a topological space X is a non-empty set of subsets of X, whose union is X. A cover c of X is said to be *open* (*closed*) if all elements of c are open (closed). A family r of subsets of X is said to be a *refinement* of a family c of subsets of X if each element of r is contained in an element of c.

In what follows, we consider two symbols, "-1" and "∞", for which we suppose that:

(1) $-1 < k < \infty$ for every $k \in \{0, 1, \dots\}$.

(2) $\infty + k = k + \infty = \infty$, $-1 + k = k + (-1) = k$ for every $k \in \{0, 1, \dots\} \cup \{-1, \infty\}$.

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We define the *order* of a family r of subsets of a space X as follows:

(a) $\mathrm{ord}(r) = -1$ if and only if r consists the empty set only.

(b) $\mathrm{ord}(r) = k$, where $k \in \{0, 1, \ldots\}$, if and only if the intersection of any $k+2$ distinct elements of r is empty and there exist $k+1$ distinct elements of r, whose intersection is not empty.

(c) $\mathrm{ord}(r) = \infty$, if and only if for every $k \in \{1, 2, \ldots\}$ there exist k distinct elements of r, whose intersection is not empty.

We denote by dim the function, calling *covering dimension*, with domain the class of all topological spaces and range the set $\{0, 1, \ldots\} \cup \{-1, \infty\}$, satisfying the following conditions:

(1) $\dim(X) \leqslant k$ if and only if for every finite open cover c of the space X there exists a finite open cover r of X, refinement of c, such that $\mathrm{ord}(r) \leqslant k$.

(2) $\dim(X) = k$, if $\dim(X) \leqslant k$ and $\dim(X) \nleqslant k-1$.

(3) $\dim(X) = \infty$, if $\dim(X) \leqslant k$ does not hold for every $k = -1, 0, 1, 2, \ldots$

In study [5], we insert a topological dimension, calling quasi covering dimension and we prove that it is always greater than or equal to the classical covering dimension.

1.1. Definition. [5] A *quasi cover* of X is a non-empty set of subsets of X, whose union is dense in X. A quasi cover c of X is said to be *open* if all elements of c are open in the space X. Moreover, two quasi covers c_1 and c_2 are said to be *similar* (in short $c_1 \sim c_2$) if their unions are the same dense subset of X.

For every topological space X the relation $\sim$ is an equivalence relation on the set of all quasi covers of X. The collection of all equivalence classes under $\sim$ will be denoted by $\mathbf{QC}(X, \sim)$.

1.2. Definition. [5] We denote by $\dim_q$ the function, calling *quasi covering dimension*, with domain the class of all topological spaces and range the set $\{0, 1, \ldots\} \cup \{-1, \infty\}$, satisfying the following conditions:

(1) $\dim_q(X) \leqslant k$ if for every finite open quasi cover c of X there exists a finite open quasi cover r of X such that $r \sim c$, r is a refinement of c, and $\mathrm{ord}(r) \leqslant k$.

(2) $\dim_q(X) = k$ if $\dim_q(X) \leqslant k$ and $\dim_q(X) \nleqslant k-1$.

(3) $\dim_q(X) = \infty$ if $\dim_q(X) \leqslant k$ does not hold for every $k = -1, 0, 1, 2, \ldots$

In this paper we shall consider only finite topological spaces. Let

$$X = \{x_1, x_2, \ldots, x_n\}$$

be a finite topological space and let $\mathbf{U}_i$ be the smallest open subset of X which contains the point x_i, for $i = 1, 2, \ldots, n$. We give some notations which will be used in the rest of our study (see [1, 2]).

The $n \times n$ matrix $T_X = (t_{ij})$, where

$$t_{ij} = \begin{cases} 1, & \text{if } x_i \in \mathbf{U}_j \\ 0, & \text{otherwise} \end{cases}$$

is called the *incidence matrix* of the space X. We denote by $c_1, c_2, \ldots, c_n$ the n columns of the matrix T_X and by $\mathbf{1}$ the $n \times 1$ matrix which has all the elements equal to one, that is

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Let $i_1, i_2, \ldots, i_m$ be distinct elements of the set $\{1, \ldots, n\}$. By $a_{i_1 i_2 \cdots i_m}$ and $b_{i_1 i_2 \cdots i_m}$ we denote respectively the $n \times 1$ matrices

$$a_{i_1 i_2 \cdots i_m} = \begin{pmatrix} a^1_{i_1 i_2 \cdots i_m} \\ a^2_{i_1 i_2 \cdots i_m} \\ \vdots \\ a^n_{i_1 i_2 \cdots i_m} \end{pmatrix} \quad \text{and} \quad b_{i_1 i_2 \cdots i_m} = \begin{pmatrix} b^1_{i_1 i_2 \cdots i_m} \\ b^2_{i_1 i_2 \cdots i_m} \\ \vdots \\ b^n_{i_1 i_2 \cdots i_m} \end{pmatrix},$$

where

$$a^i_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & \text{if } i \in \{i_1, i_2, \ldots, i_m\} \\ 0, & \text{otherwise} \end{cases}$$

and

$$b^i_{i_1 i_2 \cdots i_m} = \begin{cases} 0, & \text{if } t_{ii_1} = t_{ii_2} = \ldots = t_{ii_m} = 0 \\ 1, & \text{otherwise.} \end{cases}$$

Let

$$c_i = \begin{pmatrix} c_{1i} \\ c_{2i} \\ \vdots \\ c_{ni} \end{pmatrix} \quad \text{and} \quad c_j = \begin{pmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{nj} \end{pmatrix}$$

be two $n \times 1$ matrices. Then, by $\max(c_i)$ we denote the maximum of the set $\{c_{1i}, c_{2i}, \ldots, c_{ni}\}$ and by $c_i + c_j$ the $n \times 1$ matrix

$$c_i + c_j = \begin{pmatrix} c_{1i} + c_{1j} \\ c_{2i} + c_{2j} \\ \vdots \\ c_{ni} + c_{nj} \end{pmatrix}.$$

Also, we write $c_i \leqslant c_j$ if only if $c_{si} \leqslant c_{sj}$, for each $s = 1, \ldots, n$.

The rest of the paper is organized as follows. In section 2 we give an algorithm to compute the dimension $\dim_q$ of a space X through a characterization of open and dense subsets of X. In section 3 we present a new algorithm to compute the dimension $\dim_q$ using the notion of quasi covers. Finally, in section 4 we present remarks concerning to this dimension.

2. An algorithm to compute the dimension $\dim_q(X)$ through a characterization of open and dense subsets of X

In this section we are going to characterize the open and dense subsets of a fixed finite topological space $X = \{x_1, x_2, \ldots, x_n\}$ using matrices.

2.1. Proposition. *Let $i_1, \ldots, i_m$ be distinct elements of the set $\{1, \ldots, n\}$. Then, $\{x_{i_1}, \ldots, x_{i_m}\} = \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l}$, for some $j_1, \ldots, j_l \in \{i_1, \ldots, i_m\}$ if and only if $a_{i_1 i_2 \cdots i_m} = b_{j_1 j_2 \cdots j_l}$.*

Proof. Let $\{x_{i_1}, \ldots, x_{i_m}\} = \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l}$, for some $j_1, \ldots, j_l \in \{i_1, \ldots, i_m\}$. We prove that $a_{i_1 i_2 \cdots i_m} = b_{j_1 j_2 \cdots j_l}$. For every $i \in \{1, \ldots, n\}$ in the i-row of these matrices we have the following cases:

$$\begin{aligned}
(1)\quad a^i_{i_1 i_2 \cdots i_m} = 1 &\Leftrightarrow i \in \{i_1, \ldots, i_m\} \Leftrightarrow x_i \in \{x_{i_1}, \ldots, x_{i_m}\} \\
&\Leftrightarrow \text{there exists } r \in \{1, \ldots, l\} \text{ such that } x_i \in \mathbf{U}_{j_r} \\
&\Leftrightarrow t_{ij_r} = 1 \Leftrightarrow b^i_{j_1 j_2 \cdots j_l} = 1. \\
(2)\quad a^i_{i_1 i_2 \cdots i_m} = 0 &\Leftrightarrow i \notin \{i_1, \ldots, i_m\} \Leftrightarrow x_i \notin \{x_{i_1}, \ldots, x_{i_m}\} \\
&\Leftrightarrow x_i \notin \mathbf{U}_{j_r}, \text{ for each } r \in \{1, \ldots, l\} \\
&\Leftrightarrow t_{ij_r} = 0, \text{ for each } r \in \{1, \ldots, l\} \Leftrightarrow b^i_{j_1 j_2 \cdots j_l} = 0.
\end{aligned}$$

We conclude that $a_{i_1 i_2 \cdots i_m} = b_{j_1 j_2 \cdots j_l}$.

Conversely, assume that $a_{i_1 i_2 \cdots i_m} = b_{j_1 j_2 \cdots j_l}$, for some $j_1, \ldots, j_l \in \{i_1, \ldots, i_m\}$. We prove that $\{x_{i_1}, \ldots, x_{i_m}\} = \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l}$. Let $i \in \{i_1, \ldots, i_m\}$. Then, $a^i_{i_1 i_2 \cdots i_m} = 1$. By assumption, $b^i_{j_1 j_2 \cdots j_l} = 1$. Therefore, there exists $r \in \{1, \ldots, l\}$ such that $t_{ij_r} = 1$ or equivalently $x_i \in \mathbf{U}_{j_r}$. Hence, $\{x_{i_1}, \ldots, x_{i_m}\} \subseteq \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l}$. Let $x_i \in \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l}$. Then, there exists $r \in \{1, \ldots, l\}$ such that $x_i \in \mathbf{U}_{j_r}$ or equivalently $t_{ij_r} = 1$. Thus, $b^i_{j_1 j_2 \cdots j_l} = 1$. By assumption, $a^i_{i_1 i_2 \cdots i_m} = 1$ and, therefore, $x_i \in \{x_{i_1}, \ldots, x_{i_m}\}$. Hence, $\mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l} \subseteq \{x_{i_1}, \ldots, x_{i_m}\}$. Thus, $\{x_{i_1}, \ldots, x_{i_m}\} = \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l}$. □

2.2. Corollary. *Let $i_1, \ldots, i_m$ be distinct elements of the set $\{1, \ldots, n\}$. Then, $\{x_{i_1}, \ldots, x_{i_m}\} = \mathbf{U}_{i_r}$, for some $r \in \{1, \ldots, m\}$ if and only if $a_{i_1 i_2 \cdots i_m} = c_{i_r}$.*

Proof. Follows from Proposition 2.1 and by the fact that $b_{i_r} = c_{i_r}$, for every $r \in \{1, \ldots, m\}$. □

2.3. Proposition. *Let $j_1, \ldots, j_l$ be distinct elements of the set $\{1, \ldots, n\}$. The set $\mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l}$ is dense in X if and only if $\max(b_{j_1 j_2 \cdots j_l} + c_j) = 2$, for each $j \in \{1, \ldots, n\} \setminus \{j_1, \ldots, j_l\}$.*

Proof. Suppose that $\mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l}$ is dense in X and let $j \in \{1, \ldots, n\} \setminus \{j_1, \ldots, j_l\}$. We set $k = \max(b_{j_1 j_2 \cdots j_l} + c_j)$ and prove that $k = 2$. Clearly, $k > 0$ and by the definitions of the matrices T_X and $b_{j_1 j_2 \cdots j_l}$ we have that either $k = 1$ or $k = 2$. Since $\mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l}$ is dense in X, there exists $q \in \{1, \ldots, l\}$ such that $\mathbf{U}_{j_q} \cap \mathbf{U}_j \neq \emptyset$. Therefore, $t_{i_0 j_q} = t_{i_0 j} = 1$, for some $i_0 \in \{1, \ldots, n\}$, which means that $b^{i_0}_{j_1 j_2 \cdots j_l} + t_{i_0 j} = 1 + 1 = 2$. Thus, $k = 2$.

Conversely, let $\max(b_{j_1 j_2 \cdots j_l} + c_j) = 2$, for each $j \in \{1, \ldots, n\} \setminus \{j_1, \ldots, j_l\}$. We shall prove that the set $\mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l}$ is dense in X. Assume that the set $\mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l}$ is not dense in X. Then, there exists an open set U in X such that

$$U \cap (\mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l}) = \emptyset. \tag{2.1}$$

Therefore, there exists $\mu \in \{1, \ldots, n\}$ such that $\mathbf{U}_\mu \subseteq U$ and $x_\mu \notin \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l}$. Hence, $\mu \notin \{j_1, \ldots, j_l\}$. Since $\max(b_{j_1 j_2 \cdots j_l} + c_\mu) = 2$, there exists $i_0 \in \{1, \ldots, n\}$

such that $b^{i_0}_{j_1 j_2 \cdots j_l} = t_{i_0 \mu} = 1$. Thus, $x_{i_0} \in \mathbf{U}_{j_q} \cap \mathbf{U}_\mu$, for some $q \in \{1, \ldots, l\}$, which contradicts the relation (2.1). □

Since for every open subset $U = \{x_{i_1}, \ldots, x_{i_m}\}$ of X there exist elements $j_1, \ldots, j_l \in \{i_1, \ldots, i_m\}$ such that $U = \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_l}$, from Propositions 2.1 and 2.3 we have the following corollary.

2.4. Corollary. *Let $i_1, \ldots, i_m$ be distinct elements of the set $\{1, \ldots, n\}$. Then, the set $\{x_{i_1}, \ldots, x_{i_m}\}$ is open and dense in X if and only if the following conditions hold:*

(1) There exist $j_1, \ldots, j_l \in \{i_1, \ldots, i_m\}$ such that $a_{i_1 i_2 \cdots i_m} = b_{j_1 j_2 \cdots j_l}$.
(2) $\max(b_{j_1 j_2 \cdots j_l} + c_j) = 2$, for each $j \in \{1, \ldots, n\} \setminus \{j_1, \ldots, j_l\}$.

2.5. Example. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$. We consider on X the topology which has as a basis the family $\{\{x_1\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_1, x_3, x_4, x_5\}\}$. The incidence matrix T_X of X is the 5×5 matrix

$$T_X = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\mathbf{U}_1 = \{x_1\}$, $\mathbf{U}_2 = \{x_1, x_2\}$, $\mathbf{U}_3 = \{x_1, x_3\}$, $\mathbf{U}_4 = \{x_1, x_4\}$ and $\mathbf{U}_5 = \{x_1, x_3, x_4, x_5\}$.

For the subset $\{x_1\}$ of X we have

$$a_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = b_1 = c_1.$$

Hence, this set is open in X and by Corollary 2.2 we have that $\{x_1\} = \mathbf{U}_1$. Moreover,

$$b_1 + c_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; b_1 + c_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; b_1 + c_4 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; b_1 + c_5 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore, $\max(b_1 + c_j) = 2$, for $j = 2, 3, 4, 5$. By the Corollary 2.4 we have that the set $\{x_1\}$ is open and dense in X.

For the subset $\{x_2, x_3\}$ of X we have

$$a_{23} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Since $a_{23} \neq b_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $a_{23} \neq b_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $a_{23} \neq b_{23} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, by Proposition 2.1 the set $\{x_2, x_3\}$ is not open in X.

For the subset $\{x_1, x_3, x_4\}$ of X we have

$$a_{134} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = b_{34}.$$

Hence, this set is open in X and by Proposition 2.1 we have that $\{x_1, x_3, x_4\} = \mathbf{U}_3 \cup \mathbf{U}_4$. Moreover,

$$b_{34} + c_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ b_{34} + c_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ b_{34} + c_5 = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

Therefore, $\max(b_{34} + c_j) = 2$, for $j = 1, 2, 5$. By the Corollary 2.4 we have that the set $\{x_1, x_3, x_4\}$ is open and dense in X.

2.6. Proposition. [5] *For the space X we have*

$$\dim_q(X) = \max\{\dim(D) : D \text{ is an open and dense subset of } X\}.$$

From Corollary 2.4 we get the following proposition.

2.7. Proposition. *The quasi covering dimension $\dim_q(X)$ is equal to the maximum of all $\dim(\{x_{i_1}, \ldots, x_{i_m}\})$ with the properties:*

(1) There exist $j_1, \ldots, j_l \in \{i_1, \ldots, i_m\}$ such that $a_{i_1 i_2 \cdots i_m} = b_{j_1 j_2 \cdots j_l}$.

(2) $\max(b_{j_1 j_2 \cdots j_l} + c_j) = 2$, for each $j \in \{1, \ldots, n\} \setminus \{j_1, \ldots, j_l\}$.

In the study [2] it was presented an algorithm of polynomial order for computing the covering dimension of the space $X = \{x_1, \ldots, x_n\}$. More precisely, the algorithm consists of the following $n - 1$ steps:

2.8. Algorithm.

Step 1: Read the n columns $c_1, \ldots, c_n$ of the incidence matrix T_X of X. If some column is equal to $\mathbf{1}$, then print $\dim(X) = 0$. Otherwise, go to Step 2.

Step 2: Find the sums $c_{j_{11}} + c_{j_{21}} + \ldots + c_{j_{(n-1)1}}$, for each

$$\{j_{11}, j_{21}, \ldots, j_{(n-1)1}\} \subseteq \{1, \ldots, n\}.$$

If there exists $\{j^0_{11}, j^0_{21}, \ldots, j^0_{(n-1)1}\} \subseteq \{1, \ldots, n\}$ such that

$$c_{j^0_{11}} + c_{j^0_{21}} + \ldots + c_{j^0_{(n-1)1}} \geqslant \mathbf{1},$$

then go to Step 3. Otherwise, print

$$\dim(X) = \max(c_1 + c_2 + \ldots + c_n) - 1.$$

Step 3: Find the sums $c_{j_{12}} + c_{j_{22}} + \ldots + c_{j_{(n-2)2}}$, for each

$$\{j_{12}, j_{22}, \ldots, j_{(n-2)2}\} \subseteq \{j^0_{11}, j^0_{21}, \ldots, j^0_{(n-1)1}\}.$$

If there exists $\{j^0_{21}, j^0_{22}, \ldots, j^0_{(n-2)2}\} \subseteq \{j^0_{11}, j^0_{21}, \ldots, j^0_{(n-1)1}\}$ such that

$$c_{j^0_{12}} + c_{j^0_{22}} + \ldots + c_{j^0_{(n-2)2}} \geqslant \mathbf{1},$$

then go to Step 4. Otherwise, print

$$\dim(X) = \max(c_{j^0_{11}} + c_{j^0_{21}} + \ldots + c_{j^0_{(n-1)1}}) - 1.$$

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Step $n-2$: Find the sums $c_{j_{1(n-3)}} + c_{j_{2(n-3)}} + c_{j_{3(n-3)}}$, for each

$$\{j_{1(n-3)}, j_{2(n-3)}, j_{3(n-3)}\} \subseteq \{j^0_{1(n-4)}, j^0_{2(n-4)}, j^0_{3(n-4)}, j^0_{4(n-4)}\}.$$

If there exists $\{j^0_{1(n-3)}, j^0_{2(n-3)}, j^0_{3(n-3)}\} \subseteq \{j^0_{1(n-4)}, j^0_{2(n-4)}, j^0_{3(n-4)}, j^0_{4(n-4)}\}$ such that

$$c_{j^0_{1(n-3)}} + c_{j^0_{2(n-3)}} + c_{j^0_{3(n-3)}} \geqslant \mathbf{1},$$

then go to Step $n-1$. Otherwise, print

$$\dim(X) = \max(c_{j^0_{1(n-4)}} + c_{j^0_{2(n-4)}} + c_{j^0_{3(n-3)}} + c_{j^0_{4(n-4)}}) - 1.$$

Step $n-1$: Find the sums $c_{j_{1(n-2)}} + c_{j_{2(n-2)}}$, for each

$$\{j_{1(n-2)}, j_{2(n-2)}\} \subseteq \{j^0_{1(n-3)}, j^0_{2(n-3)}, j^0_{3(n-3)}\}.$$

If there exists $\{j^0_{1(n-2)}, j^0_{2(n-2)}\} \subseteq \{j^0_{1(n-3)}, j^0_{2(n-3)}, j^0_{3(n-3)}\}$ such that

$$c_{j^0_{1(n-2)}} + c_{j^0_{2(n-2)}} \geqslant \mathbf{1},$$

then print

$$\dim(X) = \max(c_{j^0_{1(n-2)}} + c_{j^0_{2(n-2)}}) - 1.$$

2.9. Remark. It was proved that an upper bound on the number of iterations of the Algorithm 2.8 is $\frac{1}{2}n^2 + \frac{3}{2}n - 3$.

Now, we are going to give an algorithm for computing the quasi covering dimension of the space $X = \{x_1, \ldots, x_n\}$.

2.10. Algorithm.

Step 0: Read the n columns $c_1, \ldots, c_n$ of the incidence matrix T_X of X.

Step 1: Find $k_1 = \dim(X)$ (Algorithm 2.8).

Step 2: Find the set $\mathcal{P}_1$ of all subsets $\{i_{11}, \ldots, i_{(n-1)1}\}$ of $\{1, \ldots, n\}$ with the properties:

(1) There exist $j_{11}, \ldots, j_{l1} \in \{i_{11}, \ldots, i_{(n-1)1}\}$ such that $a_{i_{11}i_{21}\cdots i_{(n-1)1}} = b_{j_{11}j_{21}\cdots j_{l1}}$.
(2) $\max(b_{j_{11}j_{21}\cdots j_{l1}} + c_j) = 2$, for each $j \in \{1, \ldots, n\} \setminus \{j_{11}, \ldots, j_{l1}\}$.

If $\mathcal{P}_1 = \emptyset$, then put $k_2 = 0$ and go to the step 3. Otherwise, use Algorithm 2.8 to find

$$k_2 = \max(\{\dim(\{x_{i_{11}}, \ldots, x_{i_{(n-1)1}}\}) : \{i_{11}, \ldots, i_{(n-1)1}\} \in \mathcal{P}_1\})$$

and go to the Step 3.

Step 3: Find the set $\mathcal{P}_2$ of all subsets $\{i_{12}, \ldots, i_{(n-2)2}\}$ of $\{1, \ldots, n\}$ with the properties:

(1) There exist $j_{12}, \ldots, j_{l2} \in \{i_{12}, \ldots, i_{(n-2)2}\}$ such that $a_{i_{12}i_{22}\cdots i_{(n-2)2}} = b_{j_{12}j_{22}\cdots j_{l2}}$.

(2) $\max(b_{j_{12}j_{22}\cdots j_{l2}} + c_j) = 2$, for each $j \in \{1, \ldots, n\} \setminus \{j_{12}, \ldots, j_{l2}\}$.

If $\mathcal{P}_2 = \emptyset$, then put $k_3 = 0$ and go to the step 4. Otherwise, use Algorithm 2.8 to find

$$k_3 = \max(\{\dim(\{x_{i_{12}}, \ldots, x_{i_{(n-2)2}}\}) : \{i_{12}, \ldots, i_{(n-2)2}\} \in \mathcal{P}_2\})$$

and go to the Step 4.

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Step n: Find the set $\mathcal{P}_{n-1}$ of all subsets $\{i_{1(n-1)}\}$ of $\{1, \ldots, n\}$ with the property $a_{i_{1(n-1)}} = b_{i_{1(n-1)}} = c_{i_{1(n-1)}}$. If $\mathcal{P}_{n-1} = \emptyset$, then put $k_n = 0$ and go to the step $n+1$. Otherwise, use Algorithm 2.8 to find

$$k_n = \max(\dim(\{x_{i_{1(n-1)}}\}) : \{i_{1(n-1)}\} \in \mathcal{P}_{n-1})$$

and go to the Step $n+1$.

Step $n+1$: Print $\dim_q(X) = \max\{k_1, k_2, \ldots, k_n\}$.

2.11. Example. Let X be the space of Example 2.5. We use Algorithm 2.10 to compute $\dim_q(X)$.

Step 0. The 5 columns of the incidence matrix T_X are

$$c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; c_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; c_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; c_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; c_5 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Step 1. Using Algorithm 2.8 we find $k_1 = \dim(X) = 1$.

Step 2. We have $\mathcal{P}_1 = \{\{1, 2, 3, 4\}, \{1, 3, 4, 5\}\}$. Using Algorithm 2.8 we find $\dim(\{x_1, x_2, x_3, x_4\}) = 2$ and $\dim(\{x_1, x_3, x_4, x_5\}) = 0$. Therefore, $k_2 = 2$.

Step 3. We have $\mathcal{P}_2 = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$. Using Algorithm 2.8 we find $\dim(\{x_1, x_2, x_3\}) = \dim(\{x_1, x_2, x_4\}) = \dim(\{x_1, x_3, x_4\}) = 1$. Therefore, $k_3 = 1$.

Step 4. We have $\mathcal{P}_3 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$. Using Algorithm 2.8 we find $\dim(\{x_1, x_2\}) = \dim(\{x_1, x_3\}) = \dim(\{x_1, x_4\}) = 0$. Therefore, $k_4 = 2$.

Step 5. We have $\mathcal{P}_4 = \{\{1\}\}$. Using Algorithm 2.8 we find $\dim(\{x_1\}) = 0$. Therefore, $k_5 = 0$.

Step 6. Print $\dim_q(X) = \max\{k_1, k_2, k_3, k_4, k_5\} = 2$.

3. An algorithm to compute the dimension $\dim_q(X)$ using the notion of quasi cover

In what follows, we consider a fixed finite topological space $X = \{x_1, x_2, \ldots, x_n\}$. For every $\mathbf{c} \in \mathbf{QC}(X, \sim)$ we denote by $\mathbf{c}(X)$ the set of all subsets $\{x_{i_1}, \ldots, x_{i_m}\}$ of X such that the family $\{\mathbf{U}_{i_1}, \ldots, \mathbf{U}_{i_m}\} \in \mathbf{c}$. Also by $\preceq_{\mathbf{c}}$ we define a relation on the set $\mathbf{c}(X)$ as follows:

$$\{x_{i_1}, \ldots, x_{i_{m_1}}\} \preceq_{\mathbf{c}} \{x_{i'_1}, \ldots, x_{i'_{m_2}}\}$$

if and only if

$$\{\mathbf{U}_{i_1}, \ldots, \mathbf{U}_{i_{m_1}}\} \subseteq \{\mathbf{U}_{i'_1}, \ldots, \mathbf{U}_{i'_{m_2}}\}.$$

This relation is a preorder on the set $\mathbf{c}(X)$.

3.1. Definition. Let $\mathbf{c} \in \mathbf{QC}(X, \sim)$. Every minimum element of $(\mathbf{c}(X), \preceq_{\mathbf{c}})$ is called a $\mathbf{c}$-*minimal family.*

3.2. Remark. (1) For the finite topological space X and for every $\mathbf{c} \in \mathbf{QC}(X, \sim)$ there exist $\mathbf{c}$-minimal families on the set $\mathbf{c}(X)$ (see Proposition 3.4).
(2) If $\{x_{i_1}, \ldots, x_{i_{m_1}}\}$ and $\{x_{i'_1}, \ldots, x_{i'_{m_2}}\}$ are two $\mathbf{c}$-minimal families, for some $\mathbf{c} \in \mathbf{QC}(X, \sim)$ then $\{\mathbf{U}_{i_1}, \ldots, \mathbf{U}_{i_{m_1}}\} = \{\mathbf{U}_{i'_1}, \ldots, \mathbf{U}_{i'_{m_2}}\}$.
(3) It is known that a finite space X is T_0 if and only if $\mathbf{U}_i = \mathbf{U}_j$ implies $x_i = x_j$ for every i, j. We note that, if the finite space X is T_0, then the relation $\preceq_{\mathbf{c}}$ is an order. Also, in this case there exists exactly one minimal family on the set $\mathbf{c}(X)$.

3.3. Proposition. *Let* $\mathbf{c} \in \mathbf{QC}(X, \sim)$. *If the family* $\{x_{i_1}, \ldots, x_{i_m}\} \in \mathbf{c}(X)$ *is not a* $\mathbf{c}$*-minimal family, then there exist* $i'_1, \ldots, i'_{m-1} \in \{i_1, \ldots, i_m\}$ *such that* $\{x_{i'_1}, \ldots, x_{i'_{m-1}}\} \in \mathbf{c}(X)$.

Proof. Suppose that the family $\{x_{i_1}, \ldots, x_{i_m}\} \in \mathbf{c}(X)$ is not $\mathbf{c}$-minimal. Then, there exists $\{x_{r_1}, \ldots, x_{r_\mu}\} \in \mathbf{c}(X)$ such that $\{x_{i_1}, \ldots, x_{i_m}\} \npreceq_{\mathbf{c}} \{x_{r_1}, \ldots, x_{r_\mu}\}$ or equivalently $\{\mathbf{U}_{i_1}, \ldots, \mathbf{U}_{i_m}\} \nsubseteq \{\mathbf{U}_{r_1}, \ldots, \mathbf{U}_{r_\mu}\}$. Let $\alpha \in \{1, \ldots, m\}$ such that $\mathbf{U}_{i_\alpha} \notin \{\mathbf{U}_{r_1}, \ldots, \mathbf{U}_{r_\mu}\}$. Since $\{\mathbf{U}_{r_1}, \ldots, \mathbf{U}_{r_\mu}\} \in \mathbf{c}$, there exists $\beta \in \{1, \ldots, \mu\}$ such that $x_{i_\alpha} \in \mathbf{U}_{r_\beta}$. By the fact that $\mathbf{U}_{i_\alpha}$ is the smallest open set of X containing the point x_{i_α} we have that $\mathbf{U}_{i_\alpha} \subseteq \mathbf{U}_{r_\beta}$. Also, since $\mathbf{U}_{i_\alpha} \notin \{\mathbf{U}_{r_1}, \ldots, \mathbf{U}_{r_\mu}\}$, we have $\mathbf{U}_{i_\alpha} \neq \mathbf{U}_{r_\beta}$. Therefore, $\mathbf{U}_{i_\alpha} \subset \mathbf{U}_{r_\beta}$. Since $\{\mathbf{U}_{i_1}, \ldots, \mathbf{U}_{i_m}\} \in \mathbf{c}$, there exists $\gamma \in \{1, \ldots, m\}$ such that $x_{r_\beta} \in \mathbf{U}_{i_\gamma}$. By the fact that $\mathbf{U}_{r_\beta}$ is the smallest open set of X containing the point x_{r_β} we have that $\mathbf{U}_{r_\beta} \subseteq \mathbf{U}_{i_\gamma}$. Hence, $\mathbf{U}_{i_\alpha} \subset \mathbf{U}_{i_\gamma}$ and, therefore, the family $\{\mathbf{U}_{i_1}, \ldots, \mathbf{U}_{i_m}\} \setminus \{\mathbf{U}_{i_\alpha}\} \in \mathbf{c}$ has $m - 1$ elements. □

3.4. Proposition. *Let* $\mathbf{c} \in \mathbf{QC}(X, \sim)$,

$$\nu = \min\{m \in \{1, 2, \ldots\} : \text{there exist } j_1, \ldots, j_m \text{ such that } \{x_{j_1}, \ldots, x_{j_m}\} \in \mathbf{c}(X)\},$$

and $\{x_{j_1}, \ldots, x_{j_\nu}\} \in \mathbf{c}(X)$. *Then,* $\{x_{j_1}, \ldots, x_{j_\nu}\}$ *is a* $\mathbf{c}$*-minimal family.*

Proof. Suppose that the family $\{x_{j_1}, \ldots, x_{j_\nu}\}$ is not $\mathbf{c}$-minimal. By Proposition 3.3, there exists an element of $\mathbf{c}(X)$ with $\nu - 1$ elements, which is a contradiction by the choice of ν. □

3.5. Proposition. *Let* $\mathbf{c} \in \mathbf{QC}(X, \sim)$ *and* $\{x_{i_1}, \ldots, x_{i_m}\}$ *be a* $\mathbf{c}$*-minimal family. If* $\mathrm{ord}(\{\mathbf{U}_{i_1}, \ldots, \mathbf{U}_{i_m}\}) = k \geqslant 0$, *then for every* $\{x_{r_1}, \ldots, x_{r_\mu}\} \in \mathbf{c}(X)$ *we have* $\mathrm{ord}(\{\mathbf{U}_{r_1}, \ldots, \mathbf{U}_{r_\mu}\}) \geqslant k$.

Proof. Let $\{x_{r_1},\dots,x_{r_\mu}\} \in \mathbf{c}(X)$. Then, $\{x_{i_1},\dots,x_{i_m}\} \preceq_{\mathbf{c}} \{x_{r_1},\dots,x_{r_\mu}\}$ and, therefore, $\{\mathbf{U}_{i_1},\dots,\mathbf{U}_{i_m}\} \subseteq \{\mathbf{U}_{r_1},\dots,\mathbf{U}_{r_\mu}\}$. Since $\mathrm{ord}(\{\mathbf{U}_{i_1},\dots,\mathbf{U}_{i_m}\}) = k$, we have $\mathrm{ord}(\{\mathbf{U}_{r_1},\dots,\mathbf{U}_{r_\mu}\} \geqslant k$. □ □

3.6. Proposition. *Let $k \in \{0,1,\dots\}$. Then, $\dim_q(X) \leqslant k$ if and only if for every $\mathbf{c} \in \mathbf{QC}(X,\sim)$ there exists $\{\mathbf{U}_{i_1},\dots,\mathbf{U}_{i_m}\} \in \mathbf{c}$ such that $\mathrm{ord}(\{\mathbf{U}_{i_1},\dots,\mathbf{U}_{i_m}\}) \leqslant k$.*

Proof. Let $\dim_q(X) \leqslant k$ and $\mathbf{c} \in \mathbf{QC}(X,\sim)$. We set

$$\nu = \min\{m \in \{1,2,\dots\} : \text{there exist } i_1,\dots,i_m \text{ such that } \{\mathbf{U}_{i_1},\dots,\mathbf{U}_{i_m}\} \in \mathbf{c}\}$$

and $c = \{\mathbf{U}_{i_1},\dots,\mathbf{U}_{i_\nu}\} \in \mathbf{c}$. Since $\dim_q(X) \leqslant k$, there exists an open quasi cover $r = \{V_1,\dots,V_\mu\}$ of X such that $r \sim c$, r is a refinement of c, and $\mathrm{ord}(r) \leqslant k$. For the proof of the proposition it suffices to prove that $c \subseteq r$. We suppose that there exists $\alpha \in \{1,\dots,\nu\}$ such that $\mathbf{U}_{i_\alpha} \notin r$. Since $r \sim c$, there exists $\beta \in \{1,\dots,\mu\}$ such that $x_{i_\alpha} \in V_\beta$. By the fact that $\mathbf{U}_{i_\alpha}$ is the smallest open set of X containing the point x_{i_α} we have that $\mathbf{U}_{i_\alpha} \subseteq V_\beta$. Also, since $\mathbf{U}_{i_\alpha} \notin r$, we have $\mathbf{U}_{i_\alpha} \neq V_\beta$. Therefore, $\mathbf{U}_{i_\alpha} \subset V_\beta$. Since r is a refinement of c, there exists $\gamma \in \{1,\dots,\nu\}$ such that $V_\beta \subseteq \mathbf{U}_{j_\gamma}$. Hence,

$$\mathbf{U}_{i_\alpha} \subset \mathbf{U}_{i_\gamma}.$$

We observe that the family $c \setminus \{\mathbf{U}_{i_\alpha}\} \in \mathbf{c}$ has $\nu-1$ elements, which is a contradiction by the choice of ν. Thus, $c \subseteq r$.

Conversely, suppose that for every $\mathbf{c} \in \mathbf{QC}(X,\sim)$ there exists $\{\mathbf{U}_{i_1},\dots,\mathbf{U}_{i_m}\} \in \mathbf{c}$ such that $\mathrm{ord}(\{\mathbf{U}_{i_1},\dots,\mathbf{U}_{i_m}\}) \leqslant k$. We prove that $\dim_q(X) \leqslant k$. Let c be an arbitrary finite open quasi cover of the space X. Then, there exists $\mathbf{c} \in \mathbf{QC}(X,\sim)$ such that $c \in \mathbf{c}$. Let $r = \{\mathbf{U}_{i_1},\dots,\mathbf{U}_{i_m}\} \in \mathbf{c}$ such that $\mathrm{ord}(\{\mathbf{U}_{i_1},\dots,\mathbf{U}_{i_m}\}) \leqslant k$. Then, $r \sim c$. It suffices to prove that the open quasi cover $\{\mathbf{U}_{i_1},\dots,\mathbf{U}_{i_m}\}$ of X is a refinement of c. Indeed, since $r \sim c$, for each $q \in \{1,\dots,m\}$ there exists $V_q \in c$ such that $x_{i_q} \in V_q$. Hence, $\mathbf{U}_{i_q} \subseteq V_q$, for every $q \in \{1,\dots,m\}$. □

3.7. Proposition. *Let $k \in \{0,1,\dots\}$. Then, $\dim_q(X) \leqslant k$ if and only if for every $\mathbf{c} \in \mathbf{QC}(X,\sim)$ there exists a $\mathbf{c}$-minimal family $\{x_{j_1},\dots,x_{j_\nu}\}$ such that $\mathrm{ord}(\{\mathbf{U}_{j_1},\dots,\mathbf{U}_{j_\nu}\}) \leqslant k$.*

Proof. Let $\dim_q(X) \leqslant k$ and $\mathbf{c} \in \mathbf{QC}(X,\sim)$. By Proposition 3.6 there exists $\{x_{i_1},\dots,x_{i_m}\} \in \mathbf{c}(X)$ with $\mathrm{ord}(\{\mathbf{U}_{i_1},\dots,\mathbf{U}_{i_m}\}) \leqslant k$. Let $\{x_{j_1},\dots,x_{j_\nu}\} \in \mathbf{c}(X)$ be a $\mathbf{c}$-minimal family (see Proposition 3.4). If $\mathrm{ord}(\{\mathbf{U}_{j_1},\dots,\mathbf{U}_{j_\nu}\}) > k$, then by Proposition 3.5, $\mathrm{ord}(\{\mathbf{U}_{i_1},\dots,\mathbf{U}_{i_m}\}) > k$, which is a contradiction. Therefore, $\mathrm{ord}(\{\mathbf{U}_{j_1},\dots,\mathbf{U}_{j_\nu}\}) \leqslant k$.

Conversely, suppose that for every $\mathbf{c} \in \mathbf{QC}(X,\sim)$ there is a $\mathbf{c}$-minimal family $\{x_{j_1},\dots,x_{j_\nu}\}$ such that $\mathrm{ord}(\{\mathbf{U}_{j_1},\dots,\mathbf{U}_{j_\nu}\}) \leqslant k$. Then, $\{\mathbf{U}_{j_1},\dots,\mathbf{U}_{j_\nu}\} \in \mathbf{c}$ and by Proposition 3.6 we have $\dim_q(X) \leqslant k$. □

3.8. Proposition. [1] *Let $c_{i_1},\dots,c_{i_m}$ be m columns of the incidence matrix T_X and $k = \max(c_{i_1}+\dots+c_{i_m})$. Then, $\mathrm{ord}(\{\mathbf{U}_{i_1},\dots,\mathbf{U}_{i_m}\}) = k-1$.*

3.9. Proposition. *For every $\mathbf{c} \in \mathbf{QC}(X,\sim)$ let $\{x_{i_1^{\mathbf{c}}},\dots,x_{i_m^{\mathbf{c}}}\} \in \mathbf{c}(X)$ be a $\mathbf{c}$-minimal family. Then,*

$$\dim_q(X) = \max\{\max(c_{i_1^{\mathbf{c}}}+\dots+c_{i_m^{\mathbf{c}}}) - 1 : \mathbf{c} \in \mathbf{QC}(X,\sim)\}.$$

Proof. Let $k_{\mathbf{c}} = \max(c_{i_1^{\mathbf{c}}} + \ldots + c_{i_m^{\mathbf{c}}})$, for every $\mathbf{c} \in \mathbf{QC}(X, \sim)$ and

$$k = \max\{k_{\mathbf{c}} - 1 : \mathbf{c} \in \mathbf{QC}(X, \sim)\}.$$

By Proposition 3.8 we have

$$\operatorname{ord}(\{\mathbf{U}_{i_1^{\mathbf{c}}}, \ldots, \mathbf{U}_{i_m^{\mathbf{c}}}\}) = k_{\mathbf{c}} - 1, \ \mathbf{c} \in \mathbf{QC}(X, \sim). \tag{3.1}$$

Therefore, by Proposition 3.7, $\dim_q(X) \leqslant k$. We prove that $\dim_q(X) = k$. Suppose that $\dim_q(X) < k$. Let $\mathbf{c}_0 \in \mathbf{QC}(X, \sim)$ such that $k = k_{\mathbf{c}_0} - 1$. By Proposition 3.6 there exists $\{\mathbf{U}_{r_1}, \ldots, \mathbf{U}_{r_\mu}\} \in \mathbf{c}_0$ such that $\operatorname{ord}(\{\mathbf{U}_{r_1}, \ldots, \mathbf{U}_{r_\mu}\}) < k$. By relation (3.1) we have $\operatorname{ord}(\{\mathbf{U}_{i_1^{\mathbf{c}_0}}, \ldots, \mathbf{U}_{i_m^{\mathbf{c}_0}}\}) = k_{\mathbf{c}_0} - 1 = k$. Therefore, by Proposition 3.5, $\operatorname{ord}(\{\mathbf{U}_{r_1}, \ldots, \mathbf{U}_{r_\mu}\}) \geqslant k$ which is a contradiction. Thus, $\dim_q(X) = k$. □

The proof of the following proposition is a straightforward verification from the definitions.

3.10. Proposition. *The quasi covers* $\{U_{i_1}, \ldots, U_{i_{k_1}}\}$ *and* $\{U_{j_1}, \ldots, U_{j_{k_2}}\}$ *of* X *are similar if and only if* $b_{i_1 i_2 \cdots i_{k_1}} = b_{j_1 j_2 \cdots j_{k_2}}$.

Using the notion of the quasi cover, Proposition 2.3 can be written as follows.

3.11. Proposition. *Let* $i_1, \ldots, i_m$ *be distinct elements of the set* $\{1, \ldots, n\}$. *The set* $\{\mathbf{U}_{i_1}, \ldots, \mathbf{U}_{i_m}\}$ *is a quasi cover of* X *if and only if* $\max(b_{i_1 i_2 \cdots i_m} + c_j) = 2$, *for each* $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$.

3.12. Proposition. *Let* $i_1, \ldots, i_m$ *be distinct elements of the set* $\{1, \ldots, n\}$ *such that* $\max(b_{i_1 i_2 \cdots i_m} + c_j) = 2$, *for each* $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$. *If for every set* $\{i'_1, \ldots, i'_{m-1}\} \subseteq \{i_1, \ldots, i_m\}$ *we have* $b_{i'_1 i'_2 \cdots i'_{m-1}} \neq b_{i_1 i_2 \cdots i_m}$, *then the family* $\{x_{i_1}, \ldots, x_{i_m}\}$ *is a* $\mathbf{c}$*-minimal family, where* $\{\mathbf{U}_{i_1}, \ldots, \mathbf{U}_{i_m}\} \in \mathbf{c}$.

Proof. By Proposition 3.11 the set $\{\mathbf{U}_{i_1}, \ldots, \mathbf{U}_{i_m}\}$ is a quasi cover of X. Let $\mathbf{c}$ be the element of $\mathbf{QC}(X, \sim)$ for which $\{\mathbf{U}_{i_1}, \ldots, \mathbf{U}_{i_m}\} \in \mathbf{c}$. Suppose that the family $\{x_{i_1}, \ldots, x_{i_m}\}$ is not a $\mathbf{c}$-minimal family. By Proposition 3.3, there exist $i'_1, \ldots, i'_{m-1} \in \{i_1, \ldots, i_m\}$ such that $\{x_{i'_1}, \ldots, x_{i'_{m-1}}\} \in \mathbf{c}(X)$. By Proposition 3.10, $b_{i'_1 i'_2 \cdots i'_{m-1}} = b_{i_1 i_2 \cdots i_m}$ which is a contradiction. □

The proof of the following proposition is straightforward verification of the Propositions 3.9 and 3.12.

3.13. Proposition. *The quasi covering dimension* $\dim_q(X)$ *is equal to the maximum of all* $\max(c_{i_1} + \ldots + c_{i_m}) - 1$ *with the properties:*

(1) $\max(b_{i_1 i_2 \cdots i_m} + c_j) = 2$, *for each* $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$.

(2) *For every* $\{i'_1, \ldots, i'_{m-1}\} \subseteq \{i_1, \ldots, i_m\}$ *we have* $b_{i'_1 i'_2 \cdots i'_{m-1}} \neq b_{i_1 i_2 \cdots i_m}$.

3.14. Algorithm.

Let $X = \{x_1, \ldots, x_n\}$ be a finite space. Our intended algorithm contains the following $n + 1$ steps:

Step 0. Read the n columns $c_1, \ldots, c_n$ of the matrix T_X.

Step 1. Find the set S_1 of all $\{i_{11}\} \subseteq \{1, \ldots, n\}$ satisfying the property:
$\max(b_{i_{11}} + c_j) = 2$, for each $j \in \{1, \ldots, n\} \setminus \{i_{11}\}$.
If $S_1 = \emptyset$, then put $k_1 = 0$ and go to the Step 2. Otherwise, put

$$k_1 = \max\{\max(c_{i_{11}}) - 1 : \{i_{11}\} \in S_1\}$$

and go to the Step 2.

Step 2. Find the set S_2 of all $\{i_{12}, i_{22}\} \subseteq \{1,\dots,n\}$ satisfying the properties:
(1) $\max(b_{i_{12}i_{22}} + c_j) = 2$, for each $j \in \{1,\dots,n\} \setminus \{i_{12}, i_{22}\}$.
(2) For every $\{i'_{12}\} \subseteq \{i_{12}, i_{22}\}$ we have $b_{i'_{12}} \neq b_{i_{12}i_{22}}$.
If $S_2 = \emptyset$, then put $k_2 = 0$ and go to the Step 3. Otherwise, put

$$k_2 = \max\{\max(c_{i_{11}} + c_{i_{22}}) - 1 : \{i_{11}, i_{12}\} \in S_2\}$$

and go to the Step 3.

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. . .
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Step $n-2$. Find the set S_{n-2} of all $\{i_{1(n-2)},\dots,i_{(n-2)(n-2)}\} \subseteq \{1,\dots,n\}$ satisfying the properties:
(1) $\max(b_{i_{1(n-2)}i_{2(n-2)}\cdots i_{(n-2)(n-2)}} + c_j) = 2$, for each $j \in \{1,\dots,n\} \setminus \{i_{1(n-2)},\dots,i_{(n-2)(n-2)}\}$.
(2) For every $\{i'_{1(n-2)},\dots,i'_{(n-3)(n-2)}\} \subseteq \{i_{1(n-2)},\dots,i_{(n-2)(n-2)}\}$ we have $b_{i'_{1(n-2)}i'_{2(n-2)}\cdots i'_{(n-3)(n-2)}} \neq b_{i_{1(n-2)}i_{2(n-2)}\cdots i_{(n-2)(n-2)}}$.
If $S_{n-2} = \emptyset$, then put $k_{n-2} = 0$ and go to the Step $n-1$. Otherwise, put

$$k_{n-2} = \max\{\max(c_{i_{1(n-2)}}+\dots+c_{i_{(n-2)(n-2)}})-1 : \{i_{1(n-2)},\dots,i_{(n-2)(n-2)}\} \in S_{n-2}\}$$

and go to the Step $n-1$.

Step $n-1$. Find the set S_{n-1} of all $\{i_{1(n-1)},\dots,i_{(n-1)(n-1)}\} \subseteq \{1,\dots,n\}$ satisfying the properties:
(1) $\max(b_{i_{1(n-1)}i_{2(n-1)}\cdots i_{(n-1)(n-1)}} + c_j) = 2$, for each $j \in \{1,\dots,n\} \setminus \{i_{1(n-1)},\dots,i_{(n-1)(n-1)}\}$.
(2) For every $\{i'_{1(n-1)},\dots,i'_{(n-2)(n-1)}\} \subseteq \{i_{1(n-1)},\dots,i_{(n-1)(n-1)}\}$ we have $b_{i'_{1(n-1)}i'_{2(n-1)}\cdots i'_{(n-2)(n-1)}} \neq b_{i_{1(n-1)}i_{2(n-1)}\cdots i_{(n-1)(n-1)}}$.
If $S_{n-1} = \emptyset$, then put $k_{n-1} = 0$ and go to the Step n. Otherwise, put

$$k_{n-1} = \max\{\max(c_{i_{1(n-1)}}+\dots+c_{i_{(n-1)(n-1)}})-1 : \{i_{1(n-1)},\dots,i_{(n-1)(n-1)}\} \in S_{n-1}\}$$

and go to the Step n.

Step n. If for every $\{i'_{1n},\dots,i'_{(n-1)n}\} \subseteq \{1,\dots,n\}$ we have $b_{i'_{1n}i'_{2n}\cdots i'_{(n-1)n}} \neq \mathbf{1}$, then put

$$k_n = \max(c_1 + \dots + c_n) - 1$$

and go to the Step $n+1$. Otherwise, put $k_n = 0$ and go to the Step $n+1$.

Step $n+1$. Print $\dim_q(X) = \max\{k_1, k_2, \dots, k_n\}$.

3.15. Example. Let X be the space of Example 2.5. We use Algorithm 3.14 to compute $\dim_q(X)$.

Step 0. The 5 columns of the incidence matrix T_X are

$$c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ c_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ c_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ c_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ c_5 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Step 1. We have $S_1 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$ and

$$k_1 = \max\{\max(c_i) - 1 : i = 1, \dots, 5\} = 0.$$

Step 2. We have $S_2 = \{\{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}\}$ and

$$\max(c_2 + c_3) - 1 = \max(c_2 + c_4) - 1 = \max(c_2 + c_5) - 1 = \max(c_3 + c_4) - 1 = 1.$$

Hence, $k_2 = 1$.

Step 3. We have $S_3 = \{\{2, 3, 4\}\}$ and $k_3 = \max(c_2 + c_3 + c_4) - 1 = 2$.

Step 4. We have $S_4 = \emptyset$ and $k_4 = 0$.

Step 5. We have $b_{2345} = \mathbf{1}$ and $k_5 = 0$.

Step 6. Print $\dim_q(X) = \max\{k_1, k_2, k_3, k_4, k_5\} = 2$.

4. Remarks on the quasi covering dimension

In this section we present some remarks with respect to quasi covering dimension and the algorithms of sections 2 and 3.

4.1. Remark. Let $A = (\alpha_{ij})$ be a $n \times n$ matrix and $B = (\beta_{ij})$ be a $m \times m$ matrix. The *Kronecker product* of A and B (see, for instance, [4]) is the $mn \times mn$ matrix

$$A \otimes B = \begin{pmatrix} \alpha_{11}B & \dots & \alpha_{1n}B \\ \vdots & \ddots & \vdots \\ \alpha_{n1}B & \dots & \alpha_{mn}B \end{pmatrix}.$$

Let $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_m\}$ be two finite spaces with incidence matrices T_X and T_Y, respectively. It is known that the incidence matrix of the space $X \times Y$ is the kronecker product $T_X \otimes T_Y$ of T_X and T_Y (see, [7]).

Here, we give an example from which we may conclude that the inequality

$$\dim_q(X \times Y) \leqslant \dim_q(X) + \dim_q(Y)$$

does not hold for every finite topological spaces X and Y.

4.2. Example. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ with the topologies

$$\tau_X = \{\emptyset, \{x_2\}, \{x_1, x_2\}, \{x_2, x_3\}, X\}$$

and

$$\tau_Y = \{\emptyset, \{y_1\}, \{y_1, y_2\}, \{y_1, y_3\}, \{y_1, y_4\}, \{y_1, y_2, y_3\}, \{y_1, y_2, y_4\}, \{y_1, y_3, y_4\}, Y\}.$$

The incidence matrices of X and Y are

$$T_X = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_Y = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, the incidence matrix $T_{X\times Y}$ of the product space $X \times Y$ is

$$T_{X\times Y} = T_X \otimes T_Y = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In study [1], we have compute that $\dim(X \times Y) = 5$. Thus, by Proposition 2.6 we have that $\dim_q(X \times Y) \geqslant 5$. Also, for the topological spaces X and Y, following one of the Algorithms 2.10 and 3.14, we have that $\dim_q(X) = 1$ and $\dim_q(Y) = 2$. From the above we may conclude that $\dim_q(X \times Y) \nleq \dim_q(X) + \dim_q(Y)$.

4.3. Remark. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite space.

(a) Algorithm 2.10: From the Step 1 up to Step n we appoint all the open and dense subsets $\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ of X and we compute their covering dimensions (based on the Algorithm 2.8). So, we have to apply the Algorithm 2.10

$$\binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{2} + \binom{n}{1} = 2^n - 1 \text{ times.}$$

(b) Algorithm 3.14: We do not need to use Algorithm 2.8. From the Step 1 up to Step n we find all the numbers $\max(c_{i_1} + \ldots + c_{i_m}) - 1$ of the subsets $\{i_1, \ldots, i_m\}$ of $\{1, \ldots, n\}$ which satisfy the conditions of Proposition 3.13. Therefore, the number of iterations the algorithm performs in Steps $1, 2, \ldots, n$ is $2^n - 1$.

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