

The sum of the largest and smallest signless Laplacian eigenvalues and some Hamiltonian properties of graphs

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Abstract

The signless Laplacian eigenvalues of a graph G are eigenvalues of the matrix $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of the degrees of the vertices in G and $A(G)$ is the adjacency matrix of G . Using a result on the sum of the largest and smallest signless Laplacian eigenvalues obtained by Das in [2], we in this note present sufficient conditions based on the sum of the largest and smallest signless Laplacian eigenvalues for some Hamiltonian properties of graphs.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. For a graph $G = (V(G), E(G))$, we use n to denote its order $|V(G)|$. A subset V_1 of the vertex set $V(G)$ is independent if no two vertices in V_1 are adjacent in G . The size of a maximum independent set is called the independence number of G and it is denoted by $\alpha(G)$. We use $G_1 \vee G_2$ to denote the the join of two disjoint graphs G_1 and G_2 . The graph consists of p isolated vertices is denoted by pK_1 . Let $D(G)$ be a diagonal matrix such that its diagonal entries are the degrees of vertices in a graph G . The signless Laplacian matrix of a graph G , denoted $Q(G)$, is defined as $D(G) + A(G)$, where $A(G)$ is the adjacency matrix of G . The eigenvalues $q_1(G) \geq q_2(G) \geq \dots \geq q_n(G)$ of $Q(G)$ are called the signless Laplacian eigenvalues of G . A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G . A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G . A graph G is called traceable if G has a Hamiltonian path.

In this note we present sufficient conditions based on the sum of the largest and smallest signless Laplacian eigenvalues for the Hamiltonian and traceable graphs. The main results are as follows.

Theorem 1.1. *Let G be a k -connected graph ($k \geq 2$) of order $n \geq 4$. If $q_1 + q_n \geq 3n - 2k - 4$, then G is Hamiltonian or G is $(k+1)K_1 \vee K_r$ with $2 \leq r \leq k$.*

Theorem 1.2. *Let G be a k -connected ($k \geq 1$) graph of order $n \geq 4$. If $q_1 + q_n \geq 3n - 2k - 6$, then G is traceable or G is $(k+2)K_1 \vee K_r$ with $1 \leq r \leq k$.*

2. Proofs

In order to prove Theorem 1.1 and Theorem 1.2, we need the following result obtained by Das as our lemma. Lemma 2.1 below is Theorem 3.2 on Page 995 in [2].

Lemma 2.1. *Let G be a connected graph on $n \geq 4$ vertices with independence number α . Then $q_1 + q_n + 2\alpha \leq 3n - 2$ with equality holding if and only if G is $\alpha K_1 \vee K_{n-\alpha}$.*

Proof of Theorem 1. Let G be a graph satisfying the conditions in Theorem 1.1. Suppose, to the contrary, that G is not Hamiltonian. Since $k \geq 2$, G has a cycle. Choose a longest cycle C in G and give an orientation on C . Since G is not Hamiltonian, there exists a vertex $u_0 \in V(G) - V(C)$. By Menger's theorem, we can find k pairwise disjoint (except for u_0) paths P_1, P_2, \dots, P_k between u_0 and $V(C)$. Let v_i be the end vertex of P_i on C , where $1 \leq i \leq k$. Without loss of generality, we assume that the appearance of v_1, v_2, \dots, v_k agrees with the orientation of C . We use v_i^+ to denote the successor of v_i along the orientation of C , where $1 \leq i \leq k$. Since C is a longest cycle in G , we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq k$ and the index $k+1$ is regarded as 1. Moreover, $S := \{u_0, v_1^+, v_2^+, \dots, v_k^+\}$ is independent (otherwise G would have cycles which are longer than C). From Lemma 2.1, we have that

$$3n - 2 = 3n - 2k - 4 + 2(k+1) \leq q_1 + q_n + 2|S| \leq q_1 + q_n + 2\alpha \leq 3n - 2.$$

From Lemma 2.1 again, we have that $q_1 + q_n = 3n - 2k - 4$, S is a maximum independent set of size $\alpha = k+1$, and G is $(k+1)K_1 \vee K_{n-(k+1)}$. Notice that G is Hamiltonian if $n - (k+1) \geq (k+1)$. Thus $n - (k+1) \leq k$. Since G is k -connected with $k \geq 2$, G must be $(k+1)K_1 \vee K_r$ with $2 \leq r \leq k$. □

Proof of Theorem 2. Let G be a graph satisfying the conditions in Theorem 1.2. Suppose, to the contrary, that G is not traceable. Choose a longest path P in G and give an orientation on P . Let x and y be the two end vertices of P . Since G is not traceable, there exists a vertex $u_0 \in V(G) - V(P)$. By Menger's theorem, we can find k pairwise disjoint (except for u_0) paths P_1, P_2, \dots, P_k between u_0 and $V(P)$. Let v_i be the end vertex of P_i on P , where $1 \leq i \leq k$. Without loss of generality, we assume that the appearance of v_1, v_2, \dots, v_k agrees with the orientation of P . Since P is a longest path in G , $x \neq v_i$ and $y \neq v_i$, for each i with $1 \leq i \leq k$, otherwise G would have paths which are longer than P . We use v_i^+ to denote the successor of v_i along the orientation of P , where $1 \leq i \leq k$. Since P is a longest path in G , we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq k-1$. Moreover, $\{u_0, v_1^+, v_2^+, \dots, v_k^+, x\}$ is independent (otherwise G would have paths which are longer than P). From Lemma 2.1, we have that

$$3n - 2 = 3n - 2k - 6 + 2(k+2) \leq q_1 + q_n + 2|S| \leq q_1 + q_n + 2\alpha \leq 3n - 2.$$

From Lemma 2.1 again, we have that $q_1 + q_n = 3n - 2k - 6$, S is a maximum independent set of size $\alpha = k+2$, and G is $(k+2)K_1 \vee K_{n-(k+2)}$. Notice that G is traceable if $n - (k+2) \geq (k+1)$. Thus $n - (k+2) \leq k$. Since G is k -connected with $k \geq 1$, G must be $(k+2)K_1 \vee K_r$ with $1 \leq r \leq k$. □

References

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