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Approximation of Modified Jakimovski-Leviatan-Beta Type Operators

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ABSTRACT. In the present paper, we define Jakimovski-Leviatan type modified operators. We study some approximation results for these operators. We also determine the order of convergence in terms of modulus of continuity, Lipschitz functions, Peetre's K-functional, second order modulus of continuity and weighted modulus of continuity.

Keywords: Jakimovski-Leviatan operators, Korovkin's theorem, Modulus of continuity, Rate of convergence, K-functional, Weighted space

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1. INTRODUCTION AND PRELIMINARIES

Appell polynomials were introduced in 1880 (see [4]). In 1969, Jakimovski and Leviatan introduced an operators P_n by using Appell polynomials [7]. The Appell polynomials are defined by the identity as follows:

$$S(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k, \tag{1.1}$$

for an analytic function in the disk $|u| < r$ $(r > 1)$ and $p_n(x) = \sum_{i=0}^{n} a_i \frac{x^{n-i}}{(n-i)!}$ $(n \in \mathbb{N})$ taken $S(u) = \sum_{n=0}^{\infty} a_n u^n$, $S(1) \neq 0$. An exponential type the class of functions considerable on the semi-axis and satisfy the property $|f(x)| \leq \kappa e^{\gamma x}$, for some finite constants κ, $\gamma > 0$ and denote the set of such functions by $E[0, \infty)$. The sequence of infinite sum of the operators P_n is convergent and well-defined which are considered by the authors as follows [7]:

$$P_n(f;x) = \frac{e^{-nx}}{S(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{1.2}$$

for all $n \in \mathbb{N}$, where $n > \frac{\alpha}{\log r}$. In case of $\frac{a_n}{S(1)} \geq 0$ for all $n \in \mathbb{N}$, Wood [20] proved that the operator P_n is positive on $[0;1)$. For more results see also [13], [11] and [6]. They established that $\lim_{n\to\infty} P_n(f;x) \to f(x)$, uniformly in each compact subset of $[0,1)$.

If $S(1) = 1$ in (1.2) we get $p_n(x) = \frac{x^n}{n!}$, and we recover the well-known classical Favard-Szász operators defined in 1950 by

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \tag{1.3}$$

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In the last quarter of twentieth century, the quantum calculus (also known as q- calculus) was studied in [8,12] (see [3,14,15,18]).

2. Construction of Operators and Auxiliary results

In this paper, we define a Beta integral type modification of Jakimovski-Leviatan operators. We also introduce modified Jakimovski-Leviatan-Stancu type operators and obtain better approximation results. For $x \in [0,\infty), \quad p_r(x) \geq 0$ and $S(1) \neq 0$, we define

$$J_n^*(f;x) = \frac{e^{-nx}}{S(1)} \sum_{r=0}^{\infty} P_r(nx) \frac{1}{B(r+1,n)} \int_0^{\infty} \frac{t^r}{(1+t)^{r+n+1}} f(t)\mathrm{d}t, \tag{2.4}$$

Lemma 2.1. *If we take $e_i = t^{i-1}$ for $i = 1,2,3$. Let $J_n^*(\,\cdot\,;\,\cdot\,)$ be the operators given by (2.4). Then for all $x \in [0,\infty), \quad p_r(x) \geq 0$ and $S(1) \neq 0$, we have the following identities:*

(1) $J_n^*(e_1;x) = 1,$
(2) $J_n^*(e_2;x) = \left(\frac{n}{n-1}\right) x + \frac{1}{n-1}\left(\frac{S'(1)}{S(1)} + 1\right),$
(3) $J_n^*(e_3;x) = \frac{n^2}{(n-2)(n-1)} x^2 + \frac{2n}{(n-2)(n-1)}\left(\frac{S'(1)}{S(1)} + 2\right) x + \frac{1}{(n-2)(n-1)}\left(\frac{S''(1)}{S(1)} + \frac{S'(1)}{S(1)} + 2\right).$

Proof. We can easily see that

$$\sum_{r=0}^{\infty} P_r(nx) = S(1)e^{nx}, \tag{2.5}$$

$$\sum_{r=0}^{\infty} rP_r(nx) = \left(S'(1) + nS(1)x\right) e^{nx}, \tag{2.6}$$

$$\sum_{r=0}^{\infty} r^2 P_r(nx) = \left(S''(1) + 2nS'(1)x + S'(1) + n^2 S(1)x^2\right) e^{nx}. \tag{2.7}$$

(1) By taking $f = e_1$

$$\begin{aligned} J_n^*(e_1;x) &= \frac{e^{-nx}}{S(1)} \sum_{r=0}^{\infty} P_r(nx) \frac{1}{B(r+1,n)} \int_0^{\infty} \frac{t^r}{(1+t)^{r+n+1}} \mathrm{d}t, \\ &= \frac{e^{-nx}}{S(1)} \sum_{r=0}^{\infty} P_r(nx) \frac{B(r+1,n)}{B(r+1,n)} \\ &= 1. \end{aligned}$$

(2) By taking $f = e_2$

$$\begin{aligned} J_n^*(e_2;x) &= \frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{1}{B(r+1,n)}\int_0^{\infty}\frac{t^{r+1}}{(1+t)^{r+n+1}}\mathrm{d}t,\\ &= \frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{B(r+2,n-1)}{B(r+1,n)}\\ &= \frac{r+1}{n-1}\frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{B(r+1,n)}{B(r+1,n)}\\ &= \frac{1}{n-1}+\frac{1}{n-1}\frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}rP_r(nx)\\ &= \frac{1}{n-1}+\frac{n}{n-1}\left(x+\frac{1}{n}\frac{S'(1)}{S(1)}\right). \end{aligned}$$

(3) By taking $f=e_3$

$$\begin{aligned} J_n^*(e_2;x) &= \frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{1}{B(r+1,n)}\int_0^{\infty}\frac{t^{r+2}}{(1+t)^{r+n+1}}\mathrm{d}t,\\ &= \frac{1}{(n-2)(n-1)}\frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)(r^2+3r+2)\\ &= \frac{2}{(n-2)(n-1)}+\frac{3}{(n-2)(n-1)}\left(\frac{S'(1)}{S(1)}+nx\right)\\ &+ \frac{1}{(n-2)(n-1)}\left(\frac{S''(1)}{S(1)}+2nx\frac{S'(1)}{S(1)}+\frac{S'(1)}{S(1)}+nx+n^2x^2\right). \end{aligned}$$

□

Lemma 2.2. *Take $\eta_j=(e_i-x)^j$ for $i=2, j=1,2$. Let $J_n^*(\,\cdot\,;\,\cdot\,)$ be the operators given by (2.4). Then for all $x\in[0,\infty)$, $p_r(x)\geq 0$ and $S(1)\neq 0$, we have the following identities:*

1° $J_n^*(\eta_1;x) = \frac{x}{n}+\frac{1}{n-1}\left(\frac{S'(1)}{S(1)}+1\right)$;

2° $J_n^*(\eta_2;x)$
$=\frac{(n+2)}{(n-2)(n-1)}x^2+\frac{2n}{(n-2)(n-1)}\left(\frac{2}{n}\left(\frac{S'(1)}{S(1)}\right)+1\right)x+\frac{1}{(n-2)(n-1)}\left(\frac{S''(1)}{S(1)}+\frac{S'(1)}{S(1)}+2\right)x.$

Let $\alpha,\beta\in\mathbb{R}$ such that $0\leq\alpha<\beta$. Then for $x\in[0,\infty)$, $p_r(x)\geq 0$, and $S(1)\neq 0$, we define

(2.8) $$J_n^{\alpha,\beta}(f;x)=\frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{1}{B(r+1,n)}\int_0^{\infty}\frac{t^r}{(1+t)^{r+n+1}}f\left(\frac{nt+\alpha}{n+\beta}\right)\mathrm{d}t,$$

Lemma 2.3. *Take $e_i=t^{i-1}$ for $i=1,2,3$. Let $J_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ be the operators given by (2.8). Then for all $x\in[0,\infty)$, $p_r(x)\geq 0$ and $S(1)\neq 0$, we have the following identities:*

(1) $J_n^{\alpha,\beta}(e_1;x) = 1$

(2) $J_n^{\alpha,\beta}(e_2;x) = \frac{n^4}{(n+\beta)(n-1)}x+\frac{n}{(n+\beta)(n-1)}\left(\frac{S'(1)}{S(1)}+1\right)+\frac{\alpha}{n+\beta}$

(3) $J_n^{\alpha,\beta}(e_3;x) = \frac{n^2}{(n+\beta)^2(n-2)(n-1)}x^2+\frac{2n^2}{(n+\beta)^2(n-1)}\left\{\frac{n}{n-2}\left(\frac{S'(1)}{S(1)}+2\right)+\alpha\right\}x$
$+\frac{n^2}{(n+\beta)^2(n-2)(n-1)}\left(\frac{S''(1)}{S(1)}+\frac{S'(1)}{S(1)}+2\right)+\frac{2n\alpha}{(n+\beta)^2(n-1)}\left(\frac{S'(1)}{S(1)}+1\right)+\frac{\alpha^2}{(n+\beta)^2}.$

3. Main Results

We obtain the Korovkin type and weighted Korovkin type approximation theorems for the operators defined by (2.8).

Let $C_B[0,\infty)$ be the set of all bounded and continuous functions on $[0,\infty)$, which is a linear normed space with

$$\|f\|_{C_B} = \sup_{x\geq 0} |f(x)|.$$

Let

$$C_\zeta[0,\infty) := \{f \in C[0,\infty) \,:\, |f(t)| \leq M(1+t)^\zeta \text{ for some } M > 0\},$$

and

$$H := \left\{f \in C[0,\infty) : \frac{f(x)}{1+x^2} \quad \text{is convergent as } x \to \infty\right\}.$$

Theorem 3.1. *Let $x \in [0,\infty)$, $\quad f \in C_\zeta[0,\infty) \cap H$ with $\zeta \geq 2$. Then for $p_r(x) \geq 0$, $\quad S(1) \neq 0$, the operators $J_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ defined by (2.8) satisfy*

$$\lim_{n\to\infty} J_n^{\alpha,\beta}(f;x) \to f(x)$$

uniformly on each compact subset of $[0,\infty)$.

Proof. The proof is based on Lemma 2.3 and well known Korovkin's theorem regarding the convergence of a sequence of linear positive operators. So it is enough to prove the conditions

$$\lim_{n\to\infty} J_n^{\alpha,\beta}((e_i;x) = x^{i-1}, \quad i = 1,2,3 \quad \text{as } n \to \infty$$

uniformly on $[0,\infty]$.
Clearly $\frac{1}{n} \to 0$, $\;(n \to \infty)$ we have

$$\lim_{n\to\infty} J_n^{\alpha,\beta}(e_2;x) = x, \quad \lim_{n\to\infty} J_n^{\alpha,\beta}(e_3;x) = x^2.$$

This completes the proof. □

In the space $[0,\infty)$, following Gadžiev [9,10,17], we recall the weighted spaces of the functions for which the analogous of the Korovkin theorem holds (see also [1,5,19]) .

Let $x \to \phi(x)$ be a continuous and strictly increasing function and $\varrho(x) = 1+\phi^2(x)$, $\lim_{x\to\infty} \varrho(x) = \infty$. Let $B_\varrho[0,\infty)$ be a set of functions defined on $[0,\infty)$ and satisfying

$$|f(x)| \leq M_f\,\varrho(x),$$

where M_f is a constant depending only on f . Its subset of continuous functions will be denoted by $C_\varrho[0,\infty)$, i.e., $C_\varrho[0,\infty) = B_\varrho[0,\infty) \cap C[0,\infty)$. It is well known that a sequence of linear positive operators $\{J_n^{\alpha,\beta}\}_{n\geq 1}$ maps $C_\varrho[0,\infty)$ into $B_\varrho[0,\infty)$ if and only if

$$|L_n(\varrho;x)| \leq K\,\varrho(x),$$

where $x \in [0,\infty)$ and K is a positive constant. Note that $B_\varrho[0,\infty)$ is a normed space with the norm

$$\|f\|_\varrho = \sup_{x\geq 0} \frac{|f(x)|}{\varrho(x)}.$$

Finally, let $C_\varrho^0[0,\infty)$ be a subset of $C_\varrho[0,\infty)$ such that the limit

$$\lim_{n\to\infty} \frac{f(x)}{\varrho(x)} = K_f$$

exists and is finite.

Let $B[0,1]$ be the space of all bounded functions on $[0,1]$ and $C[0,1]$ be the space of all functions f continuous on $[0,1]$ equipped with norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|, \quad f \in C[0,1].$$

The famous Korovkin's theorems state as follows:

Theorem 3.2 (cf. [16]). *Let $\{L_n\}_{n\geq 1}$ be the sequence of linear positive operators acting from $C[0,1]$ into $B[0,1]$. Then*

$$\lim_{n\to\infty} \|L_n(t^k; x) - x^k\|_\infty = 0 \; (k = 0, 1, 2),$$

if and only if for all $f \in C[0,1]$

$$\lim_{n\to\infty} \|L_n(f(t); x) - f\|_\infty = 0.$$

Theorem 3.3. *Let $\{J_n^{\alpha,\beta}\}_{n\geq 1}$ be the sequence of linear positive operators acting from $C_\varrho[0,\infty)$ into $B_\varrho[0,\infty)$ satisfies the conditions*

$$\lim_{n\to\infty} \|J_n^{\alpha,\beta}(\varphi^{i-1}(t); x) - \varphi^{i-1}(x)\|_\varrho = 0 \; (i = 1, 2, 3)$$

then for any function $f \in C_\varrho^0[0,\infty)$,

$$\lim_{n\to\infty} \|J_n^{\alpha,\beta}(f(t); x) - f\|_\varrho = 0.$$

Proof. For the completeness, we give some sketch of the proof for the version which will be used in our next result. Consider $\varphi(x) = x, \quad \varrho(x) = 1 + x^2$, and

$$\|J_n^{\alpha,\beta}(e_i; x) - x^{i-1}\|_\varrho = \sup_{x\geq 0} \frac{| J_n^{\alpha,\beta}(e_i; x) - x^{i-1} |}{1 + x^2}.$$

Then for $i = 1, 2, 3$ it is easily proved that

$$\lim_{n\to\infty} \|J_n^{\alpha,\beta}(e_i; x) - x^{i-1}\|_\varrho = 0.$$

Hence by using the above Theorem 3.2, for any function $f \in C_\varrho^0(\mathbb{R}^+)$, we get

$$\lim_{n\to\infty} \|J_n^{\alpha,\beta}(f(t); x) - f\|_\varrho = 0.$$

$\square$

Theorem 3.4. *Let $x \in [0,\infty), \quad f \in C_\varrho^0[0,\infty)$ with $\varrho(x) = 1 + x^2$. Then for $p_r(x) \geq 0, \quad S(1) \neq 0$, we have*

$$\lim_{n\to\infty} \|J_n^{\alpha,\beta}(f; x) - f\|_\varrho \to 0.$$

Proof. Using Theorem 3.3 for $\varphi(x) = x$ and $\varrho(x) = 1 + x^2$, we consider

$$\|J_n^{\alpha,\beta}(e_i; x) - x^{i-1}\|_\varrho = \sup_{x\geq 0} \frac{| J_n^{\alpha,\beta}(e_i; x) - x^{i-1} |}{1 + x^2},$$

for $i = 1, 2, 3$.
According to Lemma 2.3 for $i = 1$, it is obvious that $| J_n^{\alpha,\beta}(e_1; x) - 1 | \to 0$, and therefore

$$\lim_{n\to\infty} \|J_n^{\alpha,\beta}(e_1; x) - 1\|_\varrho = 0.$$

For $i = 2$

$$\sup_{x\geq 0}\frac{|\ J_n^{\alpha,\beta}(e_2;x)-t\ |}{1+x^2} \leq \left|\frac{n^2}{(n+\beta)(n-1)}-1\right|\sup_{x\geq 0}\frac{x}{1+x^2} + \left|\frac{n}{(n+\beta)(n-1)}\left(\frac{S'(1)}{S(1)}+1\right)+\frac{\alpha}{n+\beta}\right|\sup_{x\geq 0}\frac{1}{1+x^2}.$$

Therefore

$$\lim_{n\to\infty}\|J_n^{\alpha,\beta}(e_2;x)-x\|_\varrho = 0.$$

For $i = 3$

$$\begin{aligned}\sup_{x\geq 0}\frac{J_n^{\alpha,\beta}(e_3;x)-x^2\ |}{1+x^2} \leq\ & \left|\frac{n^4}{(n+\beta)^2(n-2)(n-1)}-1\right|\sup_{x\geq 0}\frac{x^2}{1+x^2}\\ +\ & \left|\frac{2n^2}{(n+\beta)^2(n-2)(n-1)}\left\{\frac{n}{n-2}\left(\frac{S'(1)}{S(1)}+2\right)+\alpha\right\}\right|\sup_{x\geq 0}\frac{x}{1+x^2}\\ +\ & \left|\frac{n^2}{(n+\beta)^2(n-2)(n-1)}\left(\frac{S''(1)}{S(1)}+\frac{S'(1)}{S(1)}+2\right)\right.\\ +\ & \left.\frac{2n\alpha}{(n+\beta)^2(n-1)}\left(\frac{S'(1)}{S(1)}\right)+\frac{\alpha^2}{(n+\beta)^2}\right|\sup_{x\geq 0}\frac{1}{1+x^2}.\end{aligned}$$

Hence we have

$$\lim_{n\to\infty}\|J_n^{\alpha,\beta}(e_3;x)-x^2\|_\varrho = 0.$$

Which completes the proof of Korovkin's type theorem. □

4. Rate of Convergence

Here we calculate the rate of convergence of operators (2.8) by means of modulus of continuity and Lipschitz type functions.

Let $f \in C_B[0,\infty]$ be the space of all bounded and uniformly continuous functions on $[0,\infty)$ and $x \geq 0$. Then for $\delta > 0$, the modulus of continuity of f denoted by $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by

(4.9) $$\omega(f,\delta) = \sup_{|t-x|\leq\delta}|f(t)-f(x)\ |, \quad t\in[0,\infty).$$

It is known that $\lim_{\delta\to 0+}\omega(f,\delta) = 0$ for $f \in C_B[0,\infty)$ and for any $\delta > 0$ one has

(4.10) $$|f(t)-f(x)| \leq \left(\frac{|t-x|}{\delta}+1\right)\omega(f,\delta).$$

Take $\mu_j = (e_i - x)^j$ for $i = 2, j = 1,2$ and in the sequel we use the following notations:

(4.11) $$\delta_n^{\alpha,\beta} = \sqrt{J_n^{\alpha,\beta}(\mu_2;x)},$$

Here

$$J_n^{\alpha,\beta}(\mu_j;x) = \begin{cases} \left(\frac{n^2}{(n+\beta)(n-1)} - 1\right)x + \frac{n}{(n+\beta)(n-1)}\left(\frac{S'(1)}{S(1)}+1\right) + \frac{\alpha}{n+\beta} \\ \text{for } j=1,\ 0<\alpha<\beta,\ \ \alpha,\beta\in\mathbb{R} \\ \left(\frac{n^4}{(n+\beta)^2(n-2)(n-1)} - \frac{2n^2}{(n+\beta)(n-1)} + 1\right)x^2 \\ +\left[\frac{2n^2}{(n+\beta)^2(n-1)}\left\{\frac{n}{n-2}\left(\frac{S'(1)}{S(1)}+2\right)+\alpha\right\}\right. \\ \left.-\frac{2n}{(n+\beta)(n-1)}\left(\frac{S'(1)}{S(1)}+1\right) + \frac{2\alpha}{n+\beta}\right]x \\ +\frac{n^2}{(n+\beta)^2(n-2)(n-1)}\left(\frac{S''(1)}{S(1)} + \frac{S'(1)}{S(1)} + 2\right) \\ +\frac{2n\alpha}{(n+\beta)^2(n-1)}\left(\frac{S'(1)}{S(1)}+1\right) + \frac{\alpha^2}{(n+\beta)^2} \\ \text{for } j=2,\ 0<\alpha<\beta,\ \ \alpha,\beta\in\mathbb{R} \end{cases}$$

when $\alpha=\beta=0$, then $\delta_n^{\alpha,\beta}$ is reduced to $\delta_n^* = \sqrt{J_n^*(\eta_2;x)}$.

Theorem 4.5. *For $x\in[0,\infty)$, $f\in C_B[0,\infty)$ the operators $J_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ defined by (2.8) satisfying:*

$$|J_n^{\alpha,\beta}(f;x) - f(x)| \leq 2\omega\left(f;\delta_n^{\alpha,\beta}\right), \tag{4.12}$$

where $n\in\mathbb{N}$, $p_r(x)\geq 0$, $S(1)\neq 0$ and $\delta_n^{\alpha,\beta}$ is defined in (4.11).

Proof. For our sequence of positive linear operators $\{J_n^{\alpha,\beta}(.;.)\}$ we have

$$\begin{aligned} J_n^{\alpha,\beta}(f;x) - f(x) &= J_n^{\alpha,\beta}(f;x) - f(x)J_n^{\alpha,\beta}(1;x) \\ &= J_n^{\alpha,\beta}(f(t)-f(x);x) \\ &\leq J_n^{\alpha,\beta}(|\,f(t)-f(x)\,|;x), \end{aligned}$$

since $J_n^{\alpha,\beta}(1;x)=1$. From (4.9) and (4.10) easily we get

$$\begin{aligned} |J_n^{\alpha,\beta}(f;x) - f(x)| &\leq J_n^{\alpha,\beta}\left(1+\frac{|\,t-x\,|}{\delta};x\right)\omega(f;\delta) \\ &= \left(1+\frac{1}{\delta}J_n^{\alpha,\beta}(|\,t-x\,|;x)\right)\omega(f;\delta). \end{aligned}$$

Cauchy-Schwarz inequality give us

$$J_n^{\alpha,\beta}(|\,t-x\,|;x) \leq J_n^{\alpha,\beta}(1;x)^{\frac{1}{2}} J_n^{\alpha,\beta}\left((t-x)^2;x\right)^{\frac{1}{2}}$$

so that

$$|\,J_n^{\alpha,\beta}(f;x) - f(x)\,| \leq \left(1+\frac{1}{\delta}J_n^{\alpha,\beta}\left((t-x)^2;x\right)^{\frac{1}{2}}\right)\omega(f;\delta). \tag{4.13}$$

Finally, putting $\delta = \delta_n^{\alpha,\beta} = \sqrt{J_n^{\alpha,\beta}(\mu_2;x)}$ we get the assertion. □

Remark 4.1. *Choosing* $\delta = \frac{1}{n+\beta}$ *in* (4.13) *we obtain the following estimate*

$$|J_n^{\alpha,\beta}(f;x) - f(x)| \leq \left(1 + (n+\beta)\delta_n^{\alpha,\beta}\right)\omega\left(f; \frac{1}{n+\beta}\right), \tag{4.14}$$

where δ_n^* *defined in* (4.11).

Remark 4.2. *For* $\alpha = \beta = 0$ *the corresponding estimate for the sequence of positive linear operators* $\{J_n^{\alpha,\beta}\}$ *is reduced to* $\{J_n^*\}$ *defined by* (2.4) *which can take the form as*

$$|J_n^*(f;x) - f(x)| \leq 2\omega\left(f; \delta_n^*\right), \tag{4.15}$$

where $\delta_n^* = \sqrt{J_n^*(\eta_2;x)}$.

Now we give the rate of convergence of the operators $J_n^{\alpha,\beta}(f;x)$ defined in (2.8) in terms of the elements of the usual Lipschitz class $Lip_M(\nu)$. Let $f \in C_B[0,\infty)$, $M > 0$ and $0 < \nu \leq 1$. The class $Lip_M(\nu)$ is defined as

$$Lip_M(\nu) = \left\{f : |f(\zeta_1) - f(\zeta_2)| \leq M|\zeta_1 - \zeta_2|^\nu \quad (\zeta_1, \zeta_2 \in [0,\infty))\right\}. \tag{4.16}$$

Theorem 4.6. *Suppose* $x \in [0,\infty)$, $f \in Lip_M(\nu)$ *with* $(M > 0,\ 0 < \nu \leq 1)$. *Then operators* $J_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ *defined by* (2.8) *satisfying:*

$$|J_n^{\alpha,\beta}(f;x) - f(x)| \leq M\left(\delta_n^{\alpha,\beta}\right)^{\nu/2},$$

where $\delta_n^{\alpha,\beta}$ *is defined in* (4.11).

Proof. Use (4.16) and apply Hölder's inequality

$$\begin{aligned}
|J_n^{\alpha,\beta}(f;x) - f(x)| &\leq |J_n^{\alpha,\beta}(f(t) - f(x);x)| \\
&\leq J_n^{\alpha,\beta}\left(|f(t) - f(x)|;x\right) \\
&\leq MJ_n^{\alpha,\beta}\left(|t-x|^\nu;x\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&|J_n^{\alpha,\beta}(f;x) - f(x)| \\
&\leq M\frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{1}{B(r+1,n)}\int_0^\infty \frac{t^r}{(1+t)^{r+n+1}}|t-x|^\nu \mathrm{d}t \\
&= M\frac{e^{-nx}}{S(1)}\left(\sum_{r=0}^{\infty}P_r(nx)\frac{1}{B(r+1,n)}\right)^{\frac{2-\nu}{2}} \\
&\times \left(P_r(nx)\frac{1}{B(r+1,n)}\right)^{\frac{\nu}{2}}\int_0^\infty \frac{t^r}{(1+t)^{r+n+1}}|t-x|^\nu \mathrm{d}t \\
&\leq M\left(\frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{1}{B(r+1,n)}\int_0^\infty \frac{t^r}{(1+t)^{r+n+1}}\mathrm{d}t\right)^{\frac{2-\nu}{2}} \\
&\times \left(\frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{1}{B(r+1,n)}\int_0^\infty \frac{t^r}{(1+t)^{r+n+1}}|t-x|^2 \mathrm{d}t\right)^{\frac{\nu}{2}} \\
&= MJ_n^{\alpha,\beta}\left(\mu_2;x\right)^{\frac{\nu}{2}}.
\end{aligned}$$

This completes the proof. □

Let

$$C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}, \tag{4.17}$$

with the norm

$$\|g\|_{C_B^2[0,\infty)} = \|g\|_{C_B[0,\infty)} + \|g'\|_{C_B[0,\infty)} + \|g''\|_{C_B[0,\infty)}, \tag{4.18}$$

where

$$\|g\|_{C_B[0,\infty)} = \sup_{x\in[0,\infty)} |\, g(x)|. \tag{4.19}$$

Theorem 4.7. *Let $x \in [0,\infty)$ and $J_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ be the operator defined by (2.8). Then for any $g \in C_B^2[0,\infty)$, we have*

$$|J_n^{\alpha,\beta}(f;x) - f(x)| \leq \frac{1}{2}\delta_n^{\alpha,\beta}(2+\delta_n^{\alpha,\beta})\|g\|_{C_B^2[0,\infty)},$$

where $n \in \mathbb{N}$, $p_r(x) \geq 0$, $S(1) \neq 0$ and $\delta_n^{\alpha,\beta}$ is defined in (4.11).

Proof. Let $g \in C_B^2[0,\infty)$. Then by using the generalized mean value theorem in the Taylor series expansion we have

$$g(t) = g(x) + g'(x)(t-x) + g''(\psi)\frac{(t-x)^2}{2},$$

which follows

$$|g(t) - g(x)| \leq M_1|t-x| + \frac{1}{2}M_2(t-x)^2,$$

where by using the result of (4.18) and (4.19) we have

$$M_1 = \sup_{x\in[0,\infty)} |g'(x)| = \|g'\|_{C_B[0,\infty)} \leq \|g\|_{C_B^2[0,\infty)},$$

$$M_2 = \sup_{x\in[0,\infty)} |g''(x)| = \|g''\|_{C_B[0,\infty)} \leq \|g\|_{C_B^2[0,\infty)},$$

again from 4.18, we have

$$|g(t) - g(x)| \leq \left(|t-x| + \frac{1}{2}(t-x)^2\right)\|g\|_{C_B^2[0,\infty)}.$$

Since

$$|J_n^{\alpha,\beta}(g,x) - g(x)| = |J_n^{\alpha,\beta}(g(t)-g(x);x)| \leq J_n^{\alpha,\beta}(|g(t)-g(x)|;x),$$

and also

$$J_n^{\alpha,\beta}\left(|t-x|;x\right) \leq J_n^{\alpha,\beta}\left((t-x)^2;x\right)^{\frac{1}{2}} = \delta_n^{\alpha,\beta}$$

we get

$$\begin{aligned} |J_n^{\alpha,\beta}(g;x) - g(x)| &\leq \left(J_n^{\alpha,\beta}(|t-x|;x) + \frac{1}{2}J_n^{\alpha,\beta}((t-x)^2;x)\right)\|g\|_{C_B^2[0,\infty)} \\ &\leq \frac{1}{2}\delta_n^{\alpha,\beta}(2+\delta_n^{\alpha,\beta})\|g\|_{C_B^2[0,\infty)}. \end{aligned}$$

This completes the proof. □

The Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf_{C_B^2[0,\infty)} \left\{ \left(\|f-g\|_{C_B[0,\infty)} + \delta \|g''\|_{C_B^2[0,\infty)} \right) : g \in \mathcal{W}^2 \right\}, \tag{4.20}$$

where

$$\mathcal{W}^2 = \left\{ g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty) \right\}. \tag{4.21}$$

There exits a positive constant $C > 0$ such that $K_2(f,\delta) \leq C\omega_2(f,\delta^{1/2}), \quad \delta > 0$, where the second order modulus of continuity is given by

$$\omega_2(f,\delta^{1/2}) = \sup_{0<h<\delta^{1/2}} \sup_{x\in\mathbb{R}^+} |f(x+2h) - 2f(x+h) + f(x)|. \tag{4.22}$$

Theorem 4.8. *Suppose $x \in [0,\infty), n \in \mathbb{N}$ and $f \in C_B[0,\infty)$. Then the operators $J_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ defined by (2.8) satisfying*

$$|J_n^{\alpha,\beta}(f;x) - f(x)| \leq 2M \left\{ \omega_2\left(f; \sqrt{\Delta_n^{\alpha,\beta}}\right) + \min(1, \Delta_n^{\alpha,\beta}) \|f\|_{C_B[0,\infty)} \right\},$$

where M is a positive constant, $p_r(x) \geq 0, \qquad S(1) \neq 0$ and $\Delta_n^{\alpha,\beta} = \frac{(2+\delta_n^{\alpha,\beta})\delta_n^{\alpha,\beta}}{4}$.

Proof. As previous we easily conclude that

$$\begin{aligned} |J_n^{\alpha,\beta}(f;x) - f(x)| &\leq |J_n^{\alpha,\beta}(f-g;x)| + |J_n^{\alpha,\beta}(g;x) - g(x)| + |f(x) - g(x)|, \\ &\leq 2\|f-g\|_{C_B[0,\infty)} + \frac{\delta_n^{\alpha,\beta}}{2}(2+\delta_n^{\alpha,\beta})\|g\|_{C_B^2[0,\infty)}, \\ &\leq 2\left(\|f-g\|_{C_B[0,\infty)} + \frac{\delta_n^{\alpha,\beta}}{4}(2+\delta_n^{\alpha,\beta})\|g\|_{C_B^2[0,\infty)} \right). \end{aligned}$$

By taking infimum over all $g \in C_B^2[0,\infty)$ and by using (4.20), we get

$$|J_n^{\alpha,\beta}(f;x) - f(x)| \leq 2K_2\left(f; \frac{\delta_n^{\alpha,\beta}(2+\delta_n^{\alpha,\beta})}{4}\right).$$

Now for an absolute constant $M > 0$ in [2] we use the relation

$$K_2(f;\delta) \leq M\{\omega_2(f;\sqrt{\delta}) + \min(1,\delta)\|f\|\}.$$

This completes the proof. ∎

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