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Annihilation of $\mathrm{tor}_{\mathbb{Z}_p}(\mathscr{G}^{\mathrm{ab}}_{K,S})$ for real abelian extensions $K/\mathbb{Q}$

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Abstract
Let K be a real abelian extension of $\mathbb{Q}$. Let p be a prime number, S the set of p-places of K and $\mathscr{G}_{K,S}$ the Galois group of the maximal $S \cup \{\infty\}$-ramified pro-p-extension of K (i.e., unramified outside p and ∞). We revisit the problem of annihilation of the p-torsion group $\mathscr{T}_K := \mathrm{tor}_{\mathbb{Z}_p}(\mathscr{G}^{\mathrm{ab}}_{K,S})$ initiated by us and Oriat then systematized in our paper on the construction of p-adic L-functions in which we obtained a canonical ideal annihilator of $\mathscr{T}_K$ in full generality (1978–1981). Afterwards (1992–2014) some annihilators, using cyclotomic units, were proposed by Solomon, Belliard–Nguyen Quang Do, Nguyen Quang Do–Nicolas, All, Belliard–Martin. In this text, we improve our original papers and show that, in general, the Solomon elements are not optimal and/or partly degenerated. We obtain, whatever K and p, an universal non-degenerated annihilator in terms of p-adic logarithms of cyclotomic numbers related to L_p-functions at $s = 1$ of *primitive characters of* K (Theorem 9.4). Some computations are given with PARI programs; the case $p = 2$ is analyzed and illustrated in degrees 2, 3, 4 to test a conjecture.

Keywords: Class field theory, Abelian p-ramification; annihilation of p-torsion modules, p-adic L-functions, Stickelberger's elements, Cyclotomic units
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1. Introduction

Let $K/\mathbb{Q}$ be a real abelian extension of Galois group G_K. Let p be a prime number, S the set of p-places of K, and $\mathscr{G}_{K,S}$ the Galois group of the maximal S-ramified in the ordinary sense (i.e., unramified outside p and ∞, whence totally real if $p = 2$) pro-p-extension of K.

We revisit the classical problem of annihilation of the so-called $\mathbb{Z}_p[G_K]$-module $\mathscr{T}_K := \mathrm{tor}_{\mathbb{Z}_p}(\mathscr{G}^{\mathrm{ab}}_{K,S})$, as dual of $\mathrm{H}^2(\mathscr{G}_{K,S}, \mathbb{Z}_p(0))$. This was initiated by us [12] (1979) and improved by Oriat [22] (1981). Then in our paper [13] (1978/79) on the construction of p-adic L-functions (via an "arithmetic Mellin transform" from the "Spiegel involution" of suitable Stickelberger elements) we obtained incidentally a canonical ideal annihilator $\mathscr{A}_K$ of $\mathscr{T}_K$ in full generality, but our purpose, contrary to the present work, was the semi-simple case with p-adic characters and the annihilation of the isotopic components; this aspect has then been outdated by the "principal theorems" of Ribet–Mazur–Wiles–Kolyvagin–Greither (refer for instance to the bibliography of [15]), and many other contributions.

Afterwards some annihilators, using cyclotomic units, were proposed by Solomon [26] (1992), Belliard–Nguyen Quang Do [5] (2005), Nguyen Quang Do–Nicolas [21] (2011), All [1] (2013), Belliard–Martin [4] (2014), using techniques of Sinnott, Rubin, Thaine, Coleman, from Iwasawa's theory.

In this text, we translate into english some parts of the above 1978–1981's papers, written in french with tedious classical techniques, then we show that, in general, the Solomon elements Ψ_K are often degenerated regarding the annihilator $\mathscr{A}_K$, even for cyclic fields, and explain the origin of this gap due to trivialization of some Euler factors.

We obtain, whatever K and p (Theorem 9.4), an universal non-degenerated annihilator $\mathscr{A}_K$, in terms of p-adic logarithms of cyclotomic numbers, perhaps the best possible regarding these classical methods, but probably too general to cover all the possible Galois structures of $\mathscr{T}_K$, which raises the question of the existence of a better theorem than Stickelberger's one.

Indeed, if the semi-simple case is now completely solved, the non-semi-simple case is far to be known. Numerical experiments show in this case that the results are far to give the precise Galois structure of $\mathscr{T}_K$ (e.g., in direction of its Fitting ideal), moreover, it seems to us that many (all ?) papers are based on the classical reasoning with Kummer's theory and Leopoldt's Spiegel involution applied to Stickelberger's elements, even translated into Iwasawa's theory, without practical analysis of the results (e.g., with extensive numerical illustrations). So, there is some difficulties to compare these various contributions.

Thus, we perform some computations given with PARI programs [23] to analyse the quality of such annihilators, which is in general not addressed by papers dealing with Iwasawa's theory. We consider in a large part the case $p=2$, illustrated in degrees 2, 3, 4 to test the Conjecture 5.7.

2. Notations and reminders on p-ramification theory

Let K be a real abelian number field of degree d, of Galois group G_K, and let $p \geq 2$ be a prime number; we denote by S the set of prime ideals of K dividing p. Let $\mathscr{G}_{K,S}$ be the Galois group of the maximal $S \cup \{\infty\}$-ramified pro-p-extension of K and let H_K^{pr} be the maximal abelian $S \cup \{\infty\}$-ramified pro-p-extension of K. To simplify, we put $\mathscr{G}_{K,S}^{\mathrm{ab}} =: \mathscr{G}_K$ and (e.g., [8, Chapter III, § (c)]):

$$\mathscr{T}_K := \mathrm{tor}_{\mathbb{Z}_p}(\mathscr{G}_K) = \mathrm{Gal}(H_K^{\mathrm{pr}}/K_\infty)$$

where $K_\infty = K\mathbb{Q}_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension of K; so:

$$\mathscr{G}_K \simeq \mathbb{Z}_p \bigoplus \mathscr{T}_K$$

since, in the abelian case, Leopoldt's conjecture is true.

We denote by F an extension of K such that H_K^{pr} is the direct compositum of K_∞ and F over K, then by $\mathscr{C}\!\ell_K^\infty$ the subgroup of the p-class group $\mathscr{C}\!\ell_K$ corresponding, by class field theory, to $\mathrm{Gal}(H_K/K_\infty \cap H_K)$, where H_K is the p-Hilbert class field. We have (where $\sim$ means "equality up to a p-adic unit"):

$$\#\mathscr{C}\!\ell_K^\infty \sim \frac{\#\mathscr{C}\!\ell_K}{[K_\infty \cap H_K : K]} \sim \#\mathscr{C}\!\ell_K \cdot \frac{[K \cap \mathbb{Q}_\infty : \mathbb{Q}]}{e_p} \cdot \frac{2}{\#(\langle -1 \rangle \cap \mathrm{N}_{K/\mathbb{Q}}(U_K))}, \tag{2.1}$$

where e_p is the ramification index of p in $K/\mathbb{Q}$ [8, Theorem III.2.6.4], and U_K is defined as follows:

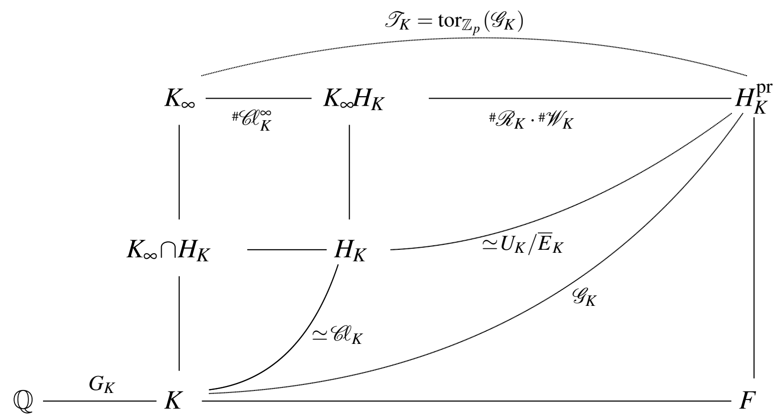
For each $\mathfrak{p} \mid p$, let $K_\mathfrak{p}$ be the $\mathfrak{p}$-completion of K and $\overline{\mathfrak{p}}$ the corresponding prime ideal of the ring of integers of $K_\mathfrak{p}$; then let:

$$U_K := \Big\{u \in \bigoplus_{\mathfrak{p}\mid p} K_\mathfrak{p}^\times,\ u = 1 + x,\ x \in \bigoplus_{\mathfrak{p}\mid p} \overline{\mathfrak{p}}\Big\} \ \ \&\ \ W_K := \mathrm{tor}_{\mathbb{Z}_p}(U_K)$$

the $\mathbb{Z}_p$-module (of $\mathbb{Z}_p$-rank $d = [K:\mathbb{Q}]$) of principal local units at p and its torsion subgroup, respectively; by class field theory this gives in the diagram:

$$\mathrm{Gal}(H_K^{\mathrm{pr}}/H_K) \simeq U_K/\overline{E}_K \ \ \&\ \ \mathrm{Gal}(H_K^{\mathrm{pr}}/K_\infty H_K) \simeq \mathrm{tor}_{\mathbb{Z}_p}\big(U_K/\overline{E}_K\big),$$

where $\overline{E}_K$ is the closure of the group E_K of p-principal global units of K (i.e., units $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p}|p} \mathfrak{p}}$):



For any field k, let μ_k be the group of roots of unity of k of p-power order. Then $W_K = \bigoplus_{\mathfrak{p}|p} \mu_{K_\mathfrak{p}}$. We have the following exact sequence defining $\mathscr{W}_K$ and $\mathscr{R}_K$ via the p-adic logarithm log ([8, Lemma III.4.2.4] or [9, Lemma 3.1 & § 5]):

$$\begin{aligned} 1 \to \mathscr{W}_K := W_K/\mu_K &\longrightarrow \mathrm{tor}_{\mathbb{Z}_p}\big(U_K/\overline{E}_K\big) \\ &\xrightarrow{\log} \mathrm{tor}_{\mathbb{Z}_p}\big(\log(U_K)/\log(\overline{E}_K)\big) =: \mathscr{R}_K \to 0. \end{aligned} \tag{2.2}$$

The group $\mathscr{R}_K$ is called the *normalized p-adic regulator of* K and makes sense for any number field (see the above references in [9] for more details and the main properties of these invariants).

It is clear that the annihilation of $\mathscr{T}_K$ mainely concerns the group $\mathscr{R}_K$ since the p-class group is in general trivial (and so for p large enough) and because the regulator may be non-trivial with large valuations and unpredictible p (see [11] for some conjectures and [10] giving programs of fast computation of the *group structure of* $\mathscr{T}_K$ *for any number field* given by means of polynomials).

Definition 2.1. *A field K is said to be p-rational if the Leopoldt conjecture is satisfied for p in K and if the torsion group $\mathscr{T}_K$ is trivial ([14, Section III, § 2], then [8, Theorem IV.3.5], [10], and bibliographies for the history and properties of p-rationality).*

This has deep consequences in Galois theory over K since $\mathscr{T}_K$ is the dual of $\mathrm{H}^2(\mathscr{G}_{K,S}, \mathbb{Z}_p(0))$ [18].

3. Kummer theory and Spiegel involution

3.1 Kummer theory

We denote by $\mathbb{Q}_n$, $n \geq 0$, the nth stage in $\mathbb{Q}_\infty$ so that $[\mathbb{Q}_n : \mathbb{Q}] = p^n$. Let $n_0 \geq 0$ be defined by $K \cap \mathbb{Q}_\infty =: \mathbb{Q}_{n_0}$.

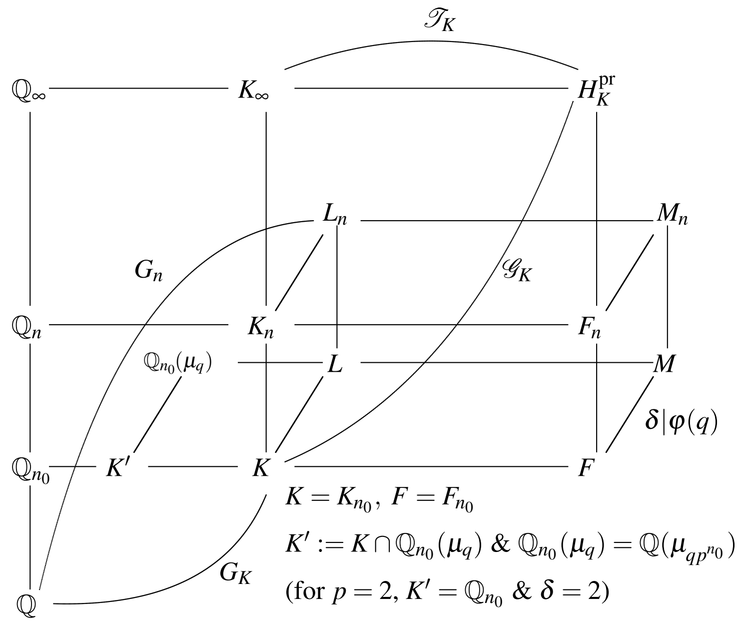
Let $n \geq n_0$. We denote by K_n the compositum $K\mathbb{Q}_n$ and by F_n the compositum $FK_n = F\mathbb{Q}_n$ (in other words, $K = K_{n_0}$, $F = F_{n_0}$). Then we have the *group isomorphism* $\mathrm{Gal}(F_n/K_n) \simeq \mathscr{T}_K$ for all $n \geq n_0$.

Put $q = p$ (resp. 4) if $p \neq 2$ (resp. $p = 2$). Let $L = K(\mu_q)$ and $M = F(\mu_q)$; then put $L_n := LK_n$ for all $n \geq n_0$.

Let $M_n := F_n(\mu_q)$ (whence $L = L_{n_0}$, $M = M_{n_0}$). For $p \neq 2$, the degrees $[L_n : K_n] = [M_n : F_n]$ are equal to a divisor δ of $p-1$ independent of $n \geq n_0$ (δ is even since K is real). For $p = 2$, $\delta = 2$. In any case, one has, for $n \geq n_0$:

$$L_n = K(\mu_{qp^n}).$$

All this is summarized by the following diagram:



Lemma 3.1. *Let f_K be the conductor of K. Then the conductor f_{L_n} of L_n ($n \geq n_0$) is equal to* $\mathrm{l.c.m.}\,(f_K, qp^n)$. *Thus for n large enough (explicit), $f_{L_n} = qp^n f'$, with $p \nmid f'$. If $p \nmid f_K$, then $f_{L_n} = qp^n f_K$ for all $n \geq n_0 + e$.*

Proof. A classical formula (see, e.g., [8, Proposition II.4.1.1]). □

Lemma 3.2. *Let p^e, $e \geq 0$, be the exponent of $\mathscr{T}_K$. Then, for all $n \geq n_0 + e$, the restriction $\mathscr{T}_K \longrightarrow \mathrm{Gal}(F_n/K_n)$ is an isomorphism of G_K-modules and $\mathscr{T}_K \simeq \mathrm{Gal}(M_n/L_n)$.*

Proof. The abelian group $\mathscr{G}_K := \mathrm{Gal}(H_K^{\mathrm{pr}}/K)$ is normal in $\mathrm{Gal}(H_K^{\mathrm{pr}}/\mathbb{Q})$, then $(\mathscr{G}_K)^{p^{n-n_0}}$ is normal; but $(\mathscr{G}_K)^{p^{n-n_0}}$ fixes F_n which is Galois over $\mathbb{Q}$. In other words, G_K, as well as $\mathrm{Gal}(K_n/\mathbb{Q})$ or $\mathrm{Gal}(K_\infty/\mathbb{Q})$, operate by conjugation in the same way since $\mathscr{G}_K$ is abelian; if F is clearly non-unique, then F_{n_0+e} is canonical, being the fixed fiel of $\big(\mathscr{G}_K\big)^{p^e}$. Then $\mathrm{Gal}(M_n/L_n) \simeq \mathrm{Gal}(F_n/K_n)$ is trivialy an somorphism of G_K-modules. □

The use of the extension F is not strictly necessary but clarifies the reasoning which needs to work at any level $n \geq n_0 + e$ to preserve Galois structures.

The extension M_n/L_n (of exponent p^e) is a Kummer extension for the "exponent" qp^n since L_n contains the group μ_{qp^n} and since $n \geq n_0 + e$.

Let $G_n := \mathrm{Gal}(L_n/\mathbb{Q})$ and let, for $n \geq n_0 + e$,

$$\mathrm{Rad}_n := \{w \in L_n^\times,\ \sqrt[qp^n]{w} \in M_n\}$$

be the radical of M_n/L_n. Then we have the group isomorphism:

$$\mathrm{Rad}_n / L_n^{\times qp^n} \simeq \mathrm{Gal}(M_n/L_n).$$

In some sense, the group $\mathrm{Rad}_n/L_n^{\times qp^n}$ does not depend on $n \geq n_0 + e$ since the canonical isomorphism $\mathrm{Gal}(M_{n+h}/L_{n+h}) \simeq \mathrm{Gal}(M_n/L_n)$ gives $L_{n+h}(\sqrt[qp^n]{\mathrm{Rad}_n}) = M_{n+h}$; the map $\mathrm{Rad}_n/L_n^{\times qp^n} \xrightarrow{p^h} \mathrm{Rad}_{n+h}/L_{n+h}^{\times qp^{n+h}}$ is an isomorphism for any $h \geq 0$. In other words, as soon as $n \geq n_0 + e$, we have:

$$\mathrm{Rad}_n \subseteq L_n^{\times qp^{n-e}}\ \ \&\ \ \mathrm{Rad}_{n+h} = \mathrm{Rad}_n^{p^h} \cdot L_{n+h}^{\times qp^{n+h}}.$$

3.2 Spiegel involution

The structures of $(\mathbb{Z}/qp^n\mathbb{Z})[G_n]$-modules of the Galois group $\mathrm{Gal}(M_n/L_n)$ and $\mathrm{Rad}_n/L_n^{\times qp^n}$ are related via the "Spiegel involution" defined as follows: let $\omega_n : G_n \longrightarrow \mathbb{Z}/qp^n\mathbb{Z}$ be the *character of Teichmüller of level n* defined by:

$$\zeta^s = \zeta^{\omega_n(s)}, \text{ for all } s \in G_n \text{ and all } \zeta \in \mu_{qp^n}.$$

The Spiegel involution is the involution of $(\mathbb{Z}/qp^n\mathbb{Z})[G_n]$ defined by:

$$x := \sum_{s\in G_n} a_s \cdot s \;\mapsto\; x^* := \sum_{s\in G_n} a_s \cdot \omega_n(s) \cdot s^{-1}.$$

Thus, if s is the Artin symbol $\left(\frac{L_n}{a}\right)$, then $\left(\frac{L_n}{a}\right)^* \equiv a \cdot \left(\frac{L_n}{a}\right)^{-1} \pmod{qp^n}$. For the convenience of the reader we prove once again the very classical:

Lemma 3.3. *Let $n \geq n_0 + e$ where $p^{n_0} = [K \cap \mathbb{Q}_\infty : \mathbb{Q}]$ and p^e is the exponent of $\mathscr{T}_K$. The annihilators A_n of $\mathrm{Gal}(M_n/L_n)$ (thus of $\mathscr{T}_K$) in $(\mathbb{Z}/qp^n\mathbb{Z})[G_n]$ are the images of the annihilators S_n of $\mathrm{Rad}_n/L_n^{\times qp^n}$ by the Spiegel involution and inversely. An annihilator A_n of $\mathscr{T}_K$ only depends on its projection $A_{K,n}$ in $(\mathbb{Z}/qp^n\mathbb{Z})[G_K]$.*

Proof. To simplify, put $\overline{\mathrm{Rad}} := \mathrm{Rad}_n/L_n^{\times qp^n}$, $\mathscr{T} := \mathrm{Gal}(M_n/L_n) \simeq \mathscr{T}_K$. Let:

$$\begin{aligned}\lambda \;:\; \overline{\mathrm{Rad}} \times \mathscr{T} &\longrightarrow \mu_{qp^n}\\ (\overline{w}, \tau) &\longmapsto \left(\sqrt[qp^n]{w}\right)^{\tau - 1};\end{aligned}$$

then λ is a non-degenerated $\mathbb{Z}/qp^n\mathbb{Z}$-bilinear form such that:

$$\lambda(\overline{w}^s, \tau) = \lambda(\overline{w}, \tau^{s^*}), \text{ for all } s \in G_n,$$

where $s^* = \omega_n(s) \cdot s^{-1}$ (see e.g., [8, Corollary I.6.2.1]).
Let $S_n = \sum_{s\in G_n} a_s \cdot s \in (\mathbb{Z}/qp^n\mathbb{Z})[G_n]$; then, for all $(\overline{w}, \tau) \in \overline{\mathrm{Rad}} \times \mathscr{T}$ we have:

$$\lambda(\overline{w}^{S_n}, \tau) = \prod_{s\in G_n} \lambda(\overline{w}^s, \tau)^{a_s} = \prod_{s\in G_n} \lambda(\overline{w}, \tau^{s^*})^{a_s} = \lambda(\overline{w}, \tau^{S_n^*}).$$

So, if S_n annihilates $\overline{\mathrm{Rad}}$, then $\lambda(\overline{w}, \tau^{S_n^*}) = 1$ for all $\overline{w}$ & τ; since λ is non-degenerated, $\tau^{S_n^*} = 1$ for all $\tau \in \mathscr{T}$. Whence the annihilation of $\mathscr{T}$ by $A_n = S_n^*$ *(without any assumption on K nor on p)*, then by the projection $A_{K,n}$ since $\mathrm{Gal}(L_n/K)$ acts trivially on $\mathrm{Gal}(M_n/L_n)$. □

Remark 3.4. *(i) As we have mention, the radical Rad_n does not depend realy on the field L_n for $n \geq n_0 + e$; so, if we consider the radical of the maximal p-ramified abelian p-extension of L_n, of exponent qp^n:*

$$\mathrm{Rad}'_n := \{w' \in L_n^\times,\ L_n(\sqrt[qp^n]{w'})/L_n \textit{ is p-ramified}\},$$

we obtain a group whose p-rank tends to infinity with n; this is due mainely to the $\mathbb{Z}_p$-rank of the compositum of the $\mathbb{Z}_p$-extensions of L_n (totally imaginary) and from the less known $\mathscr{T}_{L_n}$ which contains $\mathscr{T}_{K_n}$. But since $\mathscr{T}_K$ is annihilated by $1 - s_\infty$, $\mathrm{Rad}_n/L_n^{\times qp^n}$ is annihilated by $(1 - s_\infty)^* = 1 + s_\infty$ which means that only the "minus part" of $\mathrm{Rad}'_n/L_n^{\times qp^n}$ is needed, which eliminates the huge "plus" part containing in particular all the units. Thus Rad_n is essentially given by the "relative" S'_n-units of L_n (S'_n being the set of p-places of L_n) and generators of some "relative" p-classes of L_n.

(ii) In the case $p = 2$, let $\mathscr{T}_K^{\mathrm{res}} := \mathrm{tor}_{\mathbb{Z}_2}(\mathscr{G}_{K,S}^{\mathrm{res\,ab}})$, where $\mathscr{G}_{K,S}^{\mathrm{res}}$ is the Galois group of the maximal abelian S-ramified (i.e., unramified outside 2 but possibly complexified) pro-2-extension of K and let $\mathrm{Rad}_n^{\mathrm{res}}$ the corresponding radical $\{w \in L_n^\times,\ \sqrt[4\cdot 2^n]{w} \in M_n^{\mathrm{res}}\}$, where M_n^{res} is analogous to M_n for the restricted sense. We observe that in the restricted sense, we have the exact sequence [8, Theorem III.4.1.5] $0 \to (\mathbb{Z}/2\mathbb{Z})^d \longrightarrow \mathscr{T}_K^{\mathrm{res}} \longrightarrow \mathscr{T}_K \to 1$, then a dual exact sequence with radicals. As in [2], one may consider more general ray class fields and find results of annihilation with suitable Stickelberger or Solomon elements.

4. Stickelberger elements and cyclotomic numbers

4.1 General definitions

Let $f \geq 1$ be any modulus and let $\mathbb{Q}^f$ be the corresponding cyclotomic field $\mathbb{Q}(\mu_f)$.[1] Let L be a subfield of $\mathbb{Q}^f$.

(i) We define (where all Artin symbols are taken over $\mathbb{Q}$):

$$\mathscr{S}_{\mathbb{Q}^f} := -\sum_{a=1}^{f} \Big(\frac{a}{f} - \frac{1}{2}\Big) \cdot \Big(\frac{\mathbb{Q}^f}{a}\Big)^{-1}$$

and the restriction:

$$\mathscr{S}_L := \mathrm{N}_{\mathbb{Q}^f/L}(\mathscr{S}_{\mathbb{Q}^f}) := -\sum_{a=1}^{f} \Big(\frac{a}{f} - \frac{1}{2}\Big) \cdot \Big(\frac{L}{a}\Big)^{-1}$$

to L of $\mathscr{S}_{\mathbb{Q}^f}$, where a runs trough the integers $a \in [1, f]$ prime to f. In this case, one must precise the relation between f and the conductor f_L of L.

We know that the properties of annihilation of ideal classes need to multiply $\mathscr{S}_L$ by an element of the ideal annihilator of the group μ_f (or μ_{2f}), which is generated by f (or $2f$) and the multiplicators:

$$\delta_c := 1 - c \cdot \Big(\frac{\mathbb{Q}^f}{c}\Big)^{-1},$$

for c odd, prime to f. This shall give integral elements in the group algebra.

(ii) Then we define in the same way:

$$\eta_{\mathbb{Q}^f} := 1 - \zeta_f \ \ \& \ \ \eta_L := \mathrm{N}_{\mathbb{Q}^f/L}(1 - \zeta_f), \ f \neq 1,$$

where ζ_f is a primitive fth root of unity for which we assume the coherent definitions $\zeta_f^{m'} = \zeta_m$ if $f = m' \cdot m$.

It is well known that if f is not a prime power, then η_f is a unit, otherwise, $\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}}(1 - \zeta_f) = \ell$ if $f = \ell^r$, $\ell \geq 2$ prime, $r \geq 1$.

Definition 4.1. *Since* $\frac{f-a}{f} - \frac{1}{2} = -(\frac{a}{f} - \frac{1}{2})$, $\mathscr{S}_{\mathbb{Q}^f} = \mathscr{S}'_{\mathbb{Q}^f} \cdot (1 - s_\infty)$ *and* $\mathscr{S}_L = \mathscr{S}'_L \cdot (1 - s_\infty)$, *where* $s_\infty := \big(\frac{\mathbb{Q}^f}{-1}\big)$ *is the complex conjugation, and where:*

$$\mathscr{S}'_{\mathbb{Q}^f} := -\sum_{a=1}^{f/2} \Big(\frac{a}{f} - \frac{1}{2}\Big) \cdot \Big(\frac{\mathbb{Q}^f}{a}\Big)^{-1} \ \ \& \ \ \mathscr{S}'_L := -\sum_{a=1}^{f/2} \Big(\frac{a}{f} - \frac{1}{2}\Big) \cdot \Big(\frac{L}{a}\Big)^{-1}.$$

4.2 Norms of Stickelberger elements and cyclotomic numbers

Let $f \geq 1$ and $m \mid f$ be any modulus and let $\mathbb{Q}^f$ and $\mathbb{Q}^m \subseteq \mathbb{Q}^f$ be the corresponding cyclotomic fields. Let $\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}$ be the restriction map:

$$\mathbb{Q}[\mathrm{Gal}(\mathbb{Q}^f/\mathbb{Q})] \longrightarrow \mathbb{Q}[\mathrm{Gal}(\mathbb{Q}^m/\mathbb{Q})],$$

or the usual arithmetic norm in $\mathbb{Q}^f/\mathbb{Q}^m$. Consider as above:

$$\mathscr{S}_{\mathbb{Q}^f} := -\sum_{a=1}^{f} \Big(\frac{a}{f} - \frac{1}{2}\Big) \cdot \Big(\frac{\mathbb{Q}^f}{a}\Big)^{-1} \ \ \& \ \ \eta_{\mathbb{Q}^f} := 1 - \zeta_f \ (f \neq 1).$$

We have, respectively:

$$\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\mathscr{S}_{\mathbb{Q}^f}) = \prod_{\ell \mid f,\ \ell \nmid m} \Big(1 - \Big(\frac{\mathbb{Q}^m}{\ell}\Big)^{-1}\Big) \cdot \mathscr{S}_{\mathbb{Q}^m}, \tag{4.1}$$

$$\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\eta_{\mathbb{Q}^f}) = (\eta_{\mathbb{Q}^m})^{\prod_{\ell \mid f,\ \ell \nmid m} \left(1 - \left(\frac{\mathbb{Q}^m}{\ell}\right)^{-1}\right)} \ \text{ if } m \neq 1. \tag{4.2}$$

[1] Such modulus are conductors of the corresponding cyclotomic fields, except for an even integer not divisible by 4; but this point of view is essential to establish the functional properties of Stickelberger elements and cyclotomic numbers. So, if f is odd, we distinguish, by abuse, the notations $\mathbb{Q}^f$ and $\mathbb{Q}^{2f}$ despite their equality.

As we have explained in the previous footnote, if m is odd, then we have:

$$\mathrm{N}_{\mathbb{Q}^{2m}/\mathbb{Q}^m}(\mathscr{S}_{\mathbb{Q}^{2m}}) = \big(1-\big(\tfrac{\mathbb{Q}^m}{2}\big)^{-1}\big)\cdot \mathscr{S}_{\mathbb{Q}^m}, \quad \mathrm{N}_{\mathbb{Q}^{2m}/\mathbb{Q}^m}(\eta_{\mathbb{Q}^{2m}}) = \eta_{\mathbb{Q}^m}^{\big(1-\big(\frac{\mathbb{Q}^m}{2}\big)^{-1}\big)},$$

where the "norms" $\mathrm{N}_{\mathbb{Q}^{2m}/\mathbb{Q}^m}$ are of course the identity. For instance one verifies immediately that $\mathscr{S}_{\mathbb{Q}^6} = \frac{1}{3}(1-s_\infty)$ and $\mathscr{S}_{\mathbb{Q}^3} = \frac{1}{6}(1-s_\infty)$, but since 2 is inert in $\mathbb{Q}^3/\mathbb{Q}$, $\big(1-\big(\frac{\mathbb{Q}^3}{2}\big)^{-1}\big) = 1-s_\infty$ and one must compute $(1-s_\infty)\mathscr{S}_{\mathbb{Q}^3} = \frac{1}{6}(1-s_\infty)^2 = \frac{1}{3}(1-s_\infty)$ as expected. We have $\mathscr{S}_{\mathbb{Q}^2} = 0$ and $\mathscr{S}_{\mathbb{Q}^1} = -\frac{1}{2}$.

If L (imaginary or real), of conductor f, is an extension of k, of conductor $m \mid f$, let $\mathscr{S}_L := \mathrm{N}_{\mathbb{Q}^f/L}(\mathscr{S}_{\mathbb{Q}^f})$ and $\eta_L := \mathrm{N}_{\mathbb{Q}^f/L}(\eta_{\mathbb{Q}^f})$, then:

$$\mathrm{N}_{L/k}(\mathscr{S}_L) = \prod_{\ell\mid f,\ \ell\nmid m}\Big(1-\Big(\frac{k}{\ell}\Big)^{-1}\Big)\cdot \mathscr{S}_k,$$

$$\mathrm{N}_{L/k}(\mathscr{S}'_L) \equiv \prod_{\ell\mid f,\ \ell\nmid m}\Big(1-\Big(\frac{k}{\ell}\Big)^{-1}\Big)\cdot \mathscr{S}'_k \pmod{(1+s_\infty)\cdot\mathbb{Q}[G_k]},$$

$$\mathrm{N}_{L/k}(\eta_L) = (\eta_k)^{\prod_{\ell\mid f,\ \ell\nmid m}\big(1-\big(\frac{k}{\ell}\big)^{-1}\big)} \text{ if } m\neq 1 \text{ (i.e., } k\neq\mathbb{Q}).$$

If $f=\ell^r$, ℓ prime, $r\geq 1$, then $\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}}(\eta_{\mathbb{Q}^f}) = \ell$, otherwise $\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}}(\eta_{\mathbb{Q}^f}) = 1$.

This implies that $\mathrm{N}_{L/k}(\mathscr{S}_L)=0$ (resp. $\mathrm{N}_{L/k}(\eta_L)=1$) as soon as there exists a prime $\ell\mid f$, $\ell\nmid m$, totally split in k. In particular, if k is real, the formula is valid for the infinite place and $\mathrm{N}_{L/k}(\mathscr{S}_L)=0$ (of course, if $L\neq\mathbb{Q}$ is real, $S_L=0$).

For the classical proofs, we consider by induction the case $f=\ell\cdot m$, with ℓ prime and examine the two cases $\ell\mid m$ and $\ell\nmid m$; the case of Stickelberger elements been crucial for our purpose, we give again a proof (a similar reasoning will be detailed for the Theorem 7.2).

To simplify, put $\mathscr{S}_{\mathbb{Q}^f} =: \mathscr{S}_f$, $\mathscr{S}_{\mathbb{Q}^m} =: \mathscr{S}_m$, and consider:

$$\mathscr{S}_f = -\sum_{a=1}^{f}\Big(\frac{a}{f}-\frac{1}{2}\Big)\cdot\Big(\frac{\mathbb{Q}^f}{a}\Big)^{-1},$$

for $f=\ell\cdot m$, $\ell\nmid m$, where a runs trough the integers $a\in[1,f]$ prime to f.

Put $a=b+\lambda\cdot m$, $b\in[1,m]$, $\lambda\in[0,\ell-1]$; since a must be prime to f, b is automatically prime to m but we must exclude $\lambda_b^*\in[0,\ell-1]$ such that:

$$b+\lambda_b^*\cdot m = b'_\ell\cdot\ell,\ \ b'_\ell\in[1,m] \ (b'_\ell \text{ is necessarily prime to } m).$$

We then have:

$$\begin{aligned}
&\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\mathscr{S}_f)\\
&\quad= -\sum_{a=1}^{f}\Big(\frac{a}{f}-\frac{1}{2}\Big)\cdot\Big(\frac{\mathbb{Q}^m}{a}\Big)^{-1} = -\sum_{b,\lambda\neq\lambda_b^*}\Big(\frac{b+\lambda m}{\ell m}-\frac{1}{2}\Big)\cdot\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\\
&\quad= -\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\sum_{\lambda\neq\lambda_b^*}\Big(\frac{b}{\ell m}+\frac{\lambda}{\ell}-\frac{1}{2}\Big)\\
&\quad= -\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\Big(\frac{\ell-1}{\ell}\frac{b}{m}-\frac{\ell-1}{2}\Big) - \sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{1}{\ell}\Big(\frac{\ell(\ell-1)}{2}-\lambda_b^*\Big)\\
&\quad= -\Big(1-\frac{1}{\ell}\Big)\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{b}{m}+\frac{1}{\ell}\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\lambda_b^*.
\end{aligned}$$

Since the correspondence $b\mapsto b'_\ell$ is bijective on the set of elements prime to m in $[1,m]$, one has, with $\lambda_b^* = \frac{b'_\ell\cdot\ell-b}{m}$ and $\Big(\frac{\mathbb{Q}^m}{b}\Big)=\Big(\frac{\mathbb{Q}^m}{b'_\ell}\Big)\Big(\frac{\mathbb{Q}^m}{\ell}\Big)$:

$$\begin{aligned}
\frac{1}{\ell}\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\lambda_b^* &= \sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{b'_\ell}{m}-\frac{1}{\ell}\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{b}{m}\\
&= \Big(\frac{\mathbb{Q}^m}{\ell}\Big)^{-1}\sum_b\Big(\frac{\mathbb{Q}^m}{b'_\ell}\Big)^{-1}\frac{b'_\ell}{m}-\frac{1}{\ell}\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{b}{m}\\
&= \Big(\Big(\frac{\mathbb{Q}^m}{\ell}\Big)^{-1}-\frac{1}{\ell}\Big)\cdot\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{b}{m}.
\end{aligned}$$

Thus we obtain:

$$\begin{aligned}\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\mathscr{S}_f) &= -\Big(1-\frac{1}{\ell}\Big)\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{b}{m} + \Big(\Big(\frac{\mathbb{Q}^m}{\ell}\Big)^{-1}-\frac{1}{\ell}\Big)\cdot\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{b}{m}\\ &= -\Big(1-\Big(\frac{\mathbb{Q}^m}{\ell}\Big)^{-1}\Big)\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{b}{m}.\end{aligned}$$

But $\frac{1}{2}\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\Big(1-\Big(\frac{\mathbb{Q}^m}{\ell}\Big)^{-1}\Big)=0$; so replacing $\frac{b}{m}$ by $\frac{b}{m}-\frac{1}{2}$ we get:

$$\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\mathscr{S}_f) = \Big(1-\Big(\frac{\mathbb{Q}^m}{\ell}\Big)^{-1}\Big)\cdot\mathscr{S}_m.$$

Then it is easy to compute that if $\ell \mid m$, any $\lambda\in[0,\ell-1]$ is suitable, giving:

$$\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\mathscr{S}_f) = \mathscr{S}_m.$$

The case of cyclotomic elements η_f is exactly the same, replacing the additive setting by the multiplicative one.

4.3 Multiplicators of Stickelberger elements

The conductor of L_n, $n\ge n_0$, is $f_{L_n}=\mathrm{l.c.m.}\,(f_K,qp^n)$ (Lemma 3.1). So in general $f_{L_n}=qp^n\cdot f'$ with $p\nmid f'$, except if f_K is divisible by a large power of p in which case one must take n large enough in the practical computations (write $f_K=qp^{n_0+r}f'$, $r\ge 0$, and take $n\ge n_0+r$). In some formulas we shall abbreviate f_{L_n} by f_n.

Let c be an (odd) integer, prime to f_n, and let:

$$\mathscr{S}_{L_n}(c) := \Big(1-c\Big(\frac{L_n}{c}\Big)^{-1}\Big)\cdot\mathscr{S}_{L_n}. \tag{4.3}$$

Then $\mathscr{S}_{L_n}(c)\in\mathbb{Z}[G_n]$ as we have explain; indeed, we have:

$$\mathscr{S}_{L_n}(c) = \frac{-1}{f_n}\sum_a\Big[a\Big(\frac{L_n}{a}\Big)^{-1}-ac\Big(\frac{L_n}{a}\Big)^{-1}\Big(\frac{L_n}{c}\Big)^{-1}\Big]+\frac{1-c}{2}\sum_a\Big(\frac{L_n}{a}\Big)^{-1}.$$

Let $a'_c\in[1,f_n]$ be the unique integer such that $a'_c\cdot c\equiv a \pmod{f_n}$ and put $a'_c\cdot c = a+\lambda_a^n(c)f_n$, $\lambda_a^n(c)\in\mathbb{Z}$; then, using the bijection $a\mapsto a'_c$ in the second summation and the fact that $\Big(\frac{L_n}{a'_c}\Big)\Big(\frac{L_n}{c}\Big)=\Big(\frac{L_n}{a}\Big)$, this yields:

$$\begin{aligned}\mathscr{S}_{L_n}(c) &= \frac{-1}{f_n}\Big[\sum_a a\Big(\frac{L_n}{a}\Big)^{-1}-\sum_a a'_c\cdot c\Big(\frac{L_n}{a'_c}\Big)^{-1}\Big(\frac{L_n}{c}\Big)^{-1}\Big]+\frac{1-c}{2}\sum_a\Big(\frac{L_n}{a}\Big)^{-1}\\ &= \frac{-1}{f_n}\sum_a\Big[a-a'_c\cdot c\Big]\Big(\frac{L_n}{a}\Big)^{-1}+\frac{1-c}{2}\sum_a\Big(\frac{L_n}{a}\Big)^{-1}\\ &= \sum_a\Big[\lambda_a^n(c)+\frac{1-c}{2}\Big]\Big(\frac{L_n}{a}\Big)^{-1}\in\mathbb{Z}[G_n].\end{aligned}$$

Lemma 4.2. *We have the relations $\lambda_{f_n-a}^n(c)+\frac{1-c}{2}=-\big(\lambda_a^n(c)+\frac{1-c}{2}\big)$ for all $a\in[1,f_n]$ prime to f_n. Then:*

$$\mathscr{S}'_{L_n}(c) := \sum_{a=1}^{f_n/2}\Big[\lambda_a^n(c)+\frac{1-c}{2}\Big]\Big(\frac{L_n}{a}\Big)^{-1}\in\mathbb{Z}[G_n] \tag{4.4}$$

is such that $\mathscr{S}_{L_n}(c)=\mathscr{S}'_{L_n}(c)\cdot(1-s_\infty)$, whence $\mathscr{S}_{L_n}(c)^*=\mathscr{S}'_{L_n}(c)^*\cdot(1+s_\infty)$.

Proof. By definition, the integer $(f_n-a)'_c$ is in $[1,f_n]$ and congruent modulo f_n to $(f_n-a)\,c^{-1}\equiv -ac^{-1}\equiv -a'_c \pmod{f_n}$; thus $(f_n-a)'_c=f_n-a'_c$ and

$$\lambda_{f_n-a}^n(c)=\frac{(f_n-a)'_c\,c-(f_n-a)}{f_n}=\frac{(f_n-a'_c)\,c-(f_n-a)}{f_n}=c-1-\lambda_a^n(c),$$

whence $\lambda_{f_n-a}^n(c)+\frac{1-c}{2}=-\big(\lambda_a^n(c)+\frac{1-c}{2}\big)$ and the result. □

The multiplicator $\delta_c := \big(1-c\big(\frac{L_n}{c}\big)^{-1}\big)$ has a great importance since the image of δ_c by the Spiegel involution is $\delta_c^* := 1-\big(\frac{L_n}{c}\big) \pmod{qp^n}$; the order of the Artin symbol of c shall be crucial.

5. Annihilation of radicals and Galois groups

5.1 Annihilation of $\mathrm{Rad}_n/L_n^{\times qp^n}$

We begin with the classical property of annihilation of class groups of imaginary abelian fields by modified Stickelberger elements $\mathscr{S}_{L_n}(c) = \delta_c \cdot \mathscr{S}_{L_n}$. Before let's give two technical lemmas. Recall that $\mathscr{S}_{L_n}(c) = \mathscr{S}'_{L_n}(c)\cdot(1-s_\infty)$ and that, from § 4.2, the $\mathscr{S}_{L_n}$, $\mathscr{S}_{L_n}(c)$ and $\mathscr{S}'_{L_n}(c) \pmod{(1+s_\infty)\mathbb{Z}[G_n]}$ form coherent families in $\varprojlim_{n\geq n_0+e} \mathbb{Q}[G_n]$ for the "norm" since f_{L_n} and $f_{L_{n+h}}$ are divisible by the same prime numbers for all $h \geq 0$.

Lemma 5.1. *Let $\zeta \in \mu_{qp^n}$, $n \geq n_0+e$. If $\zeta \in \mathrm{Rad}_n$ (or $\mathrm{Rad}_n^{\mathrm{res}}$ when $p=2$) then $\zeta = 1$.*

Proof. If $\zeta \neq 1$ with $L_n(\sqrt[qp^n]{\zeta}) \subseteq M_n$ (or M_n^{res}), we would have $L_n(\sqrt[qp^n]{\zeta}) = L_{n+h}$, where $h \geq 1$ since $\sqrt[qp^n]{\zeta}$ is of order $\geq qp^{n+1}$ and since $\mu_{p^\infty} \cap L_n^\times = \mu_{qp^n}$, which is absurd because of the linear disjonction $L_{n+h} \cap M_n = L_n$ (or $L_{n+h} \cap M_n^{\mathrm{res}} = L_n$). □

Lemma 5.2. *Let $w_0 \in \mathrm{Rad}_n$ be real. Then $w_0^2 \in L_n^{\times qp^n}$.*

Proof. Since K is real, we know that $1-s_\infty$ annihilates the $(\mathbb{Z}/qp^n\mathbb{Z})[G_n]$-module $\mathrm{Gal}(M_n/L_n)$, thus $1+s_\infty$ annihilates $\mathrm{Rad}_n/L_n^{\times qp^n}$ and $w_0^{1+s_\infty} = w_0^2 \in L_n^{\times qp^n}$ (this does not work for the restricted sense since the minus part of $\mathscr{T}_K^{\mathrm{res}}$ is of order 2^d). □

Theorem 5.3. *Let p^e be the exponent of $\mathscr{T}_K := \mathrm{tor}_{\mathbb{Z}_p}\big(\mathscr{G}_{K,S}^{\mathrm{ab}}\big)$ (p-ramification in the ordinary sense). For $p=2$, let $2^{e^{\mathrm{res}}}$ be the exponent of $\mathscr{T}_K^{\mathrm{res}} := \mathrm{tor}_{\mathbb{Z}_2}\big(\mathscr{G}_{K,S}^{\mathrm{res}\,\mathrm{ab}}\big)$, where $\mathscr{G}_{K,S}^{\mathrm{res}}$ is the Galois group of the maximal S-ramified in the restricted sense (i.e., unramified outside 2 but complexified) pro-2-extension of K and let $\mathrm{Rad}_n^{\mathrm{res}}$ be the corresponding radical.*

(i) $p>2$. For all $n \geq n_0+e$, the $(\mathbb{Z}/qp^n\mathbb{Z})[G_n]$-module $\mathrm{Rad}_n/L_n^{\times qp^n}$ is annihilated by $\mathscr{S}'_{L_n}(c)$. Thus, $\mathscr{S}'_{L_n}(c)^*$ annihilates $\mathscr{T}_K$.

(ii) $p=2$, ordinary sense. The annihilation occurs with $2\,\mathscr{S}_{L_n}(c)$ and with $4\,\mathscr{S}'_{L_n}(c)$. Thus $2\,\mathscr{S}_{L_n}(c)^*$ and $4\,\mathscr{S}'_{L_n}(c)^*$ annihilate $\mathscr{T}_K$.

(iii) $p=2$, restricted sense. For all $n \geq n_0+e^{\mathrm{res}}$, the $(\mathbb{Z}/4\cdot 2^n\mathbb{Z})[G_n]$-module $\mathrm{Rad}_n^{\mathrm{res}}/L_n^{\times 4\cdot 2^n}$ is annihilated by $2\,\mathscr{S}_{L_n}(c)$; thus $2\,\mathscr{S}_{L_n}(c)^*$ annihilates $\mathscr{T}_K^{\mathrm{res}}$.

Proof. Let $w \in \mathrm{Rad}_n$; since $L_n(\sqrt[qp^n]{w})/L_n$ is p-ramified, $(w) = \mathfrak{a}^{qp^n}\cdot\mathfrak{b}$ where $\mathfrak{a}$ is an ideal of L_n, prime to p, and $\mathfrak{b}$ is a product of prime ideals $\mathfrak{p}_n$ of L_n dividing p. Let $\mathfrak{p}_n \mid \mathfrak{b}$ and consider $\mathfrak{p}_n^{\mathscr{S}_{L_n}(c)}$; one can replace $\mathscr{S}_{L_n}(c)$ by its restriction to the decomposition field k (possibly $k=\mathbb{Q}$) of p in the abelian extension $L_n/\mathbb{Q}$, which gives rise to the Euler factor $1-\left(\frac{k}{p}\right)^{-1}$ since k, of conductor prime to p, is strictly contained in L_n of conductor qp^nf' for $n \geq n_0+e$; so this factor is 0 and $\mathfrak{b}^{\mathscr{S}_{L_n}(c)} = 1$.

From the principality of the ideal $\mathfrak{a}^{\mathscr{S}_{L_n}(c)}$ (Stickelberger's theorem) there exists $\alpha_n \in L_n^\times$ and a unit ε_n of L_n such that:

$$w^{\mathscr{S}_{L_n}(c)} = \alpha_n^{qp^n}\cdot\varepsilon_n. \tag{5.1}$$

We see that $\varepsilon_n^{1+s_\infty}$ is the qp^nth power of a unit of L_n: consider $\varepsilon_n^{1+s_\infty}$ in (5.1) with the fact that $\mathscr{S}_{L_n}(c) = \mathscr{S}'_{L_n}(c)(1-s_\infty)$. Since the $\mathbb{Z}$-rank of the groups of units of L_n and L_n^+ (the maximal real subfield of L_n) are equal, a power ε_n^N of ε_n is a real unit; so $\varepsilon_n^{1-s_\infty}$ is a torsion element and $\varepsilon_n^2 = \varepsilon_n^{1+s_\infty}\varepsilon_n^{1-s_\infty}$ is equal, up to a qp^nth power, to a p-torsion element of the form $\zeta' \in \mathrm{Rad}_n$. Thus $\zeta' = 1$ (Lemma 5.1) and $\varepsilon_n^2 \in L_n^{\times qp^n}$.

(i) Case $p \neq 2$. We deduce from the above that $\varepsilon_n \in L_n^{\times p^{n+1}}$. We have $w^{\mathscr{S}'_{L_n}(c)(1-s_\infty)} = \beta_n^{p^{n+1}}$; but $\beta_n^{1+s_\infty} = 1$ (the property is also true for $p=2$ since the result is a totally positive root of unity in L_n^+, but the proof only works taking the square of the relation (5.1) using ε_n^2), and there exists $\gamma_n \in L_n^\times$ such that $\beta_n = \gamma_n^{1-s_\infty}$, and $w^{\mathscr{S}'_{L_n}(c)}\cdot\gamma_n^{-p^{n+1}} = w_0$, a real number in the radical, thus a p^{n+1}th power (Lemma 5.2) (as above, the proof for $p=2$ only works taking once again the square of this relation to get w_0^2). Other proof for any $p \geq 2$: since $\mathscr{T}_K$ is annihilated by $1-s_\infty$, $\mathrm{Rad}_n/L_n^{\times qp^n}$ is annihilated by $1+s_\infty$, thus $w^{1-s_\infty} \in w^2\cdot L_n^{\times qp^n}$ for all $w \in \mathrm{Rad}_n$, and $w^{\mathscr{S}_{L_n}(c)} = w^{2\mathscr{S}'_{L_n}(c)}$ up to $L_n^{\times qp^n}$.

(ii) Case $p=2$ in the ordinary sense (so $L_n^+ = K_n$). The result is obvious taking the square in the previous computations giving ε_n^2 instead of ε_n for the annihilation with $2\,\mathscr{S}_{L_n}(c)$, then w_0^2 for the annihilation with $4\,\mathscr{S}'_{L_n}(c)$.

(iii) Case $p=2$ in the restricted sense. The proof is in fact contained in the same relation $(w) = \mathfrak{a}^{4\cdot 2^n}\cdot\mathfrak{b}$, for all $w \in \mathrm{Rad}_n^{\mathrm{res}}$, where $\mathfrak{a}$ is an ideal of L_n, prime to 2, and $\mathfrak{b}$ is a product of prime ideals $\mathfrak{p}_n$ of L_n dividing 2, then the relation (5.1), $n \geq n_0+e$. □

5.2 Computation of $\mathscr{S}_{L_n}(c)^*$ or $\mathscr{S}'_{L_n}(c)^*$ – Annihilation of $\mathscr{T}_K$

From the expressions (4.3) and (4.4) of $\mathscr{S}_{L_n}(c)$, the image by the Spiegel involution is:

$$\mathscr{S}_{L_n}(c)^* \equiv \sum_{a=1}^{f_n} \Big[\lambda_a^n(c) + \frac{1-c}{2}\Big]\, a^{-1}\Big(\frac{L_n}{a}\Big) \pmod{qp^n},$$

which defines a coherent familly $(\mathscr{S}_{L_n}(c)^*)_n \in \varprojlim_{n\geq n_0+e} \mathbb{Z}/qp^n\mathbb{Z}[G_n]$ of annihilators of the Galois groups $\mathrm{Gal}(M_n/L_n) \simeq \mathscr{T}_K$. In the case $p \neq 2$, one may use equivalently $\mathscr{S}'_{L_n}(c)^*$ with the half summation.

Since the operation of $\mathrm{Gal}(L_n/K)$ on $\mathrm{Gal}(M_n/L_n)$ is trivial, by restriction of $\mathscr{S}_{L_n}(c)^*$ to K (see Lemma 3.3), one obtains a coherent familly of annihilators of $\mathscr{T}_K$ denoted $(\mathscr{A}_{K,n}(c))_n \in \varprojlim_{n\geq n_0+e} \mathbb{Z}/qp^n\mathbb{Z}[G_K]$, whose p-adic limit:

$$\mathscr{A}_K(c) := \lim_{n\to\infty} \mathscr{A}_{K,n}(c) = \lim_{n\to\infty} \sum_{a=1}^{f_n} \Big[\lambda_a^n(c) + \frac{1-c}{2}\Big]\, a^{-1}\Big(\frac{K}{a}\Big) \in \mathbb{Z}_p[G_K]$$

is a canonical annihilator of $\mathscr{T}_K$ that we shall link to p-adic L-functions; of course, it is sufficient to know its coefficients modulo the exponent p^e of $\mathscr{T}_K$ and in a programming point of view, the element $\mathscr{A}_{K,n_0+e}(c)$ annihilates $\mathscr{T}_K$, knowing that [10, Program I, § 3.2] gives the group structure of $\mathscr{T}_K$.

Remark 5.4. *Let* $\alpha_{L_n} := \sum_{a=1}^{f_n} a^{-1}\Big(\frac{L_n}{a}\Big) \equiv \Big[\sum_{a=1}^{f_n}\Big(\frac{L_n}{a}\Big)^{-1}\Big]^*$*; we have:*

$$\alpha_{L_n} := \sum_{a=1}^{f_n/2} a^{-1}\Big(\frac{L_n}{a}\Big) + (f_n-a)^{-1}\Big(\frac{L_n}{f_n-a}\Big) \equiv \sum_{a=1}^{f_n/2} a^{-1}\Big(\frac{L_n}{a}\Big)(1-s_\infty) \pmod{f_n}$$

which annihilates $\mathscr{T}_K$ and is such that $\mathrm{N}_{L_n/K}(\alpha_{L_n}) \equiv 0 \pmod{qp^n}$ since K is real. We shall neglect such expressions and use the symbol $\,\widetilde{\equiv}\,$, where $A \,\widetilde{\equiv}\, B \pmod{p^{n+1}}$ will mean $A = B + \mu\cdot p^{n+1} + \nu\cdot\sum_{a=1}^{f_n} a^{-1}\Big(\frac{K}{a}\Big)$, in the group algebra $\mathbb{Z}_p[G_K]$, μ,ν in $\mathbb{Z}_p$ (we put the modulus p^{n+1} instead of qp^n to cover, subsequently, the case $p=2$; moreover, p^{n+1} annihilates $\mathscr{T}_K$ since $n \geq n_0+e$). By abuse, we still denote $\mathscr{A}_K(c) = \lim_{n\to\infty}\sum_{a=1}^{f_n}\lambda_a^n(c)\,a^{-1}\big(\frac{K}{a}\big)$.

Thus, we have obtained:

Theorem 5.5. *Let c be any integer prime to $2p$ and to the conductor of K.*
Assume $n \geq n_0+e$ and let f_n be the conductor of L_n; for all $a \in [1, f_n]$, prime to f_n, let a'_c be the unique integer in $[1, f_n]$ such that $a'_c\cdot c \equiv a \pmod{f_n}$ and put $a'_c\cdot c - a = \lambda_a^n(c)\,f_n$, $\lambda_a^n(c) \in \mathbb{Z}$.
Let $\mathscr{A}_{K,n}(c) := \sum_{a=1}^{f_n}\lambda_a^n(c)\,a^{-1}\Big(\frac{K}{a}\Big)$ and put $\mathscr{A}_{K,n}(c) = \mathscr{A}'_{K,n}(c)\cdot(1+s_\infty)$ where $\mathscr{A}'_{K,n}(c) = \sum_{a=1}^{f_n/2}\lambda_a^n(c)\,a^{-1}\Big(\frac{K}{a}\Big)$. Let $\mathscr{A}_K(c) := \lim_{n\to\infty}\Big[\sum_{a=1}^{f_n}\lambda_a^n(c)\,a^{-1}\Big(\frac{K}{a}\Big)\Big]$ and put $\mathscr{A}_K(c) =: \mathscr{A}'_K(c)\cdot(1+s_\infty)$.
(i) For $p\neq 2$, $\mathscr{A}'_K(c)$ annihilates the $\mathbb{Z}_p[G_K]$-module $\mathscr{T}_K$.
(ii) For $p=2$, the annihilation is true for $2\cdot\mathscr{A}_K(c)$ and $4\cdot\mathscr{A}'_K(c)$.

In practice, when the exponent p^e is known, one can use $n = n_0+e$ and the annihilators $\mathscr{A}_{K,n}(c)$ or $\mathscr{A}'_{K,n}(c)$, the annihilator limit $\mathscr{A}_K(c)$ being related to p-adic L-functions of primitive characters, thus giving the other approach than Solomon one, that we shall obtain in Theorem 9.4.

Remark 5.6. *We have proved in a seminar report (1977) that for $p=2$, $\mathscr{S}'_{L_n}(c)$ annihilates $\mathscr{C}\!\ell_{L_n}/\mathscr{C}\!\ell^0_{L_n}$, where $\mathscr{C}\!\ell_{L_n}$ is the 2-class group of L_n and where $\mathscr{C}\!\ell^0_{L_n}$ is generated by the classes of the the invariant ideals in L_n/K_n.*
This shows that some 2-classes may give an obstruction; but Rad_n *is particular as we have explained in Remark 3.4. In [15], Greither gives suitable statements about Stickelberger's theorem for $p=2$, using the main theorems of Iwasawa's theory about the orders $\frac{1}{2}L_2(1,\chi)$ of the isotypic components.*

From this, as well as some numerical experiments, and the roles of ε_n and w_0 in the above reasonings, we may propose the following conjecture:

Conjecture 5.7. *Let $p=2$ and let K be a real abelian number field linearly disjoint from the cyclotomic $\mathbb{Z}_2$-extension. Put $\mathscr{A}_K(c) = \mathscr{A}'_K(c)\cdot(1+s_\infty)$ (see formula of Theorem 5.5). Then $\mathscr{A}'_K(c)$ annihilates $\mathscr{T}_K$.*

If there exists, in the class of $\mathscr{A}'_K(c)$ modulo $\sum_{\sigma\in G_K}\sigma$, an element of the form $2\cdot\mathscr{A}''_K(c)$, $\mathscr{A}''_K(c) \in \mathbb{Z}_p[G_K]$, one may ask if $\mathscr{A}''_K(c)$ does annihilate $\mathscr{T}_K$. We shall give a counterexample for the annihilation of $\mathscr{T}_K$ by $\mathscr{A}''_K(c)$ (see § 6.5.5), but we ignore if this may be true under some assumptions.

5.3 Experiments for cyclic cubic fields with $p \equiv 1 \pmod 3$

To simplify we suppose f_K prime. The first part of the program gives a defining polynomial. A second part computes the p-adic valuation of $\#\mathscr{T}_K$ using [10, Program I, § 3.2] and gives $\mathscr{A}_K(c) = \Lambda_0 + \Lambda_1\sigma^{-1} + \Lambda_2\sigma^{-2}$ modulo a power of p, after the choice of c, prime to $2pf_K$, with an Artin symbol of order 3; in the program p^ex is the exponent p^e of $\mathscr{T}_K$ and fn the conductor of L_n. The parameter nt must be $>$ ex.

```
{p=7;nt=8;forprime(f=7,10^4,if(Mod(f,3)!=1,next);
for(bb=1,sqrt(4*f/27),if(vf==2 & Mod(bb,3)==0,next);A=4*f-27*bb^2;
if(issquare(A,&aa)==1,if(Mod(aa,3)==1,aa=-aa);
P=x^3+x^2+(1-f)/3*x+(f*(aa-3)+1)/27;K=bnfinit(P,1);Kpn=bnrinit(K,p^nt);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
h=component(component(component(K,8),1),2);L=List;ex=0;
i=component(matsize(Hpn),2);R=0;for(k=1,i-1,co=component(Hpn,i-k+1);
if(Mod(co,p)==0,R=R+1;val=valuation(co,p);if(val>ex,ex=val);
listinsert(L,p^val,1)));Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);if(vptor>1,S0=0;S1=0;S2=0;
pN=p*p^ex;nu=(f-1)/3;fn=pN*f;z=znprimroot(f);
zz=lift(z);t=lift(Mod((1-zz)/f,2*p));c=zz+t*f;
for(a=1,fn/2,if(gcd(a,fn)!=1,next);asurc=lift(a*Mod(c,fn)^-1);
lambda=(asurc*c-a)/fn;u=Mod(lambda*a^-1,pN);
a0=lift((a*z^0)^nu);a1=lift((a*z^2)^nu);a2=lift((a*z)^nu);
if(a0==1,S0=S0+u);if(a1==1,S1=S1+u);if(a2==1,S2=S2+u));
L0=lift(S0);L1=lift(S1);L2=lift(S2);
j=Mod(y,y^2+y+1);Y=L0+j*L1+j^2*L2;nj=valuation(norm(Y),p);
print(f," ",P," vptor=",vptor," T_K=",L," A= ",L0," ",L1," ",L2," ",nj))))}
```

Let's give a partial table for $p = 7$ and 13, in which vptor $:= v_p(\#\mathscr{T}_K)$ (examples limited to vptor ≥ 2), and nj $= v_p\big(\mathrm{N}_{\mathbb{Q}(j)/\mathbb{Q}}(\Lambda_0 + \Lambda_1 \cdot j + \Lambda_2 \cdot j^2)\big)$; one sees that, as expected, all the examples give nj $=$ vptor since $\mathscr{T}_K$ is a finite $\mathbb{Z}_7[j]$-module which may be decomposed with two 7-adic characters:

```
f     P                  vptor  T_K      coefficients   nj
313   x^3+x^2-104*x+371  2      [7,7]    [41, 41, 48]   2
577   x^3+x^2-192*x+171  2      [49]     [183, 17, 280] 2
823   x^3+x^2-274*x+61   3      [343]    [761, 419, 437] 3
883   x^3+x^2-294*x+1439 2      [7,7]    [14, 0, 35]    2
1051  x^3+x^2-350*x-2608 2      [49]     [4, 247, 309]  2
1117  x^3+x^2-372*x+2565 2      [7,7]    [7, 7, 42]     2
1213  x^3+x^2-404*x+629  2      [49]     [45, 313, 268] 2
1231  x^3+x^2-410*x-1003 2      [49]     [247, 73, 273] 2
1237  x^3+x^2-412*x+1741 2      [49]     [108, 336, 128] 2
1297  x^3+x^2-432*x-1345 2      [49]     [277, 62, 14]  2
1327  x^3+x^2-442*x-344  2      [49]     [217, 340, 251] 2
1381  x^3+x^2-460*x-1739 4      [343,7]  [1738, 2186, 2361] 4
1567  x^3+x^2-522*x-4759 2      [49]     [219, 137, 78] 2
(...)
2203  x^3+x^2-734*x+408  2      [7,7]    [28, 28, 35]   2
2251  x^3+x^2-750*x-1584 2      [49]     [191, 274, 151] 2
2557  x^3+x^2-852*x+9281 3      [49,7]   [235, 3, 286]  3
```

For $f = 33199$, $P = x^3 + x^2 - 11066x + 238541$, $\mathscr{T}_K \simeq \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$, $h = 14$, and the annihilator is equivalent, modulo $1+\sigma+\sigma^2$, to $A = \sigma - 2$.

For $f = 20857$, $P = x^3+x^2-6952x+210115$, $\mathscr{T}_K \simeq \mathbb{Z}/7^2\mathbb{Z}\times\mathbb{Z}/7^2\mathbb{Z}$, $h = 1$, and the annihilator is equivalent to $A = 7^2(\sigma-3)$ where $\sigma - 3$ is invertible modulo 7.

For $f = 1381$, $\mathscr{T}_K \simeq \mathbb{Z}/7^3\mathbb{Z}\times\mathbb{Z}/7\mathbb{Z}$, $h = 1$, $A = 1738+2186\,\sigma+2361\,\sigma^2$ is equivalent to $7\cdot(448+623\,\sigma)$ and $448+623\,\sigma$ operates on $\mathscr{T}_K^7 \simeq \mathbb{Z}/7^2\mathbb{Z}$ as $\sigma - 18$ modulo 7^2 where 18 is of order 3 modulo 7^2 as expected.

For $f = 39679$, $\mathscr{T}_K \simeq \mathbb{Z}/7^3\mathbb{Z}\times\mathbb{Z}/7\mathbb{Z}\times\mathbb{Z}/7\mathbb{Z}$, $h = 7$, and one finds the annihilator $A = 7^2(\sigma-4)$ where $\sigma - 4$ is not invertible ($\mathrm{N}_{\mathbb{Q}(j)/\mathbb{Q}}(j-4) = 21$).

For $p = 13$, the same program gives the following similar results:

```
f    P                 vptor  T_K       coefficients   nj
1033 x^3+x^2-344*x+1913 2    [169]     [311, 455, 919] 2
1459 x^3+x^2-486*x+2864 2    [13,13]   [101, 88, 153] 2
1483 x^3+x^2-494*x-2197 2    [169]     [911, 1868, 1628] 2
1543 x^3+x^2-514*x+4229 2    [169]     [1598, 603, 1866] 2
1747 x^3+x^2-582*x-4141 2    [169]     [1952, 505, 155] 2
3391 x^3+x^2-1130*x+14192 3 [169,13]  [803, 1765, 283] 3
4423 x^3+x^2-1474*x+10648 2 [169]     [52, 1213, 1888] 2
4933 x^3+x^2-1644*x-1827 2  [13,13]   [92, 79, 105] 2
5011 x^3+x^2-1670*x-4083 2  [169]     [602, 1673, 869] 2
5479 x^3+x^2-1826*x+13799 2 [13,13]   [93, 158, 28] 2
7321 x^3+x^2-2440*x-45824 2 [169]     [745, 409, 1546] 2
7963 x^3+x^2-2654*x+43944 2 [169]     [1805, 794, 860] 2
9319 x^3+x^2-3106*x-67649 2 [13,13]   [26, 52, 0]    2
```

6. Experiments and heuristics about the case $p = 2$

Conjecture 5.7 gives various possibilities of annihilation, depending on the choice of $\mathscr{A}_{K,n}(c)$, $\mathscr{A}'_{K,n}(c)$ or else, and of the degree of $K/\mathbb{Q}$, odd, even, or a 2th power. We shall give some illustrations with quadratic, quartic and cubic fields.

6.1 Quadratic fields

Although the order of $\mathscr{T}_K$ is known and given by $\frac{1}{2}L_2(1,\chi)$ (for $K \neq \mathbb{Q}(\sqrt{2})$), we give the computations for the quadratic fields K of conductor $f \geq 5$ with $\mathscr{A}'_{K,n}(c)$ ($a \in [1, f_n/2]$) instead of $\mathscr{A}_{K,n}(c)$ to test the conjecture; the computation of the Artin symbols is easily given by PARI with $\mathsf{kronecker(f,a)} = \pm 1$. The modulus $f_n = \mathrm{l.c.m.}\,(f_K, 4 \cdot 2^n)$ is computed exactely and we take $n = e + 2$.

From the annihilator $A' = a_0 + a_1 \cdot \sigma$ (in $(\mathsf{L}_0, \mathsf{L}_1)$), we deduce, modulo the norm, an equivalent annihilator denoted by abuse $A' = a_1 - a_0 \in \mathbb{Z}$.

One finds $A' \equiv 2 \cdot \#\mathscr{T}_K \pmod{2^{2+e}}$ for all $f \neq 8$ (only case with $K \cap \mathbb{Q}_\infty \neq \mathbb{Q}$) in this interval; then the class group is given (be careful to take nt large enought for the computation of the structure of $\mathscr{T}_K$):

```
{p=2;nt=18;bf=5;Bf=10^4;for(f=bf,Bf,v=valuation(f,2);M=f/2^v;
if(core(M)!=M,next);if((v==1||v>3)||(v==0 & Mod(M,4)!=1)||
(v==2 & Mod(M,4)==1),next);P=x^2-f;K=bnfinit(P,1);Kpn=bnrinit(K,p^nt);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
h=component(component(component(K,8),1),2);L=List;ex=0;
i=component(matsize(Hpn),2);for(k=1,i-1,co=component(Hpn,i-k+1);
if(Mod(co,p)==0,val=valuation(co,p);if(val>ex,ex=val);
listinsert(L,p^val,1)));Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);tor=p^vptor;S0=0;S1=0;w=valuation(f,p);
pN=p^2*p^ex;fn=pN*f/2^w;if(ex==0 & w==3,fn=p*fn);
for(cc=2,10^2,if(gcd(cc,p*f)!=1 || kronecker(f,cc)!=-1,next);c=cc;break);
for(a=1,fn/2,if(gcd(a,fn)!=1,next);asurc=lift(a*Mod(c,fn)^-1);
lambda=(asurc*c-a)/fn;u=Mod(lambda*a^-1,pN);
s=kronecker(f,a);if(s==1,S0=S0+u);if(s==-1,S1=S1+u));
L0=lift(S0);L1=lift(S1);A=L1-L0;if(A!=0,A=p^valuation(A,p));
print(f," P=",P," ",L0," ",L1," A=",A," tor=",tor," T_K=",L," Cl_K=",h))}

f_K=8   P=x^2-8   (1,0)      A'=1  tor=1   T_K=[]   Cl_K=[]
(...)
f_K=508 P=x^2-508 (223,479) A'=256 tor=128 T_K=[128] Cl_K=[]
(...)
f_K=1160 P=x^2-1160 (2,6) A'=4  tor=2   T_K=[2]  Cl_K=[2,2]
f_K=1164 P=x^2-1164 (12,4) A'=8  tor=4   T_K=[4]  Cl_K=[4]
(...)
f_K=1185 P=x^2-1185 (1640,1640) A'=0 tor=1024 T_K=[2,512] Cl_K=[2]
f_K=1189 P=x^2-1189 (2,6) A'=4  tor=2   T_K=[2]  Cl_K=[2]
```

```
(...)
f_K=1196 P=x^2-1196 (4,20) A'=16 tor=8  T_K=[8]  Cl_K=[2]
f_K=1201 P=x^2-1201 (7752,3656) A'=4096 tor=2048 T_K=[2048] Cl_K=[]
(...)
f_K=1209 P=x^2-1209 (4,4)  A'=0  tor=4   T_K=[2,2] Cl_K=[2]
(...)
f_K=1217 P=x^2-1217 (16,48) A'=32 tor=16 T_K=[16] Cl_K=[]
f_K=1221 P=x^2-1221 (8,8)  A'=0  tor=8   T_K=[2,4] Cl_K=[4]
(...)
f_K=1596 P=x^2-1596 (16,16) A'=0 tor=16 T_K=[8, 2] Cl_K=[4,2]
```

Remark 6.1. *(i) For $f = 1160$, one sees that ${}^{\#}\mathscr{C}\ell_K^{\infty} = \frac{1}{2}{}^{\#}\mathscr{C}\ell_K$ (indeed, -1 is norm in $K/\mathbb{Q}$, cf. (2.1)).*

(ii) It seems that for all the conductors, A' is of the form $2^h\,(1+\sigma)$ up to a 2-adic unit, where $h \geq 0$ takes any value and can exceed the exponent.

(iii) For f prime, the annihilator of $\mathscr{T}_K$, given by the Theorem 9.4, or by any Solomon's type element, is related to its order:

$$\frac{1}{2}L_2(1,\chi) \sim \frac{1}{2}\sum_{a=1}^{f}\chi(a)\cdot\log(1-\zeta_f^a) = \frac{1}{2}\cdot\left[\log(\eta_K) - \log(\eta_K^{\sigma})\right],$$

where $\eta_K = \mathrm{N}_{\mathbb{Q}^f/K}(1-\zeta_f)$ (here the character χ is primitive modulo f since $K = k_\chi$). The following program verifies (at least for these kind of prime conductors with trivial class group) that we have $\eta_K \cdot \varepsilon = \pm\sqrt{f}$, where ε is the fundamental unit of K or its inverse (the program gives in N_0 and N_1 the conjugates of η_K and gives ε in E):

```
{f=1201;N0=1;N1=1;X=exp(2*I*Pi/f);z=znprimroot(f);E=quadunit(f);zk=1;
for(k=1,(f-1)/2,zk=zk*z^2;N0=N0*(1-X^lift(zk));N1=N1*(1-X^lift(zk*z)));
print(N0*E," ",N1/E)}
```

We find $N_0\,\varepsilon = N_1\,\varepsilon^{-1} \approx 34.65544690 = \sqrt{1201}$, which implies that:

$$\frac{1}{2}L_2(1,\chi) \sim \frac{1}{2}\,(2\log(\varepsilon)) = \log(\varepsilon).$$

A direct computation gives $\log(\varepsilon) \sim 2^{12}$ as expected since ${}^{\#}\mathscr{T}_K = 2^{11}$ with ${}^{\#}\mathscr{R}_K \sim 2^{10}$ [9, Proposition 5.2] and ${}^{\#}\mathscr{W}_K = 2$ since 2 splits in K. Same kind of result with $f = 1217$.

6.2 A familly of cyclic quartic fields of composite conductor

We consider a conductor f product of two prime numbers q_1 and q_2 such that $q_1 - 1 \equiv 2 \pmod 4$ and $q_2 - 1 \equiv 0 \pmod 8$. So there exists only one real cyclic quartic field K of conductor f which is found eliminating the imaginary and non-cyclic fields; the quadratic subfield of K is $k = \mathbb{Q}(\sqrt{q_2})$. The program is written with $\mathscr{A}'_{K,n}(c)$ and gives all information for k and K.

The following result may help to precise the annihilations (see [14, Theorem 2] or [8, Theorem IV.3.3, Exercise IV.3.3.1]):

Lemma 6.2. *Let k be a totally real number field and let K/k be a Galois p-extension with Galois group g of order p^r. Then we have the fixed point formula: ${}^{\#}\mathscr{T}_K^{g} = {}^{\#}\mathscr{T}_k \cdot p^h$, where ($\mathfrak{l} \nmid p$ being the ramified primes in K/k):*

$$h := \min\left(n_0 + r;\ldots,\nu_{\mathfrak{l}} + \varphi_{\mathfrak{l}} + \gamma_{\mathfrak{l}},\ldots\right) - (n_0 + r) + \sum_{\mathfrak{l}\nmid p} e_{\mathfrak{l}},$$

with:

- $p^{\nu_{\mathfrak{l}}} :=$ *p-part of $q^{-1}\log(\ell)$, where $\mathfrak{l} \cap \mathbb{Z} =: \ell\mathbb{Z}$,*
- $p^{\varphi_{\mathfrak{l}}} :=$ *p-part of the residue degree of ℓ in $K/\mathbb{Q}$,*
- $p^{\gamma_{\mathfrak{l}}} :=$ *p-part of the number of prime ideals $\mathfrak{L} \mid \mathfrak{l}$ in K/k,*
- $p^{e_{\mathfrak{l}}} :=$ *p-part of the ramification index of $\mathfrak{l}$ in K/k.*

In such famillies of cyclic quartic fields, $h = \sum_{\mathfrak{l}\nmid p} e_{\mathfrak{l}}$.

6.2.1 The program

In the present familly, $h = 2$ (resp. 3) if q is inert (resp. splits) in $k/\mathbb{Q}$.

```
{p=2;nt=18;forprime(qq=17,100,if(Mod(qq,8)!=1,next);Pk=x^2-qq;
k=bnfinit(Pk,1);kpn=bnrinit(k,p^nt);Hkpn=component(component(kpn,5),2);
Lk=List;i=component(matsize(Hkpn),2);
for(j=1,i-1,C=component(Hkpn,i-j+1);if(Mod(C,p)==0,
listinsert(Lk,p^valuation(C,p),1)));forprime(q=5,100,
if(valuation(q-1,2)!=2,next);f=q*qq;Q=polsubcyclo(f,4);
for(j=1,7,P=component(Q,j);K=bnfinit(P,1);C7=component(K,7);
S=component(C7,2);D=component(C7,3);
if(Mod(D,f)!=0 || S!=[4,0] || component(polgalois(P),2)!=-1,next);break);
Cl=component(component(component(K,8),1),2);Kpn=bnrinit(K,p^nt);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn=component(component(Kpn,5),2);L=List;ex=0;
i=component(matsize(Hpn),2);for(k=1,i-1,co=component(Hpn,i-k+1);
if(Mod(co,p)==0,val=valuation(co,p);if(val>ex,ex=val);
listinsert(L,p^val,1)));Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);if(vptor>0,S0=0;S1=0;S2=0;S3=0;
pN=p^2*p^ex;fn=pN*f;dqq=(qq-1)/4;dq=(q-1)/2;
z=znprimroot(q);zz=znprimroot(qq);for(cc=3,f,if(gcd(cc,p*f)!=1,next);
cz=lift((cc*z)^dq);czz=lift((cc*zz)^dqq);if(cz!=1 || czz!=1,next);
c=cc;break);cm1=Mod(c,fn)^-1;for(a=1,fn/2,if(gcd(a,fn)!=1,next);
asurc=lift(a*cm1);lambda=(asurc*c-a)/fn;u=Mod(lambda*a^-1,pN);
aqq0=lift((a*zz^0)^dqq);aqq1=lift((a*zz^1)^dqq);
aqq2=lift((a*zz^2)^dqq);aqq3=lift((a*zz^3)^dqq);
aq0=lift((a*z^0)^dq);aq1=lift((a*z^1)^dq);
if(aqq0==1 & aq0==1,S0=S0+u);if(aqq0==1 & aq1==1,S2=S2+u);
if(aqq1==1 & aq0==1,S1=S1+u);if(aqq1==1 & aq1==1,S3=S3+u);
if(aqq2==1 & aq0==1,S2=S2+u);if(aqq2==1 & aq1==1,S0=S0+u);
if(aqq3==1 & aq0==1,S3=S3+u);if(aqq3==1 & aq1==1,S1=S1+u));
L0=lift(S0);L1=lift(S1);L2=lift(S2);L3=lift(S3);Y=Mod(y,y^2+1);
ni=L0+Y*L1+Y^2*L2+Y^3*L3;Nni=valuation(norm(ni),2));V0=1;V1=1;V2=1;V3=1;
if(L0!=0,V0=2^valuation(L0,2));if(L1!=0,V1=2^valuation(L1,2));
if(L2!=0,V2=2^valuation(L2,2));if(L3!=0,V3=2^valuation(L3,2));
print();F=component(factor(f),1);
print("f=",F," Cl=",Cl," P=",P," tor=",2^vptor," Nni=",2^Nni);
print("A=",V0,"*",L0/V0," ",V1,"*",L1/V1," ",V2,"*",L2/V2," ",V3,"*",L3/V3);
print("q=",q," qq=",qq," T_k=",Lk," T_K=",L)))}

f=[5, 17] h=[2] P=x^4-x^3-23*x^2+x+86 tor=16 Nni=16
 A=[2*5, 4*1, 2*1, 1*0] q=5 qq=17 T_k=List([2]) T_K=[4, 2, 2]
f=[13, 17] h=[2] P=x^4-x^3-57*x^2+x+664 tor=32 Nni=32
 A=[2*1, 2*1, 2*3, 2*3] q=13 qq=17 T_k=[2] T_K=[4, 4, 2]
f=[17, 29] h=[2] P=x^4-x^3-125*x^2+x+3452 tor=16 Nni=16
 A=[4*3, 2*1, 1*0, 2*1] q=29 qq=17 T_k=[2] T_K=[4, 2, 2]
f=[17, 37] h=[10] P=x^4-x^3-159*x^2+x+5662 tor=16 Nni=16
 A=[4*1, 2*3, 8*1, 2*7] q=37 qq=17 T_k=[2] T_K=[4, 2, 2]
f=[17, 53] h=[2, 2] P=x^4-x^3-227*x^2+x+11714 tor=32 Nni=32
 A=[2*1, 2*5, 2*3, 2*7] q=53 qq=17 T_k=[2] T_K=[4, 4, 2]
f=[17, 61] h=[2] P=x^4-x^3-261*x^2+x+15556 tor=16 Nni=16
 A=[2*1, 8*1, 2*5, 4*3] q=61 qq=17 T_k=[2] T_K=[4, 2, 2]
f=[5, 41] h=[2] P=x^4-x^3-56*x^2-100*x+160 tor=256 Nni=32
 A=[2*13, 2*45, 2*59, 2*27] q=5 qq=41 T_k=[16] T_K=[32, 4, 2]
f=[13, 41] h=[2] P=x^4-x^3-138*x^2-264*x+1472 tor=256 Nni=32
 A=[2*13, 2*27, 2*51, 2*5] q=13 qq=41 T_k=[16] T_K=[32, 4, 2]
f=[29, 41] h=[2] P=x^4-x^3-302*x^2-592*x+8032 tor=1024 Nni=128
 A=[4*21, 4*5, 4*15, 4*15] q=29 qq=41 T_k=[16] T_K=[32, 8, 4]
f=[37, 41] h=[2] P=x^4-x^3-384*x^2-756*x+13280 tor=256 Nni=32
 A=[2*57, 2*7, 2*47, 2*33] q=37 qq=41 T_k=[16] T_K=[32, 4, 2]
f=[41, 53] h=[2] P=x^4-x^3-548*x^2-1084*x+27712 tor=512 Nni=64
 A=[4*23, 8*15, 4*5, 8*7] q=53 qq=41 T_k=[16] T_K=[32, 4, 4]
f=[41, 61] h=[2, 2] P=x^4-x^3-630*x^2-1248*x+36896 tor=8192 Nni=1024
```

```
 A=[32*3, 16*7, 1*0, 16*7] q=61 qq=41 T_k=[16] T_K=[32, 16, 16]
f=[5, 73] h=[2] P=x^4-x^3-100*x^2+187*x+1389 tor=8 Nni=8
 A=[1*5, 1*9, 1*15, 1*3] q=5 qq=73 T_k=[2] T_K=[4, 2]
f=[13, 73] h=[2] P=x^4-x^3-246*x^2+479*x+11171 tor=8 Nni=8
 A=[1*7, 1*13, 1*13, 1*15] q=13 qq=73 T_k=[2] T_K=[4, 2]
f=[29, 73] h=[2] P=x^4-x^3-538*x^2+1063*x+58767 tor=8 Nni=8
 A=[1*5, 1*7, 1*15, 1*5] q=29 qq=73 T_k=[2] T_K=[4, 2]
f=[37, 73] h=[2] P=x^4-x^3-684*x^2+1355*x+96581 tor=128 Nni=128
 A=[1*0, 16*1, 8*1, 8*1] q=37 qq=73 T_k=[2] T_K=[8, 8, 2]
f=[53, 73] h=[10] P=x^4-x^3-976*x^2+1939*x+200241 tor=8 Nni=8
 A=[1*15, 1*15, 1*5, 1*13] q=53 qq=73 T_k=[2] T_K=[4, 2]
f=[61, 73] h=[2] P=x^4-x^3-1122*x^2+2231*x+266087 tor=16 Nni=16
 A=[8*1, 2*3, 1*0, 2*1] q=61 qq=73 T_k=[2] T_K=[4, 2, 2]
f=[5, 89] h=[2, 2] P=x^4-x^3-122*x^2-217*x+1699 tor=16 Nni=16
 A=[1*0, 2*1, 8*1, 2*3] q=5 qq=89 T_k=[2] T_K=[4, 2, 2]
f=[13, 89] h=[2] P=x^4-x^3-300*x^2-573*x+13625 tor=8 Nni=8
 A=[1*1, 1*7, 1*11, 1*13] q=13 qq=89 T_k=[2] T_K=[4, 2]
f=[29, 89] h=[2] P=x^4-x^3-656*x^2-1285*x+71653 tor=8 Nni=8
 A=[1*11, 1*5, 1*1, 1*15] q=29 qq=89 T_k=[2] T_K=[4, 2]
f=[37, 89] h=[2] P=x^4-x^3-834*x^2-1641*x+117755 tor=8 Nni=8
 A=[1*9, 1*15, 1*3, 1*5] q=37 qq=89 T_k=[2] T_K=[4, 2]
f=[53, 89] h=[2] P=x^4-x^3-1190*x^2-2353*x+244135 tor=16 Nni=16
 A=[4*1, 2*5, 4*1, 2*7] q=53 qq=89 T_k=[2] T_K=[4, 2, 2]
f=[61, 89] h=[2] P=x^4-x^3-1368*x^2-2709*x+324413 tor=8 Nni=8
 A=[1*1, 1*9, 1*11, 1*11] q=61 qq=89 T_k=[2] T_K=[4, 2]
f=[5, 97] h=[2] P=x^4-x^3-133*x^2-479*x+36 tor=16 Nni=16
 A=[2*5, 8*1, 2*1, 4*3] q=5 qq=97 T_k=[2] T_K=[4, 2, 2]
f=[13, 97] h=[10] P=x^4-x^3-327*x^2-1255*x+2558 tor=16 Nni=16
 A=[4*1, 2*7, 8*1, 2*3] q=13 qq=97 T_k=[2] T_K=[4, 2, 2]
f=[29, 97] h=[2] P=x^4-x^3-715*x^2-2807*x+16914 tor=16 Nni=16
 A=[2*3, 8*1, 2*3, 4*3] q=29 qq=97 T_k=[2] T_K=[4, 2, 2]
f=[37, 97] h=[2] P=x^4-x^3-909*x^2-3583*x+28748 tor=16 Nni=16
 A=[4*3, 2*7, 1*0, 2*3] q=37 qq=97 T_k=[2] T_K=[4, 2, 2]
f=[53, 97] h=[2] P=x^4-x^3-1297*x^2-5135*x+61728 tor=64 Nni=64
 A=[8*3, 4*7, 16*1, 4*7] q=53 qq=97 T_k=[2] T_K=[8, 4, 2]
f=[61, 97] h=[2] P=x^4-x^3-1491*x^2-5911*x+82874 tor=32 Nni=32
 A=[2*7, 2*5, 2*5, 2*7] q=61 qq=97 T_k=[2] T_K=[4, 4, 2]
```

6.2.2 The case $f = 5\cdot 73$

One may try to find a contradiction to Conjecture 5.7 with the $\mathscr{A}'_{K,n}(c)$ given by the above data. One sees that $\frac{1}{2}\mathscr{A}'_{K,n}(c)$ is not always in $\mathbb{Z}[G_K]$, but modulo the norm we have an annihilator of the form $2\cdot\mathscr{A}''_{K,n}(c)$, and similarly we may ask under what condition $\mathscr{A}''_{K,n}(c)$ annihilates $\mathscr{T}_K$.

For $f = 5\cdot 73$, $P = x^4 - x^3 - 100x^2 + 187x + 1389$, for which we have $\mathscr{T}_K \simeq \mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, $\mathscr{T}_k \simeq \mathbb{Z}/2\mathbb{Z}$, $\mathsf{Cl} = 2$, $\mathscr{A}'_{K,n}(c) = 5 + 9\sigma + 15\sigma^2 + 3\sigma^3$, giving:

$$\mathscr{A}''_{K,n}(c) = \tfrac{1}{2}\big[5 + 9\sigma + 15\sigma^2 + 3\sigma^3 - 3\,(1+\sigma+\sigma^2+\sigma^3)\big] \equiv 1 - \sigma + 2\sigma^2 \pmod 4$$

without obvious contradiction since $\#\mathscr{T}_K^{g} = 8$ (i.e., $\mathscr{T}_K^{g} = \mathscr{T}_K$) and $\#\mathscr{T}_K^{G_K} = 4$ (Lemma 6.2). Moreover, we deduce from this that $\mathrm{N}_{K/k}(\mathscr{T}_K) = \mathscr{T}_k$.

6.3 Cyclic cubic fields of prime conductors

The following program gives, for $p = 2$ and for cyclic cubic fields of prime conductor f, the group structure of $\mathscr{T}_K$ in L (from [10, § 3.2]; recall that in all such programs, the parameter $\mathtt{nt}$ must be large enough regarding the exponent of $\mathscr{T}_K$), then the (conjectural) annihilator $\mathscr{A}'_{K,n}(c)$, reduced modulo $1+\sigma+\sigma^2$; it is equal, up to an invertible element, to a power of 2 (2 is inert in $\mathbb{Q}(j)$):

```
{p=2;nt=12;forprime(f=10^4,2*10^4,if(Mod(f,3)!=1,next);P=polsubcyclo(f,3);
```

```
K=bnfinit(P,1);Kpn=bnrinit(K,p^nt);C5=component(Kpn,5);
Hpn0=component(C5,1);Hpn=component(C5,2);L=List;ex=0;
i=component(matsize(Hpn),2);for(k=1,i-1,co=component(Hpn,i-k+1);
if(Mod(co,p)==0,val=valuation(co,p);if(val>ex,ex=val);
listinsert(L,p^val,1)));Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);if(vptor>2,S0=0;S1=0;S2=0;pN=p^2*p^ex;
D=(f-1)/3;fn=pN*f;z=znprimroot(f);zz=lift(z);t=lift(Mod((1-zz)/f,p));
c=zz+t*f;for(a=1,fn/2,if(gcd(a,fn)!=1,next);asurc=lift(a*Mod(c,fn)^-1);
lambda=(asurc*c-a)/fn;u=Mod(lambda*a^-1,pN);
a0=lift((a*z^0)^D);a1=lift((a*z^2)^D);a2=lift((a*z)^D);
if(a0==1,S0=S0+u);if(a1==1,S1=S1+u);if(a2==1,S2=S2+u));
L0=lift(S0);L1=lift(S1);L2=lift(S2);L1=L1-L0;L2=L2-L0;
A=gcd(L1,L2);A=2^valuation(A,2);print(f," ",P," ", A," ",L)))}

f     P                    A   L
10399 x^3+x^2-3466*x+7703  4   [4,4]
10513 x^3+x^2-3504*x-80989 8   [8,8]
10753 x^3+x^2-3584*x-76864 4   [4,4]
10771 x^3+x^2-3590*x-26728 4   [4,4]
10903 x^3+x^2-3634*x+26248 8   [8,8]
10939 x^3+x^2-3646*x-46592 16  [16,16]
10957 x^3+x^2-3652*x-39364 4   [4,4]
11149 x^3+x^2-3716*x+39228 4   [2,2,2,2]
(...)
12757 x^3+x^2-4252*x+103001 4  [4,4]
13267 x^3+x^2-4422*x+96800 16  [16,16]
13297 x^3+x^2-4432*x+94064 4   [4,4]
13309 x^3+x^2-4436*x+100064 4  [4,4]
13591 x^3+x^2-4530*x-63928 8   [8,8]
```

6.4 Cyclic quartic fields of prime conductors

Let's give the same program for prime conductors $f \equiv 1 \pmod 8$, with the annihilator $\mathscr{A}_{K,n}(c)$:

```
{p=2;nt=18;d=4;forprime(f=5,500,if(Mod(f,2*d)!=1,next);P=polsubcyclo(f,d);
K=bnfinit(P,1);Kpn=bnrinit(K,p^nt);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);L=List;ex=0;
i=component(matsize(Hpn),2);for(k=1,i-1,co=component(Hpn,i-k+1);
if(Mod(co,p)==0,val=valuation(co,p);if(val>ex,ex=val);
listinsert(L,p^val,1)));Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);if(vptor>1,S0=0;S1=0;S2=0;S3=0;
pN=p^2*p^ex;D=(f-1)/d;fn=pN*f;z=znprimroot(f);zz=lift(z);
t=lift(Mod((1-zz)/f,p));c=zz+t*f;for(a=1,fn,if(gcd(a,fn)!=1,next);
asurc=lift(a*Mod(c,fn)^-1);lambda=(asurc*c-a)/fn;u=Mod(lambda*a^-1,pN);
a0=lift((a*z^0)^D);a1=lift((a*z^1)^D);a2=lift((a*z^2)^D);a3=lift((a*z^3)^D);
if(a0==1,S0=S0+u);if(a1==1,S1=S1+u);if(a2==1,S2=S2+u);if(a3==1,S3=S3+u));
L0=lift(S0);L1=lift(S1);L2=lift(S2);L3=lift(S3);Y=Mod(y,y^2+1);
ni=L0+Y*L1+Y^2*L2+Y^3*L3;Nni=valuation(norm(ni),2);
print(f," ",P," ",L0," ",L1," ",L2," ",L3," ",L," ",2^Nni)))}
```

One gets the following examples (with vptor > 1 and where 2^{Nni} is the norm of $L_0 - L_2 + (L_1 - L_3)\sqrt{-1}$ with $\mathscr{A}_{K,n}(c) = L_0 + L_1\sigma + L_2\sigma^2 + L_3\sigma^3$, given in A $= [\mathrm{L0}, \mathrm{L1}, \mathrm{L2}, \mathrm{L3}]$); then the list L gives the structure of $\mathscr{T}_K$:

```
f   P                      A                 L       2^Nni
17  x^4+x^3-6*x^2-x+1      [4, 6, 0, 6]      [4]     16
41  x^4+x^3-15*x^2+18*x-4  [90, 28, 102, 100] [32] 16
73  x^4+x^3-27*x^2-41*x+2  [4, 4, 0, 0]      [2,2,2] 32
89  x^4+x^3-33*x^2+39*x+8  [4, 4, 0, 0]      [2,2,2] 32
97  x^4+x^3-36*x^2+91*x-61 [8, 10, 12, 2]    [4]     16
113 x^4+x^3-42*x^2-120*x-64 [16, 28, 8, 12] [2,2,8] 64
```

```
137  x^4+x^3-51*x^2-214*x-236 [26, 8, 30, 16] [16] 16
193  x^4+x^3-72*x^2-205*x-49 [6, 0, 14, 12] [4]    16
233  x^4+x^3-87*x^2+335*x-314 [4, 0, 0, 4] [2,2,2] 32
241  x^4+x^3-90*x^2-497*x-739 [6, 0, 6, 4] [4]     16
257  x^4+x^3-96*x^2-16*x+256 [28, 20, 20, 60] [2,4,16] 128
281  x^4+x^3-105*x^2+123*x+236 [4, 4, 0, 0] [2,2,2] 32
313  x^4+x^3-117*x^2+450*x-324 [78, 12, 106, 108] [32] 16
337  x^4+x^3-126*x^2+316*x+104 [28, 12, 28, 28] [2,8,8] 256
353  x^4+x^3-132*x^2+684*x-928 [112, 60, 80, 68] [2,2,32] 64
401  x^4+x^3-150*x^2-25*x+625 [14, 4, 6, 8] [4]    16
409  x^4+x^3-153*x^2-230*x+548 [22, 8, 26, 24] [8] 16
433  x^4+x^3-162*x^2+839*x-1003 [2, 4, 10, 0] [4]  16
449  x^4+x^3-168*x^2-477*x+335 [10, 4, 10, 8] [4]  16
457  x^4+x^3-171*x^2+1114*x-2044 [76, 10, 28, 30] [32] 16
```

6.5 Detailed example of annihilation

The case of the cyclic quartic field K of conductor $f = 3433$ is particularly interesting:

6.5.1 Numerical data

We have $P = x^4 + x^3 - 1287x^2 - 12230x + 3956$ and $\mathscr{T}_K \simeq \mathbb{Z}/2^7\mathbb{Z}$, knowing that the quadratic subfield $k = \mathbb{Q}(\sqrt{3433})$ is such that $\mathscr{T}_k \simeq \mathbb{Z}/2^6\mathbb{Z}$:

```
{P=x^4+x^3-1287*x^2-12230*x+3956;K=bnfinit(P,1);p=2;nt=18;
Kpn=bnrinit(K,p^nt);Hpn=component(component(Kpn,5),2);L=List;
i=component(matsize(Hpn),2);for(k=1,i-1,c=component(Hpn,i-k+1);
if(Mod(c,p)==0,listinsert(L,p^valuation(c,p),1)));print("Structure: ",L)}
Structure: List([128])

{P=x^2-3433;K=bnfinit(P,1);p=2;nt=18;Kpn=bnrinit(K,p^nt);
Hpn=component(component(Kpn,5),2);L=List;i=component(matsize(Hpn),2);
for(k=1,i-1,c=component(Hpn,i-k+1);if(Mod(c,p)==0,
listinsert(L,p^valuation(c,p),1)));print("Structure: ",L)}
Structure: List([64])
```

The class group of K is trivial and its three fundamental units are:

```
[227193/338*x^3-6613325/338*x^2-93274465/338*x+14925255/169,
34349/169*x^3+1388772/169*x^2+10559389/169*x-3491425/169,
70276336974818125/338*x^3-677429229869394661/338*x^2
     -8323827298356088143/338*x+13065197272033438434/169]
```

6.5.2 Annihilation from $\mathscr{A}_{K,n}(c)$

We have computed $\mathscr{A}_{K,n}(c)$ and obtained:

$$\mathscr{A}_{K,n}(c) =: A_K \equiv 8\cdot 13 + 2\cdot 21\,\sigma + 16\cdot 7\,\sigma^2 + 2\cdot 23\,\sigma^3 \pmod{2^7}.$$

Let h be a group generator of $\mathscr{T}_K$ (order 2^7) and let h_0 be a generator of $\mathscr{T}_k$ (order 2^6); it is easy to prove that one may suppose $h^2 = j_{K/k}(h_0)$ (injectivity of the transfer map $j_{K/k}$) and $h_0^{\sigma^2} = h_0$. We put $j_{K/k}(h_0) =: h_0$ for simplicity. Then it follows that

$$h^{A_K} = h_0^{4\cdot 13 + 21\,\sigma + 8\cdot 7\,\sigma^2 + 23\,\sigma^3} = 1.$$

Since $h_0^{\sigma^2} = h_0$, we obtain $h^{A_K} = h_0^{(4\cdot 13+8\cdot 7)+(21+23)\,\overline{\sigma}} = h_0^{4\cdot 27+4\cdot 11\,\overline{\sigma}} = 1$; giving, modulo the norm $1+\overline{\sigma}$, $h_0^{4\cdot(27-11)} = h_0^{2^6} = 1$, as expected.

6.5.3 Annihilation from $\mathscr{A}'_{K,n}(c)$

There is (by accident ?) no numerical obstruction for an annihilation by $A'_K := \mathscr{A}'_{K,n}(c)$, with the same program replacing "$\mathsf{for}(\mathsf{a}=1,\mathsf{fn},\ldots)$" by "$\mathsf{for}(\mathsf{a}=1,\mathsf{fn}/2,\ldots)$". Then it follows that the program gives $h^{A'_K} = h^{4\cdot 13+21\,\sigma+8\cdot 15\,\sigma^2+23\,\sigma^3} = 1$. Since the restriction of A'_K to k is A'_k (no Euler factors), we get:

$$h_0^{A'_k} = h_0^{4\cdot 13+8\cdot 15+(21+23)\cdot\overline{\sigma}} = h_0^{4\cdot 43+4\cdot 11\,\overline{\sigma}} = 1$$

which is equivalent, modulo the norm, to the annihilation by $4\cdot 43-4\cdot 11 = 2^7$ for a cyclic group of order 2^6.

Now we may return to the annihilation of h; since $h^{1+\sigma^2}\in j_{K/k}(\mathscr{T}_k)$ we put $h^{1+\sigma^2} = h_0^t$. Then, with $u=13$, $v=21$, $w=15$, $z=23$, we have:

$$\begin{aligned} h^{4u+v\sigma+8w\sigma^2+z\sigma^3} &= h_0^{2u+4w\sigma^2}h^{(v+z\sigma^2)\sigma}\\ &= h_0^{2u+4w+23t\,\sigma}h^{(v-z)\sigma} = h_0^{2\cdot 43+23t\,\sigma}h^{-2\,\sigma}\\ &= h_0^{2\cdot 43+(23t-1)\,\sigma} = h_0^{2\cdot 43-23t+1} = h_0^{87-23t} = 1 \end{aligned}$$

so necessarily $87-23t\equiv 0 \pmod{2^6}$, giving $t\equiv 1 \pmod{2^6}$. So we can write:

$$h^{1+\sigma^2} = j_{K/k}(h_0).$$

6.5.4 Direct study of the G_K-module structure of $\mathscr{T}_K$

We consider $\mathscr{T}_K$ only given with the following information: h is a group generator such that $h^2 = h_0$, a generator of $j_{K/k}(\mathscr{T}_K)$; $h^\sigma = h^x$, $x\in\mathbb{Z}/2^7\mathbb{Z}$, whence $h_0^\sigma = h_0^x = h_0^{-1}$ giving $x\equiv -1 \pmod{2^6}$. The relation $h^{\sigma^2+1} = h^{x^2+1} = h^2 = h_0$ gives again $t=1$ in the previous notation $h^{\sigma^2+1} = h_0^t$. Moreover, $h^{\sigma^2-1} = h^{x^2-1} = 1$, which is in accordance with Lemma 6.2 and gives $\mathscr{T}_K^g = \mathscr{T}_K$.

If we take into account these theoretical informations for the "annihilators" A_K and A'_K we find no contradiction, but we do not know if $x=-1$ or $x=-1+2^6$ (modulo 2^7). The prime 2 splits in k, is inert in K/k and the class number of K is 1; so we have $\mathscr{W}_K\simeq\mathscr{W}_k\simeq\mathbb{Z}/2\mathbb{Z}$ and $\mathscr{T}_K = \mathrm{tor}_{\mathbb{Z}_2}\big(U_K/\overline{E}_K\big)$; then the result about x depends on the exact sequence (2.2):

$$1\to\mathbb{Z}/2\mathbb{Z}\longrightarrow\mathscr{T}_K\simeq\mathbb{Z}/2^7\mathbb{Z}\xrightarrow{\ \log\ }\mathrm{tor}_{\mathbb{Z}_2}\big(\log\big(U_K\big)/\log(\overline{E}_K)\big) =: \mathscr{R}_K\to 0,$$

knowing the units and then the structure of the regulator $\mathscr{R}_K$.

6.5.5 About the case $f_K = 233$

The field K is defined by the polynomial $P = x^4+x^3-87x^2+335x-314$ for which $\mathscr{T}_K\simeq(\mathbb{Z}/2\mathbb{Z})^3$ and $\mathscr{T}_k\simeq\mathbb{Z}/2\mathbb{Z}$.

In this case an annihilator is $A_K = 4\cdot(1+\sigma^3)$, which shows that $A'_K = 2\cdot(1+\sigma^3)$ is also suitable. Then $A''_K = \frac{1}{2}A'_K$ should be equivalent to $1-\sigma$.

Since 2 splits completely in K, we have $\mathscr{T}_K = \mathscr{W}_K\simeq(\mathbb{Z}/2\mathbb{Z})^3$ and in the same way, $\mathscr{T}_k = \mathscr{W}_k\simeq\mathbb{Z}/2\mathbb{Z}$, for which the Galois structures are well-known: in particular, $1-\sigma$ *does not annihilate* $\mathscr{T}_K$ (the class of $(1,-1,1,-1)$ is invariant). Another proof: use Lemma 6.2 giving here $\#\mathscr{T}_K^{G_K} = 2$.

7. p-adic measures and annihilations

To establish (in Section 9) a link with the values of p-adic L-functions, $L_p(s,\chi)$, at $s=1$, we shall refer to [13, Section II] using the point of view of explicit p-adic measures (from pseudo-measures in the sense of [24]) with a Mellin transform for the construction of $L_p(s,\chi)$ and the application to some properties of the λ invariants of Iwasawa's theory.

But since we only need the value $L_p(1,\chi)$, instead of $L_p(s,\chi)$, for $s\in\mathbb{Z}_p$, we can simplify the general setting, using a similar computation of $\mathscr{S}_{L_n}(c)^*$, directly in $\mathbb{Z}[G_n]$, given by Oriat in [22, Proposition 3.5].

7.1 Definition of $\mathscr{A}_{L_n}$ and $\mathscr{A}_{L_n}(c)$

Let $n\ge n_0+e$, where $\mathscr{T}_K^{p^e} = 1$, and put $\varphi_n := \varphi(qp^n) = (p-1)\cdot p^n$ if $p\ne 2$, $\varphi_n = 2^{n+1}$ otherwise.

We consider (where c is odd and prime to f_n and where a runs trough the integers in $[1,f_n]$, prime to f_n):

$$\mathscr{A}_{L_n} := \frac{-1}{f_n\varphi_n}\sum_a a^{\varphi_n}\Big(\frac{L_n}{a}\Big)\ \ \&\ \ \mathscr{A}_{L_n}(c) := \Big[1-c^{\varphi_n}\Big(\frac{L_n}{c}\Big)\Big]\mathscr{A}_{L_n}. \qquad (7.1)$$

For now, these elements, or more precisely their restrictions to K, are not to be confused with the restrictions $\mathscr{A}_{K,n}(c)$ of $\mathscr{S}_{L_n}(c)^*$ defined in § 5.2, even we shall prove that they are indeed equal; but such an expression is more directly associated to L_p-functions. Then:

$$\begin{aligned}\mathscr{A}_{L_n}(c) &= \Big[1 - c^{\varphi_n}\Big(\frac{L_n}{c}\Big)\Big]\frac{-1}{f_n\varphi_n}\sum_a a^{\varphi_n}\Big(\frac{L_n}{a}\Big)\\ &\mathrel{\overset{\equiv}{\equiv}} \frac{-1}{f_n\varphi_n}\Big[\sum_a a^{\varphi_n}\Big(\frac{L_n}{a}\Big) - \sum_a a^{\varphi_n}c^{\varphi_n}\Big(\frac{L_n}{a}\Big)\Big(\frac{L_n}{c}\Big)\Big]\\ &\text{(in the same way, use } a'_c \text{ such that}\\ &\qquad\qquad a'_c\cdot c \equiv a \pmod{f_n},\ 1\le a'_c \le f_n)\\ &\mathrel{\overset{\equiv}{\equiv}} \frac{-1}{f_n\varphi_n}\Big[\sum_a a^{\varphi_n}\Big(\frac{L_n}{a}\Big) - \sum_a a'^{\,\varphi_n}_c c^{\varphi_n}\Big(\frac{L_n}{a'_c}\Big)\Big(\frac{L_n}{c}\Big)\Big]\\ &\mathrel{\overset{\equiv}{\equiv}} \frac{1}{f_n\varphi_n}\sum_a\Big[(a'_c\cdot c)^{\varphi_n} - a^{\varphi_n}\Big]\Big(\frac{L_n}{a}\Big).\end{aligned}$$

Lemma 7.1. *We have* $(a'_c\cdot c)^{\varphi_n} - a^{\varphi_n} \equiv 0 \pmod{f_n\varphi_n}$.

Proof. By definition, $a'_c\cdot c = a + \lambda^n_a(c) f_n$ with $\lambda^n_a(c)\in\mathbb{Z}$. Consider:

$$\begin{aligned}A &:= \frac{(a'_c\cdot c)^{\varphi_n} - a^{\varphi_n}}{f_n\varphi_n}\\ &= \frac{[a^{\varphi_n} + \lambda^n_a(c) f_n\varphi_n a^{\varphi_n-1} + \lambda^n_a(c)^2 f_n^2\frac{\varphi_n(\varphi_n-1)}{2}a^{\varphi_n-2}+\cdots] - a^{\varphi_n}}{f_n\varphi_n}\\ &\equiv \lambda^n_a(c)a^{\varphi_n-1} + \lambda^n_a(c)^2 f_n\frac{(\varphi_n-1)}{2}a^{\varphi_n-2}\\ &\equiv \lambda^n_a(c)a^{\varphi_n-1} \equiv \lambda^n_a(c)a^{-1} \pmod{p^{n+1}},\end{aligned}$$

since $a^{\varphi_n}\equiv 1 \pmod{qp^n}$.

When $p=2$, one must take into account the term $\lambda^n_a(c) f_n\frac{\varphi_n-1}{2}a^{\varphi_n-2}\sim\frac{1}{2}\lambda^n_a(c)f_n$, in which case the congruence is with the modulus p^{n+1} (which is sufficient since for $n\ge n_0+e$, this modulus annihilates $\mathscr{T}_K$ for any p). □

We have obtained for all $n\ge n_0+e$:

$$\mathscr{A}_{L_n}(c) \mathrel{\overset{\equiv}{\equiv}} \sum_{a=1}^{f_n}\lambda^n_a(c)\cdot a^{-1}\Big(\frac{L_n}{a}\Big) \mathrel{\overset{\equiv}{\equiv}} \mathscr{S}_{L_n}(c)^*, \tag{7.2}$$

thus giving again, by restriction to K, the annihilator $\mathscr{A}_{K,n}(c)\in\mathbb{Z}_p[G_K]$ of $\mathscr{T}_K$ such that (for all $n\ge n_0+e$) $\mathscr{A}_{K,n}(c)\mathrel{\overset{\equiv}{\equiv}}\sum_{a=1}^{f_n}\lambda^n_a(c)\,a^{-1}\Big(\frac{K}{a}\Big)$.

7.2 Normic properties of the $\mathscr{A}_{L_n}$ – Euler factors

Theorem 7.2. *[13, Proposition II.2 (iv)]. Let K be of conductor $f = m\ell$ where m is the conductor of a subfield k of K and where $\ell\ne p$ is a prime number. For $n\ge n_0$, let $L_n := K(\mu_{qp^n})$ and the analogous field l_n for k, of conductors f_n and m_n, respectively; recall that $\varphi_n = \varphi(qp^n)$.*

Let $\mathscr{A}_{L_n} := \frac{-1}{f_n\varphi_n}\sum_a^{f_n} a^{\varphi_n}\Big(\frac{L_n}{a}\Big)$ *and* $\mathscr{A}_{l_n} := \frac{-1}{m_n\varphi_n}\sum_b^{m_n} b^{\varphi_n}\Big(\frac{l_n}{b}\Big)$. *Then:*

$$\mathrm{N}_{L_n/l_n}(\mathscr{A}_{L_n}) \mathrel{\overset{\equiv}{\equiv}} \Big(1-\ell^{\varphi_n}\frac{1}{\ell}\Big(\frac{l_n}{\ell}\Big)\Big)\mathscr{A}_{l_n},\ \ \textit{resp.,}\ \ \mathrm{N}_{L_n/l_n}(\mathscr{A}_{L_n}) \mathrel{\overset{\equiv}{\equiv}} \mathscr{A}_{l_n},$$

if $\ell\nmid m$, resp., $\ell\mid m$ (congruences modulo $p^{n+1}\mathbb{Z}_p[G_n] + (1-s_\infty)\mathbb{Z}_p[G_n]$).

Proof. Suppose first that $\ell\nmid m$, so $f_n = lm_n$.[2] Put $a = b+\lambda\, m_n$, $\lambda\in[0,\ell-1]$, $b\in[1,m_n]$ prime to m_n; since $a\in[1,f_n]$ is prime to f_n, b is prime to m_n and $\lambda\ne\lambda^*_b$ such that $b+\lambda^*_b m_n =: b'_\ell\cdot\ell$, $b'_\ell\in\mathbb{Z}$. Thus $a^{\varphi_n} = (b+\lambda\, m_n)^{\varphi_n}\equiv b^{\varphi_n} + b^{\varphi_n-1}\lambda\, m_n\varphi_n$

[2] For $\ell=2$ and m odd, $f=2m$ is not a conductor stricto sensu, but the following computations are exact and necessary with the modulus m_n and $f_n = 2m_n$; then if $f = 2^k\cdot m$ (m odd, $k\ge 2$), the second case of the theorem applies and shall give the Euler factor $\big(1-2^{\varphi_n}\frac{1}{2}\big(\frac{l_n}{2}\big)\big)\mathrel{\overset{\equiv}{\equiv}}\big(1-\frac{1}{2}\big(\frac{l_n}{2}\big)\big)$. If $p\mid f$ and $p\nmid m$, there is no Euler factor for p since m_n and f_n are divisible by p; in other words, these computations and the forthcoming ones are, by nature, not "primitive" at p.

(mod $m_n\varphi_n p^{n+1}$). Then:

$$\begin{aligned}
\mathrm{N}_{L_n/l_n}(\mathscr{A}_{L_n}) &\cong \frac{-1}{\ell m_n\varphi_n}\cdot\sum_{b,\,\lambda\neq\lambda_b^*}\Big[b^{\varphi_n}+b^{\varphi_n-1}\lambda\, m_n\varphi_n\Big]\Big(\frac{l_n}{b}\Big)\\
&\cong \frac{-(\ell-1)}{\ell m_n\varphi_n}\sum_b b^{\varphi_n}\Big(\frac{l_n}{b}\Big)-\frac{1}{\ell}\sum_{b,\,\lambda\neq\lambda_b^*}b^{\varphi_n-1}\lambda\Big(\frac{l_n}{b}\Big)\\
&\cong \Big(1-\frac{1}{\ell}\Big)\mathscr{A}_{l_n}-\frac{1}{\ell}\sum_{b,\,\lambda\neq\lambda_b^*}b^{\varphi_n-1}\lambda\Big(\frac{l_n}{b}\Big)\\
&\cong \Big(1-\frac{1}{\ell}\Big)\mathscr{A}_{l_n}-\frac{1}{\ell}\sum_b b^{\varphi_n-1}\Big(\frac{l_n}{b}\Big)\Big(\sum_{\lambda\neq\lambda_b^*}\lambda\Big)\\
&\cong \Big(1-\frac{1}{\ell}\Big)\mathscr{A}_{l_n}-\frac{1}{\ell}\sum_b b^{\varphi_n-1}\Big(\frac{l_n}{b}\Big)\Big(\frac{\ell(\ell-1)}{2}-\lambda_b^*\Big).
\end{aligned}$$

We remark that $\lambda_b^*=\lambda_b^n(\ell)$ is relative to the writing $b'_\ell\cdot\ell=b+\lambda_b^n(\ell)\,m_n$ and that $b^{\varphi_n-1}\equiv b^{-1}\pmod{p^{n+1}}$, whence using $\sum_b b^{-1}\big(\frac{l_n}{b}\big)\cong 0$:

$$\mathrm{N}_{L_n/l_n}(\mathscr{A}_{L_n})\cong\Big(1-\frac{1}{\ell}\Big)\mathscr{A}_{l_n}+\frac{1}{\ell}\sum_b\lambda_b^*\cdot b^{-1}\Big(\frac{l_n}{b}\Big).$$

But as we know (see relations 7.1 and (7.2)), $\sum_b\lambda_b^*\,b^{-1}\Big(\frac{l_n}{b}\Big)\cong\mathscr{A}_{l_n}(\ell)$; so $\mathrm{N}_{L_n/l_n}(\mathscr{A}_{L_n})\cong\Big(1-\frac{1}{\ell}\Big)\mathscr{A}_{l_n}+\frac{1}{\ell}\mathscr{A}_{l_n}(\ell)$: since $\mathscr{A}_{l_n}(\ell)\cong\Big(1-\ell^{\varphi_n}\Big(\frac{l_n}{\ell}\Big)\Big)\mathscr{A}_{l_n}$, we get $\mathrm{N}_{L_n/l_n}(\mathscr{A}_{L_n})\cong\Big(1-\ell^{\varphi_n}\frac{1}{\ell}\Big(\frac{l_n}{\ell}\Big)\Big)\mathscr{A}_{l_n}$.

The case $\ell\mid m$ is obtained more easily from the same computations. □

Of course, for all $h\geq 0$ we get:

$$\mathrm{N}_{L_{n+h}/L_n}(\mathscr{A}_{L_{n+h}})\cong\mathscr{A}_{L_n},$$

which expresses the coherence of the family $\big(\mathscr{A}_{L_n}\big)_n$ in the cyclotomic tower.

Corollary 7.3. *(i) Let K/k be an extension of fields of conductors f_K and f_k, respectively. Multiplying by $\Big[1-c^{\varphi_n}\Big(\frac{l_n}{c}\Big)\Big]=\mathrm{N}_{L_n/l_n}\Big[1-c^{\varphi_n}\Big(\frac{L_n}{c}\Big)\Big]$ to get elements in the algebras $(\mathbb{Z}/p^{n+1}\mathbb{Z})[\mathrm{Gal}(L_n/\mathbb{Q})]$ and $(\mathbb{Z}/p^{n+1}\mathbb{Z})[\mathrm{Gal}(l_n/\mathbb{Q})]$, one obtains $\mathrm{N}_{L_n/l_n}(\mathscr{A}_{L_n}(c))\cong\prod_{\ell\mid f_K,\ell}$ $\frac{1}{\ell}\Big(\frac{l_n}{\ell}\Big)\Big)\mathscr{A}_{l_n}(c)$.*

(ii) Let $\mathscr{A}_{K,n}(c)$ and $\mathscr{A}_{k,n}(c)$ be the restrictions of $\mathscr{A}_{L_n}(c)$ and $\mathscr{A}_{l_n}(c)$ to K and k, respectively; then $\mathrm{N}_{K/k}(\mathscr{A}_{K,n}(c))\cong\prod_{\ell\mid f_K,\,\ell\nmid pf_k}\Big(1-\frac{1}{\ell}\Big(\frac{k}{\ell}\Big)\Big)\cdot\mathscr{A}_{k,n}(c)$.

(iii) The family $(\mathscr{A}_{K,n})_n=(\mathrm{N}_{L_n/K}(\mathscr{A}_{L_n}))_n$ defines a pseudo-measure denoted $\mathscr{A}_K$ by abuse, such that the measure $(\mathscr{A}_{K,n}(c))_n$ defines the element $\mathscr{A}_K(c)=\Big(1-\Big(\frac{K}{c}\Big)\Big)\cdot\mathscr{A}_K\in\mathbb{Z}_p[G_K]$ and gives the main formula:

$$\mathrm{N}_{K/k}(\mathscr{A}_K(c))\cong\prod_{\ell\mid f_K,\,\ell\nmid pf_k}\Big(1-\frac{1}{\ell}\Big(\frac{k}{\ell}\Big)\Big)\cdot\mathscr{A}_k(c).$$

Remark 7.4. *(i) In a numerical point of view, we only need a minimal value of n, and we shall write (e.g., for $n=e$ when $K\cap\mathbb{Q}_\infty=\mathbb{Q}$):*

$$\mathscr{A}_{K,e}(c)\cong\sum_{\sigma\in G_K}\Big[\sum_{a,\,(\frac{K}{a})=\sigma}\lambda_a^e(c)\,a^{-1}\Big]\cdot\sigma=:\sum_{\sigma\in G_K}\Lambda_\sigma^e(c)\cdot\sigma.$$

Then the next step shall be to interprete the limit, $\Lambda_\sigma(c)$, of the coefficients $\Lambda_\sigma^n(c)=\sum_{a,\,(\frac{K}{a})=\sigma}\lambda_a^n(c)\,a^{-1}$, for $n\to\infty$, giving an equivalent annihilator, but with a more canonical interpretation.

(ii) In [12, 13, 22, 26, 28, 29, 5, 19, 27, 21, 1, 2, 4], some limits are expressed by means of p-adic logarithms of cyclotomic numbers/units of $\mathbb{Q}^f$ as expressions of the values at $s=1$ of the p-adic L-functions of K (for instance, in [29, Theorem 2.1] a link between Stickelberger elements and cyclotomic units is given following Iwasawa and Coleman). But these results are obtained with various non-comparable techniques; this will be discussed later.

(iii) In the relation $\mathscr{A}_K(c):=\Big[1-\Big(\frac{K}{c}\Big)\Big]\mathscr{A}_K$, the choice of c must be such that the integers $1-\chi(c)$ be of minimal p-adic valuation for the characters χ of K. But $1-\chi(c)$ is invertible if and only if $\chi(c)$ is not a root of unity of p-power order.

8. Remarks about Solomon's annihilators

We shall give two examples: one giving the same annihilator as our's, and another giving a Solomon annihilator in part degenerated, contrary to $\mathscr{A}_K(c)$.

8.1 Cubic field of conductor 1381 and Solomon's Ψ_K

We have (see the previous table of § 5.3) $P = x^3 + x^2 - 460x - 1739$ and the classical program gives the class number in h, the group structure of $\mathscr{T}_K$ (in L) and the units in E:

```
{P=x^3+x^2-460*x-1739;K=bnfinit(P,1);p=7;nt=8;Kpn=bnrinit(K,p^nt);r=1;
Hpn=component(component(Kpn,5),2);C8=component(K,8);E=component(C8,5);
h=component(component(C8,1),1);L=List;i=component(matsize(Hpn),2);
for(k=1,i-1,c=component(Hpn,i-k+1);if(Mod(c,p)==0,
listinsert(L,p^valuation(c,p),1)));print(L);print("h=",h," ",L," E=",E)}

h=1  List([343, 7])
E=[245/13*x^2-4606/13*x-21522/13, 147/13*x^2+3479/13*x+11272/13]
```

So, the class group is trivial, $\mathscr{T}_K = \mathscr{R}_K \simeq \mathbb{Z}/7^3\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$ and the cyclotomic units are the fundamental units. Then we shall use a definition of the automorphism σ to define the Galois operation on the units:

```
{P=x^3 + x^2 - 460*x - 1739;print(nfgaloisconj(P))}
[x, -1/13*x^2 - 2/13*x + 302/13,  1/13*x^2 - 11/13*x - 315/13]
```

From $\varepsilon = \frac{245}{13}x^2 - \frac{4606}{13}x - \frac{21522}{13}$ and $\sigma : x \mapsto -\frac{1}{13}x^2 - \frac{2}{13}x + \frac{302}{13}$, one gets:

```
Mod(245/13*(-1/13*x^2 - 2/13*x + 302/13)^2 -
4606/13*(-1/13*x^2 - 2/13*x + 302/13) - 21522/13,P)=
Mod(147/13*x^2 + 3479/13*x + 11259/13, x^3 + x^2 - 460*x - 1739)
```

which is ε^{σ} and the units are, on the $\mathbb{Q}$-base $\{1, x, x^2\}$:

$\varepsilon = \varepsilon_1 = \frac{245}{13}x^2 - \frac{4606}{13}x - \frac{21522}{13}$,
$\varepsilon^{\sigma} = \varepsilon_2 = \frac{147}{13}x^2 + \frac{3479}{13}x + \frac{11259}{13}$,
$\varepsilon^{\sigma^2} = \varepsilon_3 = -\frac{392}{13}x^2 + \frac{1127}{13}x + \frac{175948}{13}$.

The second unit given by PARI is $\frac{147}{13}x^2 + \frac{3479}{13}x + \frac{11272}{13} = -\varepsilon^{-\sigma^2}$. The order of ε modulo $p = 7$ is 114. We compute $A_i := \varepsilon_i^{114}$ modulo 7^6, $i = 1,2,3$), then $L_i := A_i - 1$:

```
{P=x^3+x^2-460*x-1739;
E1=Mod(245/13*x^2-4606/13*x-21522/13,P+Mod(0,7^6));
E2=Mod(147/13*x^2+3479/13*x+11259/13,P+Mod(0,7^6));
E3=Mod(-392/13*x^2+1127/13*x+175948/13,P+Mod(0,7^6));
L1=E1^114-1;L2=E2^114-1;L3=E3^114-1;
print(lift(L1)," ",lift(L2)," ",lift(L3))}
```

$L_1 = 17542x^2 + 48608x + 81879 = 7^2(358x^2 + 992x + 1671) = 7^2\alpha_1$,
$L_2 = 62867x^2 + 833x + 33761 = 7^2(1283x^2 + 17x + 689) = 7^2\alpha_2$,
$L_3 = 37240x^2 + 68208x + 2009 = 7^2(760x^2 + 1392x + 41) = 7^2\alpha_3$,
giving $\frac{1}{7}\log(\varepsilon_i) \equiv 7\alpha_i - \frac{1}{2}7^3\alpha_i^2 \pmod{7^4}$:
$\frac{1}{7}\log(\varepsilon) \equiv 791x^2 + 2142x + 378 = 7(113x^2 + 306x + 54) \pmod{7^4}$,
$\frac{1}{7}\log(\varepsilon^{\sigma}) \equiv 2121x^2 + 119x + 364 = 7(303x^2 + 17x + 52) \pmod{7^4}$,
$\frac{1}{7}\log(\varepsilon^{\sigma^2}) \equiv 1890x^2 + 140x + 1659 = 7(270x^2 + 20x + 237) \pmod{7^4}$.

So, the Solomon annihilator $\frac{1}{p}\sum_{\sigma \in G_K} \log(\varepsilon^{\sigma}) \cdot \sigma^{-1}$ of $\mathscr{T}_K$ is (modulo 7^3 and up to a 7-adic unit):

$$\Psi_K \equiv 7 \cdot \big[15x^2 + 12x + 5 + (9x^2 + 17x + 3)\sigma^{-1} + (25x^2 + 20x + 41)\sigma^{-2}\big].$$

Since the norm is a trivial annihilator, we can replace Ψ_K by

$$\begin{aligned}\Psi'_K &= \Psi_K - 7\cdot(15x^2+12x+5)(1+\sigma^{-1}+\sigma^{-2})\\ &\equiv 7\cdot\big[(43x^2+5x+47)\,\sigma^{-1}+(10x^2+8x+36)\big]\,\sigma^{-2}\pmod{7^3}.\end{aligned}$$

Then, $43x^2+5x+47$ is invertible p-adically (its norm is prime to 7) which gives the equivalent annihilator:

$$7\cdot\big[\sigma+(10x^2+8x+36)\cdot(43x^2+5x+47)^{-1}\equiv\sigma+31\pmod{7^2}\big]$$

equivalent to the annihilator defined by $7\cdot(\sigma-18)$ modulo 7^3.

Our annihilator, given by the previous table, is $1738+2186\,\sigma^{-1}+2361\,\sigma^{-2}$ equivalent to $448+623\,\sigma\equiv 7\cdot(\sigma-18)\pmod{7^3}$.
So $\sigma-18$ is an annihilator for the submodule $\mathscr{T}_K^7\simeq\mathbb{Z}/7^2\mathbb{Z}$, which is coherent since 18 is of order 3 modulo 7^3.
The perfect identity of the two results shows that no information has been lost for this particular case, whatever the method (but in the case of cyclic fields of prime degree, there is not any Euler factor).

8.2 Cyclic quartic field of conductor $37\cdot 45161$ and Solomon's Ψ_K

Let K be a real cyclic quartic field of conductor f such that the quadratic subfield k has conductor $m\mid f$, with for instance $f=\ell m$, ℓ prime split in $k/\mathbb{Q}$. We take $p\equiv 1\pmod 4$, $p\nmid f$.
Put $\eta_f:=1-\zeta_f$, $\eta_m:=1-\zeta_m$, $\eta_K:=\mathrm{N}_{\mathbb{Q}^f/K}(\eta_f)$, $\eta_k:=\mathrm{N}_{\mathbb{Q}^m/k}(\eta_m)$.
Then we have the Solomon annihilator:

$$\Psi_K=\frac{1}{p}\sum_{\sigma\in G_K}\log(\eta_K^\sigma)\cdot\sigma^{-1}.$$

Since, from the formula (4.2) (which applies since $m\neq 1$), one has $\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\eta_f)=\eta_m^{(1-(\frac{\mathbb{Q}^m}{\ell})^{-1})}$, i.e., $\mathrm{N}_{K/k}(\eta_K)=\eta_k^{(1-(\frac{k}{\ell})^{-1})}=1$, we get (with $G_K=\{1,\sigma,\sigma^2,\sigma^3\}$):

$$\begin{aligned}\Psi_K &= \frac{1}{p}\big(\log(\eta_K)+\log(\eta_K^\sigma)\cdot\sigma^{-1}+\log(\eta_K^{\sigma^2})\cdot\sigma^{-2}+\log(\eta_K^{\sigma^3})\cdot\sigma^{-3}\big)\\ &= \frac{1}{p}\big(\log(\eta_K)+\log(\eta_K^\sigma)\cdot\sigma^{-1}-\log(\eta_K)\cdot\sigma^{-2}-\log(\eta_K^\sigma)\cdot\sigma^{-3}\big)\end{aligned}$$

So, in this particular situation, one has:

$$\Psi_K=\frac{1}{p}\big(\log(\eta_K)+\log(\eta_K^\sigma)\cdot\sigma^{-1}\big)\cdot(1-\sigma^2). \tag{8.1}$$

Suppose that $\mathscr{T}_K$ is equal to the transfer of $\mathscr{T}_k$ (many examples are available), then $\mathscr{T}_K$ is annihilated by $(1-\sigma^2)$, whatever the structure of $\mathscr{T}_k\simeq\mathscr{T}_K$; but one expects annihilators A_K such that $\mathrm{N}_{K/k}(A_K)=A_k$ be a non-trivial annihilator of $\mathscr{T}_k$.
For instance, define K by $x=\sqrt{\ell\sqrt{m}\,\frac{\sqrt{m}+a}{2}}$ where $m=a^2+b^2$, $b=2b'$. This gives the polynomial $P=x^4-\ell m x^2+\ell^2 m b'^2$.
The following program gives many examples with non-trivial $\mathscr{T}_k$ (with m prime, $p=5$):

```
{p=5;forprime(m=1,10^5,if(Mod(m,20)!=1,next);P=x^2-m;K=bnfinit(P,1);nt=12;
Kpn=bnrinit(K,p^nt);Hpn=component(component(Kpn,5),2);L=List;
i=component(matsize(Hpn),2);R=0;for(k=1,i-1,c=component(Hpn,i-k+1);
if(Mod(c,p)==0,R=R+1;listinsert(L,p^valuation(c,p),1)));if(R>0,
print("m=",m," structure",L)))}
```

For $m=45161$, one obtains $\mathscr{T}_k\simeq\mathbb{Z}/5^5\mathbb{Z}$; then $a=205$, $b'=28$. Now we find some primes ℓ with the following program:

```
{p=5;m=45161;bprim=28;forprime(ell=7,10^3,if(Mod(ell,4)!=1,next);
if(kronecker(m,ell)!=1,next);P=x^4-ell*m*x^2+ell^2*m*bprim^2;
K=bnfinit(P,1);nt=12;Kpn=bnrinit(K,p^nt);Hpn=component(component(Kpn,5),2);
L=List;i=component(matsize(Hpn),2);
for(k=1,i-1,c=component(Hpn,i-k+1);if(Mod(c,p)==0,
listinsert(L,p^valuation(c,p),1)));
print("ell=",ell," m=",m," P=",P," structure",L))}
```

giving the following examples (for which $\mathscr{T}_k$ is a direct factor in $\mathscr{T}_K$):

```
ell=13 P=x^4-587093*x^2+5983651856 structure [3125]
ell=17 P=x^4-767737*x^2+10232398736 structure [3125,5,5]
ell=37 P=x^4-1670957*x^2+48471120656 structure [3125]
ell=997 P=x^4-45025517*x^2+35194105312016 structure [3125,25]
```

We consider the case $\ell = 37$, $P = x^4 - 1670957x^2 + 48471120656$ for wich PARI gives the following information that may be used by the reader:

```
nfgaloisconj(x^4-1670957*x^2+48471120656)=
[-x, x, -1/212380*x^3 + 43593/5740*x, 1/212380*x^3 - 43593/5740*x]

{P=x^4-1670957*x^2+48471120656;K=bnfinit(P,1);p=5;nt=8;Kpn=bnrinit(K,p^nt);
r=1; Hpn=component(component(Kpn,5),2);C8=component(K,8);E=component(C8,5);
h=component(component(C8,1),1);L=List;i=component(matsize(Hpn),2);R=0;
for(k=1,i-1,c=component(Hpn,i-k+1);if(Mod(c,p)==0,R=R+1;
listinsert(L,p^valuation(c,p),1)));print("h=",h," ",L);print("E=",E)}

h=2  List([3125])
```

Now, consider the annihilator $\mathscr{A}_{K,n}(c) =: A_K$; since $\mathscr{T}_K \simeq \mathscr{T}_k$, we get $\mathscr{T}_K^{A_K} \simeq \mathscr{T}_k^{\mathrm{N}_{K/k}(A_K)}$, where (see Corollary 7.3):

$$\mathrm{N}_{K/k}(\mathscr{A}_{K,n}(c)) \stackrel{\cdot}{\equiv} \Big(1 - \frac{1}{\ell}\Big(\frac{k}{\ell}\Big)\Big)\mathscr{A}_{k,n}(c).$$

Then $\ell = 37 \equiv 2 \pmod 5$ splits in k and $1 - \frac{1}{\ell}\Big(\frac{k}{\ell}\Big) = 1 - \frac{1}{\ell}$ is invertible modulo 5.

So A_K acts on $\mathscr{T}_K$ as $\mathscr{A}_{k,n}(c)$ on $\mathscr{T}_k$; we can use the program for quadratic fields and $p > 2$ (of course the bounds bf, Bf may be arbitrary):

```
{p=5;nt=8;bf=45161;Bf=45161;for(f=bf,Bf,v=valuation(f,2);M=f/2^v;
if(core(M)!=M,next);if((v==1||v>3)||(v==0 & Mod(M,4)!=1)||
(v==2 & Mod(M,4)==1),next);P=x^2-f;K=bnfinit(P,1);Kpn=bnrinit(K,p^nt);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
h=component(component(component(K,8),1),2);L=List;ex=0;
i=component(matsize(Hpn),2);for(k=1,i-1,co=component(Hpn,i-k+1);
if(Mod(co,p)==0,val=valuation(co,p);if(val>ex,ex=val);
listinsert(L,p^val,1)));Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);tor=p^vptor;S0=0;S1=0;pN=p*p^ex;fn=pN*f;
for(cc=2,10^2,if(gcd(cc,p*f)!=1 || kronecker(f,cc)!=-1,next);c=cc;break);
for(a=1,fn/2,if(gcd(a,fn)!=1,next);asurc=lift(a*Mod(c,fn)^-1);
lambda=(asurc*c-a)/fn;u=Mod(lambda*a^-1,pN);
s=kronecker(f,a);if(s==1,S0=S0+u);if(s==-1,S1=S1+u));
L0=lift(S0);L1=lift(S1);A=L1-L0;if(A!=0,A=p^valuation(A,p));
print(f," P=",P," ",L0," ",L1," A=",A," tor=",tor," T_K=",L," Cl_K=",h))}
```

giving the annihilator $A_k \equiv 10185 + 3935\,\overline{\sigma} \pmod{5^6}$ where $\overline{\sigma}$ generates $\mathrm{Gal}(k/\mathbb{Q})$; then, A_k is equivalent, modulo the norm, to the integer $10185 - 3935 \equiv 2 \cdot 5^5 \pmod{5^6}$, which is perfect since $\mathscr{T}_k \simeq \mathbb{Z}/5^5\mathbb{Z}$.

The class group of k being trivial, the fundamental unit ε is the cyclotomic one and is such that $\varepsilon^4 = 1 + 5^6 \cdot \alpha$, α prime to 5, which confirms that:

$$\Psi_k \sim \frac{1}{5}(\log(\varepsilon) + \log(\varepsilon^{\overline{\sigma}}) \cdot \overline{\sigma}) = \frac{1}{5}\log(\varepsilon)(1 - \overline{\sigma}) \tag{8.2}$$

equivalent (modulo the norm) to $\frac{2}{5}\log(\varepsilon)$ and $\Psi_k = A_k$ as expected. Meanwhile, the Solomon annihilator Ψ_K does not give Ψ_k by restriction, but 0.

9. About the annihilator $\mathscr{A}_K(c)$ and the primitive $L_p(1,\chi)$

9.1 Galois characters v.s. Dirichlet characters

Let f_K be the conductor of K. In most formulas, the characters χ of K must be primitive of conductor $f_\chi \mid f_K$, whence Dirichlet characters on $(\mathbb{Z}/f_\chi\mathbb{Z})^\times$ such that $\chi\big[\big(\frac{\mathbb{Q}^{f_\chi}}{a}\big)\big]$ makes sense for $a \in \mathbb{Z}$, prime to f_χ, but not necessarily for $\chi\big[\big(\frac{\mathbb{Q}^{f_K}}{a}\big)\big]$ if a prime ℓ divides both a and f_K but not f_χ. This is an obstruction to consider them as Galois characters over $\mathbb{Z}_p[G_K]$ for instance, whence defined on $(\mathbb{Z}/f_K\mathbb{Z})^\times$; so we shall introduce the corresponding Galois character of G_K, denoted $\psi_\chi =: \psi$. A Galois character ψ of G_K is also a character of $G_n = \mathrm{Gal}(L_n/\mathbb{Q})$ whose kernel fixes K, so $\psi(a)$ ($a \in [1, f_n]$ prime to f_n) is the image by ψ of the Artin symbol $\left(\frac{L_n}{a}\right)$ whence of $\left(\frac{K}{a}\right)$.

Any non-primitive writing $\psi(\mathscr{A}_K)$, for $\mathscr{A}_K \in \mathbb{Z}_p[G_K]$, may introduce a product of Euler factors. Indeed, let k_χ be the subfield fixed by the kernel of $\psi = \psi_\chi$ (then χ is a primitive character of k_χ but not necessarily of K); then, $\psi(\mathscr{A}_K) = \psi(\mathrm{N}_{K/k_\chi}(\mathscr{A}_K)) = \chi(\mathscr{E}_{k_\chi}) \cdot \chi(\mathscr{A}_{k_\chi})$ in which $\chi(\mathscr{E}_{k_\chi})$ may be non-invertible (or 0).

9.2 Expression of $\psi(\mathscr{A}_K(c))$

Let ψ be any Galois character of K considered as Galois character of $\mathrm{Gal}(L_n/\mathbb{Q})$, for $n \ge n_0 + e$. We then have the following result about the computation of the annihilator $\mathscr{A}_K(c) =: \sum_{\sigma \in G_K} \Lambda_\sigma(c) \cdot \sigma$ (given explicitely by the Theorem 5.5), without any hypothesis on K and p:

Lemma 9.1. *The expression $\psi(\mathscr{A}_K(c))$ is the product of the multiplicator $1 - \psi\big(\big(\frac{L_\infty}{c}\big)\big)$ by the non-primitive value $L_p(1,\psi)$. In other words, one has:*

$$\begin{aligned}\psi(\mathscr{A}_K(c)) &= (1-\psi(c)) \cdot L_p(1,\psi)\\ &= (1-\psi(c)) \cdot \prod_{\ell \mid f_K,\, \ell \nmid p f_\chi} \big(1-\chi(\ell)\ell^{-1}\big)\, L_p(1,\chi).\end{aligned}$$

Proof. This comes from the classical construction of p-adic L-functions [13, Propositions II.2, II.3, Définition II.3, II.4, and Remarques II.3, II.4], then [7, page 292]. Thus we obtain, using the computations of the § 7.1, the link between the limit (for $n \to \infty$):

$$\psi(\mathscr{A}_K(c)) = \sum_{\sigma \in G_K} \Lambda_\sigma(c) \cdot \psi(\sigma) \ \ \text{(cf. Remark 7.4 (i))},$$

of $\psi(\mathscr{A}_{L_n}(c)) = \psi(\mathscr{A}_{K,n}(c)) = \sum_{\sigma \in G_K} \Lambda^n_\sigma(c)\, \psi(\sigma)$, and the value at $s = 1$ of the L_p-function of the *primitive character* χ associated to ψ. □

Remark 9.2. *Note that in the various calculations in § 7.1, $\varphi_n = \varphi(qp^n)$ when $n \to \infty$ plays the role of $1-s$ when $s \to 1$ in the construction of p-adic L_p-functions by reference to Bernoulli numbers.*

For all primitive Dirichlet character $\chi \ne 1$ of K, of modulus f_χ (or pf_χ if $p \nmid f_\chi$), and for all $p \ge 2$, we have the classical formulas of the value at $s = 1$ of the p-adic L-functions (see for instance [30, Theorem 5.18]), where $\tau(\chi) = \sum_{(a,f_\chi)=1} \chi(a)\zeta^a_{f_\chi}$ is the primitive Gauss sum of χ:

$$L_p(1,\chi) = -\Big(1 - \frac{\chi(p)}{p}\Big) \cdot \frac{\tau(\chi)}{f_\chi} \sum_{a \in [1,f_\chi],\,(a,f_\chi)=1} \chi^{-1}(a) \log(1-\zeta^a_{f_\chi}),$$

where the Euler factor $1-\chi(p)p^{-1}$ illustrates the fact that for L_p-functions, any character χ is considered modulo pf_χ when $p \nmid f_\chi$.

From the Coates formula [6] and classical computations (see also some details in [11, § 2.2]) we recall that ${}^\#\mathscr{T}_K \sim [K \cap \mathbb{Q}_\infty : \mathbb{Q}] \cdot \prod_{\chi \ne 1} \frac{1}{2} L_p(1,\chi)$ (up to a p-adic unit), thus ${}^\#\mathscr{T}_K \sim \prod_{\chi \ne 1} \frac{1}{2} L_p(1,\chi)$ if $K \cap \mathbb{Q}_\infty = \mathbb{Q}$ (i.e., $n_0 = 0$). Moreover, we know that in the semi-simple case, one obtains the orders of the isotypic components of $\mathscr{T}_K$ by means of the $\frac{1}{2} L_p(1,\chi)$; but the whole Galois structure of $\mathscr{T}_K$ is more precise that the set of those given by the components $\mathscr{T}_K^{e_\theta}$, where the e_θ are the corresponding p-adic idempotents.

Remark 9.3. *Let χ be a primitive Dirichlet character of conductor $f_\chi \ne 1$. We define the "modified Solomon element" of $\mathbb{Z}_p[G_{k_\chi}]$:*

$$\Psi_{k_\chi} := -\Big(1 - \frac{\chi(p)}{p}\Big) \cdot \frac{\tau(\chi)}{f_\chi} \sum_{\tau \in G_{k_\chi}} \log(\eta^\tau_{k_\chi}) \cdot \tau^{-1}.$$

Whence $L_p(1,\chi)=\chi(\Psi_{k_\chi})$ ($\chi\neq 1$ primitive). Put:

$$C_\chi := -\Big(1-\frac{\chi(p)}{p}\Big)\cdot\frac{\tau(\chi)}{f_\chi}.$$

When $p\nmid f_\chi$, $\tau(\chi)$ is invertible and $C_\chi\cdot\log(\eta_{k_\chi}^\tau)\sim\frac{1}{p}\cdot\log(\eta_{k_\chi}^\tau)\sim\Psi_{k_\chi}$ (the original Solomon element); when $p\mid f_\chi$, the factor $\frac{1}{p}$ in C_χ is replaced, ahead the logarithms, by the quotient $\frac{1}{\tau(\chi)}$ having the suitable p-valuations. For instance, if d is prime and p unramified, $\frac{1}{p}\sum_{\sigma\in G_K}\log(\eta_K^\sigma)\cdot\sigma^{-1}$ annihilates $\mathscr{T}_K$.

9.3 The annihilator $\mathscr{A}_K(c)$ and the Ψ_{k_χ}

The following statement does not assume any hypothesis on K and p and gives again the known results of annihilation (e.g., semi-simple case, but also the point of view of [22]):

Theorem 9.4. *Let K be a real abelian number field, of degree d, of Galois group G_K and of conductor f_K. Let $\mathscr{A}_K(c)=\lim_{n\to\infty}\mathscr{A}_{K,n}(c)\in\mathbb{Z}_p[G_K]$ annihilating $\mathscr{T}_K$ (cf. Theorem 5.5). Then we have (where each χ is the primitive Dirichlet character associated to the Galois character ψ of G_K):*

$$\mathscr{A}_K(c)=\frac{1}{d}\sum_{\sigma\in G_K}\Big[\sum_{\psi\neq 1}\psi^{-1}(\sigma)(1-\psi(c))\cdot\prod_{\ell\mid f_K,\,\ell\nmid pf_\chi}\Big(1-\frac{\chi(\ell)}{\ell}\Big)\cdot\chi(\Psi_{k_\chi})\Big]\cdot\sigma,$$

with $\Psi_{k_\chi}=-\Big(1-\frac{\chi(p)}{p}\Big)\frac{\tau(\chi)}{f_\chi}\sum_{\tau\in G_{k_\chi}}\log\big(\mathrm{N}_{\mathbb{Q}^{f_\chi}/k_\chi}(1-\zeta_{f_\chi})^\tau\big)\cdot\tau^{-1}.$

Thus, $\mathscr{T}_K$ is annihilated by the ideal $\mathfrak{A}_K$ of $\mathbb{Z}_p[G_K]$ generated by the $\mathscr{A}_K(c)$, $c\in\mathbb{Z}$, prime to $2pf_K$.

Proof. For all Galois character ψ of G_K, Lemma 9.1 leads to the identity:

$$\begin{aligned}\psi(\mathscr{A}_K(c))&=\sum_{\sigma\in G_K}\Lambda_\sigma(c)\cdot\psi(\sigma)\\&=(1-\psi(c))\cdot\prod_{\ell\mid f_K,\,\ell\nmid pf_\chi}\big(1-\chi(\ell)\ell^{-1}\big)\cdot L_p(1,\chi)\\&=(1-\psi(c))\cdot\prod_{\ell\mid f_K,\,\ell\nmid pf_\chi}\big(1-\chi(\ell)\ell^{-1}\big)\cdot\chi(\Psi_{k_\chi})\end{aligned}$$

with $\psi_1(\mathscr{A}_K(c))=0$ for the unit character ψ_1.

Since the matrix $\big(\psi(\sigma)\big)_{\psi,\sigma}$ is invertible with inverse $\frac{1}{d}\Big(\psi^{-1}(\sigma)\Big)_{\sigma,\psi}$, this yields $\Lambda_\sigma(c)=\frac{1}{d}\sum_\psi\psi^{-1}(\sigma)\psi(\mathscr{A}_K(c))=\frac{1}{d}\sum_\psi\psi^{-1}(\sigma)(1-\psi(c))\cdot L_p(1,\psi)$. Whence the result using the expression of $L_p(1,\psi)$ in Lemma 9.1. □

9.4 A cyclic quartic field K of conductor $37\cdot 45161$

We recall from § 8.2 that $m=45161$ is totally ramified in K, that $\ell=37$ splits in the quadratic subfield $k=\mathbb{Q}(\sqrt{m})$ and is ramified in K/k; then $p=5$ totally splits in K. We have $\mathscr{T}_k\simeq\mathbb{Z}/5^5\mathbb{Z}$.

Denote the four characters by ψ_1, ψ_2, ψ_4 & ψ_4^{-1} (orders 1,2, 4, respectively) and let $G_K=\{1,\sigma^2,\sigma,\sigma^{-1}\}$ with σ of order 4. We shall put $\psi_4(\sigma)=i$, and so on by conjugation and the relation $\psi_2=\psi_4^2$.

Then, using the modified Solomon elements Ψ_k, Ψ_K (expressions (8.1), (8.2)):

$$\Psi_k=5^5\cdot u\quad\&\quad\Psi_K=\frac{v}{5}\big(\log(A)+\log(B)\,\sigma\big)(1-\sigma^2),$$

where u and v are p-adic units, A & $B=A^\sigma$ are the two independent units of K of relative norm 1.

We have to compute the coefficients $\psi^{-1}(\sigma)(1-\psi(c))$, which gives the array:

	ψ_1	ψ_2	ψ_4	ψ_4^{-1}
1	0	$1\cdot 2$	$1\cdot(1-i)$	$1\cdot(1+i)$
σ^2	0	$1\cdot 2$	$-1\cdot(1-i)$	$-1\cdot(1+i)$
σ	0	$-1\cdot 2$	$-i\cdot(1-i)$	$i\cdot(1+i)$
σ^{-1}	0	$-1\cdot 2$	$i\cdot(1-i)$	$-i\cdot(1+i)$

Then the terms $\prod_{\ell\mid f_K,\,\ell\nmid pf_\chi}\big(1-\chi(\ell)\ell^{-1}\big)\cdot\chi(\Psi_{k_\chi})$ have the following values, depending on the character ψ in the summation of the theorem:

- $5^5 \cdot u$ for ψ_2, since $1-\chi_2(\ell)\ell^{-1} = 1-37^{-1} \sim 1$,
- $\frac{2v}{5}\big(\log(A)+i\log(B)\big)$ & $\frac{2v}{5}\big(\log(A)-i\log(B)\big)$, for ψ_4 & ψ_4^{-1}.

We obtain, up to a p-adic unit, using the coefficients of the above array:

$$\begin{aligned}
&\mathscr{A}_K(c) = \\
&\Big[\tfrac{v}{5}\big[\log(A)+\log(B)\big]+5^5\cdot u\Big]+\Big[\tfrac{v}{5}\big[-\log(A)-\log(B)\big]+5^5\cdot u\Big]\cdot\sigma^2+ \\
&\Big[\tfrac{v}{5}\big[-\log(A)+\log(B)\big]-5^5\cdot u\Big]\cdot\sigma+\Big[\tfrac{v}{5}\big[\log(A)-\log(B)\big]-5^5\cdot u\Big]\cdot\sigma^{-1} \\
&= 5^5 u\cdot(1-\sigma)(1+\sigma^2) \\
&\qquad\qquad + v\Big[\tfrac{1}{5}\big[\log(A)+\log(B)\big]-\tfrac{1}{5}\big[\log(A)-\log(B)\big]\cdot\sigma\Big]\cdot(1-\sigma^2).
\end{aligned}$$

We give A, one of the two units of relative norm 1 (the other being $B = A^\sigma$):

```
377216797578975495402206020260112295002483855252847326395960961891321756
935656033880097414072613343385538964199960251752277854265043908282068622
071287/424760*x^3 -
863005972214749996449837366815586234260744443520807110375190268414267539
937539821074892103868728835668111842347981799323725052575447796376125480
7708541/7585*x^2 -
301058401703043815651487372068244675606729686675124486738439428208587682
003249385550605088262234049232685807258542997079887400411162925713036023
300228411/11480*x +
137753779960320144069066397981124894126287808388246384703621136571725449
454295610577594731673630502306081901547245942649393930683936045056394190
29007385081/410
```

So it is easy to compute A^4-1, congruent modulo 5^8 to:

$$5\cdot\alpha = 317056x^3+260605x^2+260934x+182595,$$

whence $\log(A)\sim 5\cdot\alpha$. The decompositions into prime ideals of 5 (which is totally split in $K/\mathbb{Q}$) and of $5\cdot\alpha$ give respectively for the 5-places:

```
[[5,[-3,-2,2,2]~,1,1,[3,4,1,1]~]1] [[5,[-3,0,2,-2]~,1,1,[2,0,4,1]~]1]
[[5,[-1,-2,-2,-2]~,1,1,[1,1,1,1]~]1] [[5,[0,-1,-2,2]~,1,1,[2,2,4,1]~]1]

[[5,[-3,-2,2,2]~,1,1,[3,4,1,1]~]2] [[5,[-3,0,2,-2]~,1,1,[2,0,4,1]~]1]
[[5,[-1,-2,-2,-2]~,1,1,[1,1,1,1]~]2] [[5,[0,-1,-2,2]~,1,1,[2,2,4,1]~]1]
```

Dividing by 5, we find that $\frac{1}{5}\log(A)\sim\pi_1\cdot\pi_2$ then $\frac{1}{5}\log(A^\sigma)\sim(\pi_1\cdot\pi_2)^\sigma =: \pi_3\cdot\pi_4$, where the π_i are integers with valuation 1 at the four prime ideals dividing 5; thus the coefficient:

$$\begin{aligned}
U-V\,\sigma &= \tfrac{1}{5}\log(A\,B)-\tfrac{1}{5}\log(A\,B^{-1}) \\
&\sim u\,\pi_1\cdot\pi_2+u'\,\pi_3\cdot\pi_4-(u\,\pi_1\cdot\pi_2-u'\,\pi_3\cdot\pi_4)\cdot\sigma,
\end{aligned}$$

of $1-\sigma^2$ in $\mathscr{A}_K(c)$ is such that:

$$U^2+V^2\equiv 2\,(u^2\,\pi_1^2\cdot\pi_2^2+u'^2\,\pi_3^2\cdot\pi_4^2) \pmod 5$$

is 5-adically invertible. So $\mathscr{A}_K(c)=5^5u(1-\sigma)(1+\sigma^2)+w(1-\sigma^2)$, u,w invertible. This gives the optimal annihilation of both $\mathscr{T}_k$ (since $\mathscr{T}_K = j_{K/k}(\mathscr{T}_k)$), and the relative factor $\mathscr{T}_K^* = 1$, as kernel of the relative norm $1+\sigma^2$ in K/k, since the operation is given by $U-V\,\sigma$ which is invertible.

9.5 A cyclic quartic field K of conductor $2^2\cdot 16212\cdot 677$

Let $K=\mathbb{Q}(x)$ where $x=\sqrt{677\,\frac{1621+39\sqrt{1621}}{2}}$. This field is also defined by $P = x^4-1097417x^2+18573782725$. The conjugates of x are given by:

```
nfgaloisconj(P)=[-x, x, -1/132015*x^3+1571/195*x, 1/132015*x^3-1571/195*x]
```

We still consider the case $p=5$. The prime $\ell=677$ splits in the quadratic subfield $k=\mathbb{Q}(\sqrt{1621})$, the ramified prime 2 does not split in k; the class number of k is 1 and that of K is 4, so we obtain a trivial 5-class group and the following group structures giving, here, a non-trivial relative $\mathscr{T}_K^*$:

$$\mathscr{T}_k \simeq \mathbb{Z}/5^2\mathbb{Z}, \;\; \mathscr{T}_K \simeq \mathbb{Z}/5^2\mathbb{Z} \times \mathbb{Z}/5^3\mathbb{Z}.$$

In k, the cyclotomic unit is the fundamental unit and is given by:

$$\varepsilon = \frac{119806883557}{26403}x^2 - \frac{3042847629386}{39};$$

we compute that $\frac{1}{5}\cdot\log(\varepsilon) \sim 5^2 \sim \Psi_k$ as expected since $\mathscr{T}_k = \mathscr{R}_k$.

The cyclotomic units A and $B=A^{\sigma}$ of K, of relative norm 1, are too large to be given here, but we can work with some representatives modulo a large power of 5. As in the previous example, we have to compute (up to 5-adic units since the Euler factors for 2 and 677 are invertible):

$$\Big[\frac{1}{5}\big[\log(A)+\log(B)\big] - \frac{1}{5}\big[\log(A)-\log(B)\big]\cdot\sigma\Big]\cdot(1-\sigma^2). \tag{9.1}$$

We see that $\log(A)$ is of the form $5\cdot\alpha$, where α is a 5-adic unit, and that $\frac{1}{5}\big[\log(A)-\log(B)\big]$ and $\frac{1}{5}\big[\log(A)+\log(B)\big]$ are 5-adically invertible, so we consider for instance:

$$C := \frac{\log(A)+\log(B)}{\log(A)-\log(B)} \equiv 13\cdot5^2x^3+5^3x^2+19\cdot5^2x+57 \pmod{5^4}$$

and we verify that, despite the denominators 5, $\frac{3}{5}\cdot x^3 - \frac{1}{5}\cdot x$ is an integer of K (congruent to x^{σ} modulo 5 as given by $nfgaloisconj(P)$) so that:

$$C \equiv 5^3\cdot3\cdot\Big(\frac{3}{5}x^3-\frac{1}{5}x+x^2\Big)+57 \pmod{5^4}.$$

Since the exponent of $\mathscr{T}_K$ is 5^3, we obtain that the coefficient $U-V\cdot\sigma$ (in (9.1)) is equal to $(57-\sigma)\cdot(1-\sigma^2)$; thus the whole annihilator is:

$$\mathscr{A}_K(c) \equiv 5^2\cdot u\cdot(1-\sigma)(1+\sigma^2)+v\cdot(57-\sigma)\cdot(1-\sigma^2) \pmod{5^4}.$$

So, on the factor $\mathscr{T}_k$ the annihilator $\mathscr{A}_K(c)$ acts as the order 5^2 of $\mathscr{T}_k$, and on the relative submodule $\mathscr{T}_K^*$, it acts as $57-\sigma$, which is very satisfactory since 57 is of order 4 modulo 5^3 (note that $57^2+1=5^3\cdot26$).

These examples show that $\mathscr{A}_K(c)$ takes into account the whole structure of $\mathscr{T}_K$; but when the Euler factor is not a p-adic unit because of a prime $\ell\equiv1 \pmod{p}$ which splits in k and is ramified in K/k, the annihilation is probably not optimal.

It should be usefull to know if the annihilators, given more recently in the literature, have best properties or not in this point of view, which is not easy since numerical tests are absent (to our knowledge).

9.6 Ideal of annihilation for arbitrary real abelian number fields

We do not make any assumption on p and G_K, nor on the decomposition of the primes $\ell \mid f_K$ in the real abelian extension $K/\mathbb{Q}$.

If $K/\mathbb{Q}$ is cyclic, one can choose c (prime to $2pf_K$) such that for all $\psi\neq1$, $1-\psi(c)$ is non-zero with minimal p-adic valuation; this valuation is 0 as soon as d is not divisible by p, taking $\big(\frac{K}{c}\big)$ as a generator of G_K. Since in the non-cyclic case, this is impossible, we can consider the augmentation ideal $\mathscr{I}_K = \big\langle 1-\big(\frac{K}{c}\big),\, c \text{ prime to } 2pf_K\big\rangle_{\mathbb{Z}[G_K]}$ of G_K and the ideal:

$$\mathscr{I}_K\cdot\mathscr{A}_K$$

which annihilates $\mathscr{T}_K$. It is clear, from Corollary 7.3, that the pseudo-measure $\mathscr{A}_K$ does not depend on $\mathscr{I}_K$ and that any choice of $\delta_K\in\mathscr{I}_K$ is such that $\delta_K\mathscr{A}_K\in\mathbb{Z}_p[G_K]$.

In a p-group G_K of p-rank r, $\delta_K=\sum_{i=1}^r\lambda_i\cdot(1-\sigma_i)$, where the generators σ_i are suitable Artin symbols of integers c_i prime to $2pf_K$; then the characters ψ may be written $\psi=\prod_{i=1}^r\psi_i$, with obvious definition of the ψ_i, so that $\psi(\delta_K)=\sum_{i=1}^r\psi(\lambda_i)\cdot(1-\psi_i(\sigma_i))=\sum_{i=1}^r\psi(\lambda_i)\cdot(1-\xi_i)$, where the ξ_i are roots of unity of p-power order. So we can minimize the p-adic valuations of the $\psi(\delta_K)$ to obtain the best annihilator.

For instance, if K is the compositum of two cyclic cubic fields and $p=3$, whatever the choice of $\delta_K=\lambda_1(1-\sigma_1)+\lambda_2(1-\sigma_2)$, λ_1,λ_2 prime to 3, where σ_1, σ_2 are two generators of G_K, then $\psi(\delta_K)\sim1-j$ for 6 characters and $\psi(\delta_K)\sim3$ for 2 other characters $\psi\neq1$. So the result depends on the structures of the $\mathscr{T}_k$ of the 4 cubic subfields k of K.

Remark 9.5. *(i) Let k be a subfield of K and let $j_{K/k}$ be the "transfer map" $\mathcal{T}_k \to \mathcal{T}_K$. Then, for $\delta_K\mathscr{A}_K$, we get:*

$$(j_{K/k}(\mathcal{T}_k))^{\delta_K\mathscr{A}_K} = j_{K/k}(\mathcal{T}_k^{\mathrm{N}_{K/k}(\delta_K\mathscr{A}_K)}) \simeq \mathcal{T}_k^{\mathrm{N}_{K/k}(\delta_K\mathscr{A}_K)} = \mathcal{T}_k^{\mathscr{E}_k\cdot\delta_k\mathscr{A}_k};$$

indeed, this comes from the injectivity of the transfer since the Leopoldt conjecture is true in abelian extensions (see e.g., [8, Theorem IV.2.1]); then if the product of Euler factors $\mathscr{E}_k := \prod_{\ell\mid f_K,\,\ell\nmid p f_k}\left(1-\frac{1}{\ell}\left(\frac{k}{\ell}\right)\right)$ is invertible (i.e., $\chi(\mathscr{E}_k)$ prime to p for all χ), this means that there is no loss of information by using the annihilation of $\mathcal{T}_K$ by the $\delta_K\mathscr{A}_K$, instead of that of $\mathcal{T}_k$ by the $\delta_k\mathscr{A}_k$; otherwise, it is not possible to eliminate the Euler factors "hidden" in $\delta_K\mathscr{A}_K$ when they are non-invertible (although they are never zero) unless to restrict ourselves to the use of the $\delta_k\mathscr{A}_k$ for $\mathcal{T}_k$, at the cost of a weaker information on the global Galois structure of $\mathcal{T}_K$.

(ii) The G_K-module $\mathcal{T}_K$ gives rise to the following submodules or quotients-modules which have interesting arithmetical meaning and are of course annihilated by the $\delta_K\mathscr{A}_K$: [3]

- *The submodule $\mathscr{C}\ell_K^{\infty} := \mathrm{Gal}(K_\infty H_K/K_\infty)$ isomorphic to a sub-module of $\mathscr{C}\ell_K$. Note that if p is unramified in $K/\mathbb{Q}$ and if (for $p=2$) -1 is not a local norm at 2, then $\mathscr{C}\ell_K^{\infty} \simeq \mathscr{C}\ell_K$ (cf. (2.1)), which explains that, in general, one says that the p-class group is annihilated by the annihilators of $\mathcal{T}_K$.*
- *The module $\mathscr{W}_K$ and the normalized p-adic regulator $\mathscr{R}_K$ defining the exact sequence (2.2).*
- *The Bertrandias–Payan module $\mathscr{B}\mathscr{P}_K := \mathcal{T}_K/\mathscr{W}_K$ for which the fixed field H_K^{bp} by $\mathscr{W}_K$ in $H_K^{\mathrm{pr}}/K_\infty$ is the compositum of the p-cyclic extensions of K which are embeddable in p-cyclic extensions of arbitrary large degree.*

Then some "logarithmic objects" defined and studied by Jaulent (see [16], [17, § 2.3, Schéma] and [3]), in a theoretical and computational point of view:

- *The logarithmic class group $\widetilde{\mathscr{C}\ell}_K := \mathrm{Gal}(H_K^{\mathrm{lc}}/K_\infty)$ (H_K^{lc} is the maximal abelian locally cyclotomic pro-p-extension of K), defining the exact sequence $1 \to \widetilde{\mathscr{C}\ell}_K^{[p]} \to \widetilde{\mathscr{C}\ell}_K \to \mathscr{C}\ell_K^{S\infty} \to 1$ ($\mathscr{C}\ell_K^{S} := \mathscr{C}\ell_K/c\ell_K(S)$ is the p-group of S-classes of K and $\widetilde{\mathscr{C}\ell}_K^{[p]}$ the subgroup generated by S).*
- *The "logarithmic regulator" $\widetilde{\mathscr{R}}_K$ as quotient of the group of "semi-local logarithmic units" by the "global logarithmic units".*

10. Conclusion

This elementary study, especially with the help of numerical computations, shows that the broad generalizations of $\mathbb{Z}_p[G_K]$-annihilations, that come from values of partial ζ-functions, with various base fields (see, e.g., [19, 20, 21, 25] among many others), may be difficult to analyse, owing to the fact that the results are not so efficients (especially in the non semi-simple and/or the non-cyclic cases), and that some degeneracies may occur because of Euler factors as soon as the p-adic pseudo-measures that are used are of "Stickelberger's type" like Solomon's elements or cyclotomic units.

Moreover, Iwasawa's techniques give more elegant formalism but do not avoid the question of Euler factors.

Depending on whether one deals with imaginary or real fields, the suitable object to be annihilated is not defined in an unique way as shown by the context of the present paper about the G_K-module $\mathcal{T}_K$. Moreover, roughly speaking, some objects are relative to the values $L_p(0,\chi)$ (order of some component of the p-class group of some non-real "mirror field"), while some other are relative to the values $L_p(1,\chi)$ (groups $\mathcal{T}_K$), and it is well known that the points "$s=0$" and "$s=1$" are mysteriously independent, giving sometimes abundant "Siegel zeros" near 1, as explained by Washington in many papers (see [11] and its bibliography), whence an unpredictible order of magnitude of the annihilators.

11. Note

All the programs of the paper may be found at:
https://www.dropbox.com/s/jb5nfc3l8gcn630/Georges%20Gras%20–%20Annihilation%20%28programs%29.pdf?dl=0

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