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On the Lifts of $F(p,-(p-q))$ −Structure Satisfying $F^p - F^{p-q} = 0$, $(F \neq 0; p, q$ odd; $I)$ on Tangent and Cotangent Bundle

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Abstract

This paper consists of three main sections. In the first part, we obtain the complete lifts of the $F(p,-(p-q))$ −structure satisfying $F^p - F^{p-q} = 0$, $(F \neq 0; p, q$ odd; $I)$ on tangent bundle. We have also obtained the integrability conditions by calculating the Nijenhuis tensors of the complete lifts of $F(p,-(p-q))$ −structure. Later, we get the conditions of to be the almost holomorphic vector field with respect to the complete lifts of $F(p,-(p-q))$ −structure. Finally, we obtained the results of the Tachibana operator applied to the vector fields with respect to the complete lifts of $F(p,-(p-q))$ −structure on tangent bundle. In the second part, all results obtained in the first section investigated according to the horizontal lifts of $F(p,-(p-q))$ −structure in tangent bundle $T(M^n)$. In the final section, all results obtained in the first and second section were investigated according to the horizontal lifts of the $F(p,-(p-q))$ −structure in cotangent bundle $T^*(M^n)$.

Keywords: Integrability; Tachibana operators; lifts; Sasakian metric; tangent bundle; cotangent bundle

Tanjant ve Kotanjant Demet Üzerinde F^p-F^(p-q)=0, (F≠0;p,q odd; I) şartını sağlayan F(p,-(p-q))-Yapısının Liftleri

Özet

Bu makale üç ana bölüm içerir. İlk bölüm içerisinde tanjant demet üzerinde $F^p - F^{p-q} = 0$, $(F \neq 0; p, q$ odd; $I)$ şartını sağlayan $F(p,-(p-q))$ −yapısının tam lifleri elde edildi. Ayrıca $F(p,-(p-q))$ −yapısının tam liflerinin Nijenhuis tensörü hesaplanarak integrallenebilme şartları da elde idildi. Daha sonra $F(p,-(p-q))$ −yapısının tam liflerine göre hemen hemen holomorfik olma şartları elde edildi. Son olarak tanjant demet üzerinde $F(p,-(p-q))$ −yapısının tam liftlerine göre vektör alanlarına uygulanan Tachibana operatörünün sonuçları elde edildi. İkinci bölüm içerisinde tanjant demet $T(M^n)$ içerisende $F(p,-(p-q))$ −yapısının yatay liftlerine göre ilk bölüm içerisinde araştırılan tüm sonuçlar elde edilmiştir. En son bölümde ise kotanjant demet $T^*(M^n)$ içersinde $F(p,-(p-q))$ −yapısının yatay liftlere göre ilk ve ikinci bölümde araştırılan tüm sonuçları elde edildi.

Anahtar Kelimeler: İntegrallenebilirlik; Tachibana operatörleri; liftler; Sasaki metriği; tanjant demet; kotanjant demet.

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Introduction

The investigation of the integrability of tensorial structures on manifolds and extension to the tangent or cotangent bundle, whereas the defining tensor field satisfies a polynomial identity has been an active research topic in the last 50 years, initiated by the fundamental works of Kentaro Yano and his collaborators, see for example (Yano & Ishihara, 1973). Also, the idea of F −structure manifold on a differentiable manifold developed by Yano (1963), Ishihara and Yano (1964), Goldberg (1971) and among others. Moreover, Yano and Patterson (1967a , 1967b) studied the horizontal and complete lifts from a differentiable manifold M^n of class C^∞ to its cotangent bundles. Later, Upadhyay and Gupta (1976) obtained some integrability conditions of $F(K,-(K-2))$ −structure, satisfying $F^K - F^{K-2} = 0$, where F is a tensor field of type $(1,1)$.

In 1985, Upadhyay and Grag have obtained some integrability conditions of $F(p,-(p-q))$ −structure satisfying $F^p - F^{p-q} = 0$, $(F \neq 0; p, q$ odd; $I)$, where F is a tensor of type $(1,1)$.

This paper consist of three main sections. In the first part, we obtain the complete lifts of the $F(p,-(p-q))$ −structure satisfying $F^p - F^{p-q} = 0$, $(F \neq 0; p, q$ odd; $I)$ on tangent bundle. We have also obtained the integrability conditions by calculating the Nijenhuis tensors of the complete lifts of $F(p,-(p-q))$ −structure. Later, we get the conditions of to be the almost holomorphic vector field with respect to the complete lifts of $F(p,-(p-q))$ −structure. Finally, we obtained the results of the Tachibana operator applied to the vector fields with respect to the complete lifts of $F(p,-(p-q))$ −structure on tangent bundle.

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Key words and phrases: Integrability; Tachibana operators; lifts; Sasakian metric; tangent bundle; cotangent bundle.

In the second part, all results obtained in the first section investigated according to the horizontal lifts of $F(p,-(p-q))$ −structure in tangent bundle $T(M^n)$. In finally section, all results obtained in the first and second section were investigated according to the horizontal lifts of the $F(p,-(p-q))$ −structure in cotangent bundle $T^*(M^n)$.

Let M^n be n −differentiable manifold of class C^∞, equipped with a $(1,1)$ tensor field F $(F \neq 0, I)$ and of class C^∞ satisfying

$$F^p - F^{p-q} = 0(2\text{rank}F - \text{rank}F^{p-q}) = \dim M^n, \tag{1.1}$$

operators s and t have been defined as follows:

$$s = F^{p-q} : t : I - F^{p-q} \tag{1.2}$$

I denoting identity operator and $p > q$ and q is any odd integral number.
In view of equations (1.1) and (1.2), we have (Upadhyay & Grag, 1985).

$$s^2 = s, t^2 = t \text{ and } s + t = I, \tag{1.3}$$

where k is same integral value such that $kq = p$, i.e.

For a tensor field $F(\neq 0)$ of type $(1,1)$ satisfying (1.1) the operators s and t defined by (1.2), when applied to the tangent space of M^n at a point, are complementary projection operators.

Let S and T complementary distributions corresponding to the projection operators s and t respectively. Let the rank of F be constant and be equal to r, then from (1.1) we have

$$\dim S = (2r - n) \text{ and } \dim T = (2n - 2r) \tag{1.4}$$

Here dimension T is even but dim S is not necessarily even. Obviously $n \leq 2r \leq 2n$. Such a structure has been called a generalised $F(p, -(p-q))$ −structure of rank r and the manifold M^n with this structure a '$\xi(p, -(p-q))$ −manifold'.

In the manifold M^n endowed with $F(p, -(p-q))$ −structure, the (1,1) tensor field ψ given by $\psi = s - t = -I + 2F^{p-q}$ gives an almost product structure (Upadhyay & Grag, 1985).

1.1 Complete Lift of $F(p, -(p-q))$ −Structure on Tangent Bundle

Let M^n be an n −dimensional differentiable manifold of class C^∞ and $T_P(M^n)$ the tangent space at a point p of M^n and

$$T(M^n) = \underset{p\in M^n}{U} T_P(M^n) \tag{1.5}$$

is the tangent bundle over the manifold M^n.

Let us denote by $T_s^r(M^n)$, the set of all tensor fields of class C^∞ and of type (r, s) in M^n and $T(M^n)$ be the tangent bundle over M^n. The complete lift of F^C of an element of $T_1^1(M^n)$ with local components F_i^h has components of the form (Yano & Patterson, 1967b)

$$F^C = \begin{bmatrix} F_i^h & 0 \\ \delta_i^h & F_i^h \end{bmatrix}. \tag{1.6}$$

Now we obtain the following results on the complete lift of F satisfying $F^p - F^{p-q} = 0$, $(F \neq 0; p, q$ odd; $I)$.

Let $F, G \in T_1^1(M^n)$. Then we have (Yano & Patterson, 1967b)

$$(FG)^C = F^C G^C. \tag{1.7}$$

Replacing G by F in (1.7) we obtain

$$(FF)^C = F^C F^C \text{or} (F^2)^C = (F^C)^2. \tag{1.8}$$

Now putting $G = F^4$ in (1.7) since G is (1,1) tensor field therefore F^4 is also (1,1) so we obtain $(FF^4)^C = F^C(F^4)^C$ which in view of (1.8) becomes

$$(F^5)^C = (F^C)^5. \tag{1.9}$$

and so on. Taking complete lift on both sides of equation $F^p - F^{p-q} = 0$ we get

$$(F^p)^C - (F^{p-q})^C = 0$$

which in consequence of equation (1.9) gives

$$(F^C)^p - (F^C)^{p-q} = 0. \tag{1.10}$$

Let F satisfying (1,1) be an F −structure of rank r in M^n. Then the complete lifts $s^C = (F^{p-q})^C$ of s and $t^C = I - (F^{p-q})^C$ of t are complementary projection tensors in $T(M^n)$. Thus there exist in $T(M^n)$ two complementary distributions S^C and T^C determined by s^C and t^C, respectively.

Proposition 1 The (1,1) tensor field $\tilde{\psi}$ given by $\tilde{\psi} = s^C - t^C = -I + 2(F^{p-q})^C$ gives an almost product structure on $T(M^n)$.

Proof. For $s^C = (F^{p-q})^C$ and $t^C = I - (F^{p-q})^C$, we have

$$\begin{aligned}\tilde{\psi}^2 &= I - 4(F^{p-q})^C + 4(F^{2p-2q})^C \\ &= I - 4(F^{p-q})^C + 4(F^p F^{p-2q})^C \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

$$= I - 4(F^{p-q})^C + 4(F^pF^{p-kq})^C$$
$$= I - 4(F^{p-q})^C + 4(F^p)^C(F^{p-p})^C$$
$$= I - 4(F^{p-q})^C + 4(F^{p-q})^C$$
$$= I$$

1.2 Horizontal Lift of $F(p, -(p-q))$ −Structure on Tangent Bundle

Let F_i^h be the component of F at A in the coordinate neighbourhood U of M^n. Then the horizontal lift F^H of F is also a tensor field of type $(1,1)$ in $T(M^n)$ whose components $\tilde{F}_B^A$ in $\pi^{-1}(U)$ are defined as

$$F^H = F^C - \gamma(\nabla F) = \begin{pmatrix} F_i^h & 0 \\ -\Gamma_t^h F_i^t + \Gamma_i^t F_t^h & F_i^h \end{pmatrix}.$$

Let F, G be two tensor fields of type $(1,1)$ on the manifold M. If F^H denotes the horizontal lift of F, we have

$$(FG)^H = F^H G^H. \tag{1.11}$$

Taking F and G identical, we get

$$(F^H)^2 = (F^2)^H. \tag{1.12}$$

Multiplying both sides by F^H and making use of the same (1.12) , we get

$$(F^H)^3 = (F^3)^H$$

Thus it follows that

$$(F^H)^4 = (F^4)^H, (F^H)^5 = (F^5)^H \tag{1.13}$$

and so on. Taking horizontal lift on both sides of equation $F^p - F^{p-q} = 0$ we get

$$(F^p)^H - (F^{p-q})^H = 0$$

view of (1.13), we can write

$$(F^H)^p - (F^H)^{p-q} = 0. \tag{1.14}$$

2. Main Results

2.1 The Nijenhuis Tensor $N_{(F^p)^C(F^p)^C}(X^C, Y^C)$ of the Complete Lift F^p on Tangent Bundle $T(M^n)$

Definition 1 Let F be a tensor field of type $(1,1)$ admitting $\mathrm{F}^{\mathrm{p}} - \mathrm{F}^{\mathrm{p-q}} = 0$, $(\mathrm{F} \neq 0; \mathrm{p}, \mathrm{q}$ odd; I) in M^{n}. The Nijenhuis tensor of a $(1,1)$ tensor field F of M^{n} is given *by*

$$N_F = [FX, FY] - F[X, FY] - F[FX, Y] + F^2[X, Y] \tag{2.1}$$

for any $X, Y \in \Im_0^1(M^n)$ (Çayır, 2015; Salimov, 2013; Salimov & Çayır, 2013). The condition of $N_F(X,Y) = N(X,Y) = 0$ is essential to integrability condition in these structures. The Nijenhuis tensor N_F is defined local coordinates by

$$N_{ij}^k \partial_k = (F_i^s \partial_s^k F_j^k - F_j^l \partial_l F_i^k - \partial_i F_j^l F_l^k + \partial_j F_i^s F_s^k)\partial_k,$$

where $X = \partial_i, Y = \partial_j, F \in \Im_1^1(M^n)$.

Definition 2 Let X and Y be any vector fields on a Riemannian manifold $(\mathrm{M}^{\mathrm{n}}, \mathrm{g})$, we have (Yano & Ishihara, 1973).

$$
\begin{aligned}
&[X^H, Y^H] = [X, Y]^H - (R(X, Y)u)^V, \\
&[X^H, Y^V] = (\nabla_X Y)^V, \\
&[X^V, Y^V] = 0,
\end{aligned}
\tag{2.2}
$$

where R is the Riemannian curvature tensor of g defined by

$$
R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \tag{2.3}
$$

In particular, we have the vertical spray u^V and the horizontal spray u^H on $T(M^n)$ defined by

$$
u^V = u^i(\partial_i)^V = u^i \partial_{\bar{i}}, u^H = u^i(\partial_i)^H = u^i \delta_i, \tag{2.4}
$$

where $\delta_i = \partial_i - u^j \Gamma_{ji}^s \partial_{\bar{s}}$. u^V is also called the canonical or Liouville vector field on $T(M^n)$.

Theorem 1 The Nijenhuis tensor $N_{(F^p)^C(F^p)^C}(X^C, Y^C)$ of the complete lift of F^p vanishes if the Nijenhuis tensor of F^{p-q} is zero.

Proof. In consequence of Definition 1 the Nijenhuis tensor of $(F^p)^C$ is given by

$$
\begin{aligned}
N_{(F^p)^C(F^p)^C}(X^C, Y^C) &= [(F^p)^C X^C, (F^p)^C Y^C] - (F^p)^C[(F^p)^C X^C, Y^C] \\
&\quad -(F^p)^C[X^C, (F^p)^C Y^C] + (F^p)^C(F^p)^C[X^C, Y^C] \\
&= \{[(F^{p-q}X)^C, (F^{p-q}Y)^C] - (F^{p-q})^C[(F^{p-q}X)^C, Y^C] \\
&\quad -(F^{p-q})^C[X^C, (F^{p-q}Y)^C] + (F^{p-q})^C(F^{p-q})^C[X^C, Y^C]\} \\
&= \{[F^{p-q}X, F^{p-q}Y] - F^{p-q}[F^{p-q}X, Y] - F^{p-q}[X, F^{p-q}Y] \\
&\quad +(F^{p-q})^2[X, Y]\}^C \\
&= N_{F^{p-q}}(X, Y)^C
\end{aligned}
$$

Theorem 2 The Nijenhuis tensor $N_{(F^p)^C(F^p)^C}(X^C, Y^V)$ of the complete lift of F^p vanishes if the Nijenhius tensor F^{p-q} is zero.

Proof.

$$
\begin{aligned}
N_{(F^p)^C(F^p)^C}(X^C, Y^V) &= [(F^p)^C X^C, (F^p)^C Y^V] - (F^p)^C[(F^p)^C X^C, Y^V] \\
&\quad -(F^p)^C[X^C, (F^p)^C Y^V] + (F^p)^C(F^p)^C[X^C, Y^V] \\
&= \{[(F^{p-q}X)^C, (F^{p-q}Y)^V] - (F^{p-q})^C[(F^{p-q}X)^C, Y^V] \\
&\quad -(F^{p-q})^C[X^C, (F^{p-q}Y)^V] + ((F^{p-q})^2)^C[X, Y]^V\} \\
&= \{[F^{p-q}X, F^{p-q}Y]^V - (F^{p-q}[F^{p-q}X, Y])^V \\
&\quad -(F^{p-q}[X, F^{p-q}Y])^V - ((F^{p-q})^2[X, Y])^V\} \\
&= N_{F^{p-q}}(X, Y)^V
\end{aligned}
$$

Theorem 3 The Nijenhuis tensor $N_{(F^p)^C(F^p)^C}(X^V, Y^V)$ of the complete lift of F^p vanishes.
Proof. Thus $[X^V, Y^V] = 0$ for all $X, Y \in \Im_0^1(M^n)$, easily we get

$$
N_{(F^p)^C(F^p)^C}(X^V, Y^V) = 0.
$$

2.2 The Purity Conditions of Sasakian Metric with Respect to $(F^p)^C$ on $T(M^n)$

Definition 3 The Sasaki metric ${}^S g$ is a (positive definite) Riemannian metric on the tangent bundle $T(M^n)$ which is derived from the given Riemannian metric on M as follows:

$$
{}^S g(X^H, Y^H) = g(X, Y), \tag{2.5}
$$

$$^{S}g(X^H,Y^V) = ^{S}g(X^V,Y^H) = 0,$$
$$^{S}g(X^V,Y^V) = g(X,Y)$$

for all $X,Y \in \Im_0^1(M^n)$.

Theorem 4 The Sasaki metric ^{S}g is pure with respect to $(F^p)^C$ if $\nabla F^{p-q} = 0$ and $F^{p-q} = I$, where I=identity tensor field of type $(1,1)$.

Proof. $S(\tilde{X},\tilde{Y}) = ^{S}g((F^p)^C\tilde{X},\tilde{Y}) - ^{S}g(\tilde{X},(F^p)^C\tilde{Y})$ if $S(\tilde{X},\tilde{Y}) = 0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form X^V,Y^V or X^H,Y^H then $S = 0$.

$i)$

$$\begin{aligned} S(X^V,Y^V) &= ^{S}g((F^p)^C X^V,Y^V) - ^{S}g(X^V,(F^p)^C Y^V) \\ &= ^{S}g((F^{p-q}X)^V,Y^V) - ^{S}g(X^V,(F^{p-q}Y)^V) \\ &= \left(g(F^{p-q}X,Y)\right)^V - \left(g(X,F^{p-q}Y)\right)^V \end{aligned}$$

$ii)$

$$\begin{aligned} S(X^V,Y^H) &= ^{S}g((F^p)^C X^V,Y^H) - ^{S}g(X^V,(F^p)^C Y^H) \\ &= -^{S}g(X^V,(F^{p-q}Y)^H + (\nabla_\gamma F^{p-q})Y^H) \\ &= -^{S}g\left(X^V,(\nabla_\gamma F^{p-q})Y^H\right) \\ &= -^{S}g\left(X^V,\left(((\nabla F^{p-q})u)Y\right)^V\right) \\ &= -g\left(X,((\nabla F^{p-q})u)Y\right)^V \end{aligned}$$

$iii)$

$$\begin{aligned} S(X^H,Y^H) &= ^{S}g((F^p)^C X^H,Y^H) - ^{S}g(X^H,(F^p)^C Y^H) \\ &= ^{S}g((F^{p-q})^C X^H,Y^H) - ^{S}g(X^H,(F^{p-q})^C Y^H) \\ &= ^{S}g((F^{p-q}X)^H + (\nabla_\gamma F^{p-q})X^H,Y^H) \\ &\quad -^{S}g(X^H,(F^{p-q}Y)^H + (\nabla_\gamma F^{p-q})Y^H) \\ &= g\left((F^{p-q}X),Y\right)^V - g\left(X,(F^{p-q}Y)\right)^V \end{aligned}$$

Definition 4 Let $\varphi \in \Im_1^1(M^n)$, and $\Im(M^n) = \sum_{r,s=0}^{\infty} \Im_s^r(M^n)$ be a tensor alebra over R. A map $\phi_\varphi|_{r+s\rangle 0}: \overset{*}{\Im}(M^n) \to \Im(M^n)$ is called as Tachibana operator or $\phi_\varphi$operator on M^n if

$a)$ ϕ_φ is linear with respect to constant coefficient,

$b)$ $\phi_\varphi: \overset{*}{\Im}(M^n) \to \Im_{s+1}^r(M^n)$ for all r and s,

$c)$ $\phi_\varphi(K \overset{C}{\otimes} L) = (\phi_\varphi K) \otimes L + K \otimes \phi_\varphi L$ for all $K,L \in \overset{*}{\Im}(M^n)$,

$d)$ $\phi_{\varphi X}Y = -(L_Y\varphi)X$ for all $X,Y \in \Im_0^1(M^n)$, where L_Y is the Lie derivation with respect to Y (see Çayır, 2016a, Çayır ve Köseoğlu, 2016, Kobayashi & Nomizu, 1963),

$e)$
$$\begin{aligned} (\phi_{\varphi X}\eta)Y &= (d(\iota_Y\eta))(\varphi X) - (d(\iota_Y(\eta o\varphi)))X + \eta((L_Y\varphi)X) \\ &= \phi X(\iota_Y\eta) - X(\iota_{\varphi Y}\eta) + \eta((L_Y\varphi)X) \end{aligned}$$

for all $\eta \in \Im_1^0(M^n)$ and $X,Y \in \Im_0^1(M^n)$, where $\iota_Y\eta = \eta(Y) = \eta \overset{C}{\otimes} Y, \overset{*}{\Im}{}_s^r(M^n)$ the module of all pure tensor fields of type (r,s) on M^n with respect to the affinor field, $\overset{C}{\otimes}$ is a tensor product with a contraction C (Çayır, 2015; Çayır, 2016b; Salimov, 2013)(see Salimov & Çayır; 2013, for applied to pure tensor field).

Remark 1 If $r = s = 0$, then from c), d) and e) of Definition4 we have $\phi_{\varphi X}(\iota_Y\eta) = \phi X(\iota_Y\eta) - X(\iota_{\varphi Y}\eta)$ for $\iota_Y\eta \in \Im_0^0(M^n)$, which is not a well-defined ϕ_φ −operator. Different choices of Y and η leading to same function $f = \iota_Y\eta$ do get the same values. Consider $M^n = R^2$ with standard coordinates x, y. Let $\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consider the function $f = 1$. This may be written in many different ways as $\iota_Y\eta$. Indeed taking $\eta = dx$, we may choose $Y = \frac{\partial}{\partial_x}$ or $Y = \frac{\partial}{\partial_x} + x\frac{\partial}{\partial_y}$. Now the right-hand side of $\phi_{\varphi X}(\iota_Y\eta) = \phi X(\iota_Y\eta) - X(\iota_{\varphi Y}\eta)$ is $(\phi X)1 - 0 = 0$ in the first case, and $(\phi X)1 - Xx = -Xx$ in the second case. For $X = \frac{\partial}{\partial_x}$, the latter expression is $-1 \neq 0$. Therefore, we put $r + s > 0$ (Salimov, 2013).

Remark 2 From d) of Definition4 we have

$$\phi_{\varphi X}Y = [\varphi X, Y] - \varphi X, Y].$$

By virtue of

$$fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

for any $f, g \in \Im_0^0(M^n)$, we see that $\phi_{\varphi X}Y$ is linear in X, but not Y (Salimov; 2013).

Theorem 5 Let ϕ_φ be the Tachibana operator and the structure $(F^p)^C - (F^{p-q})^C = 0$ defined by Definition 4 and (1.10), respectively. If $L_Y F^{p-q} = 0$, then all results with respect to $(F^p)^C$ are zero, where $X, Y \in \Im_0^1(M)$, the complete lifts $X^C, Y^C \in \Im_0^1(T(M))$ and the vertical lift $X^V, Y^V \in \Im_0^1(T(M))$.

$i)\phi_{(F^p)^C X^C}Y^C = -\left((L_Y F^{p-q})X\right)^C$
$ii)\phi_{(F^p)^C X^C}Y^V = -\left((L_Y F^{p-q})X\right)^V$
$iii)\phi_{(F^p)^C X^V}Y^C = -\left((L_Y F^{p-q})X\right)^V$
$iv)\phi_{(F^p)^C X^V}Y^V = 0$

Proof. $i)$

$$\begin{aligned}\phi_{(F^p)^C X^C}Y^C &= -(L_{Y^C}(F^p)^C)X^C \\ &= -L_{Y^C}(F^{p-q}X)^C + (F^{p-q})^C L_{Y^C}X^C \\ &= -\left((L_Y F)X\right)^C\end{aligned}$$

$ii)$

$$\begin{aligned}\phi_{(F^p)^C X^C}Y^V &= -(L_{Y^V}(F^p)^C)X^C \\ &= -L_{Y^V}(F^p)^C X^C + (F^p)^C L_{Y^V}X^C \\ &= -L_{Y^V}(F^{p-q}X)^C + (F^{p-q})^C L_{Y^V}X^C \\ &= -\left((L_Y F)X\right)^V\end{aligned}$$

$iii)$

$$\begin{aligned}\phi_{(F^p)^C X^V}Y^C &= -(L_{Y^C}(F^p)^C)X^V \\ &= -L_{Y^C}(F^p)^C X^V + (F^p)^C L_{Y^C}X^V \\ &= -L_{Y^C}(F^{p-q}X)^V + (F^{p-q})^C L_{Y^C}X^V\end{aligned}$$

$$= -\big((L_Y F^{p-q})X\big)^V$$

iv)

$$\phi_{(F^p)^C X^V} Y^V = -(L_{Y^V}(F^p)^C)X^V$$
$$= -L_{Y^V}(F^p)^C X^V + (F^p)^C L_{Y^V} X^V$$
$$= 0$$

Theorem 6 If $L_Y F^{p-q} = 0$ for $Y \in M$, then its complete lift Y^C to the tangent bundle is an almost holomorphic vector field with respect to the structure $(F^p)^C - (F^{p-q})^C = 0$.

Proof.

i)

$$(L_{Y^C}(F^p)^C)X^C = L_{Y^C}(F^p)^C X^C - (F^p)^C L_{Y^C} X^C$$
$$= L_{Y^C}(F^{p-q}X)^C - (F^{p-q})^C L_{Y^C} X^C$$
$$= \big((L_Y F^{p-q})X\big)^C$$

ii)

$$(L_{Y^C}(F^p)^C)X^V = L_{Y^C}(F^p)^C X^V - (F^p)^C L_{Y^C} X^V$$
$$= L_{Y^C}(F^{p-q}X)^V - (F^{p-q})^C L_{Y^C} X^V$$
$$= \big((L_Y F^{p-q})X\big)^V$$

2.3 The Structure $(F^p)^H - (F^{p-q})^H = 0$ on Tangent Bundle $T(M^n)$

Theorem 7 The Nijenhuis tensor $N_{(F^p)^H(F^p)^H}(X^H, Y^H)$ of the horizontal lift of F^p vanishes if the Nijenhuis tensor of the F^{p-q} is zero and $\{-(\hat{R}(F^{p-q}X, F^{p-q}Y)u) + (F^{p-q}(\hat{R}(F^{p-q}X, Y)u)) + \big(F^{p-q}(R(X, F^{p-q}Y)u)\big) - ((F^{p-q})^2(\hat{R}(X,Y)u))\}^V = 0$.

Proof.

$$N_{(F^p)^H(F^p)^H}(X^H, Y^H) = [(F^p)^H X^H, (F^p)^H Y^H] - (F^p)^H[(F^p)^H X^H, Y^H]$$
$$-(F^p)^H[X^H, (F^p)^H Y^H] + (F^p)^H (F^p)^H [X^H, Y^H]$$
$$= ([F^{p-q}X, F^{p-q}Y] - (F^{p-q})[F^{p-q}X, Y]$$
$$-(F^{p-q})[X, F^{p-q}Y] - (F^{p-q})(F^{p-q})[X,Y])^H$$
$$-(\hat{R}(F^{p-q}X, F^{p-q}Y)u)^V + (F^{p-q}(\hat{R}(F^{p-q}X, Y)u))^V$$
$$+(F^{p-q}(\hat{R}(X, F^{p-q}Y)u))^V - ((F^{p-q})^2(\hat{R}(X,Y))u)^V$$
$$= \big(N_{F^{p-q}F^{p-q}}(X,Y)\big)^H - (\hat{R}(F^{p-q}X, F^{p-q}Y)u)^V$$
$$+(F^{p-q}(\hat{R}(F^{p-q}X, Y)u))^V + (F^{p-q}(\hat{R}(X, F^{p-q}Y)u))^V$$
$$-((F^{p-q})^2(\hat{R}(X,Y)u))^V.$$

If $N_{F^{p-q}F^{p-q}}(X,Y) = 0$ and $\{-(\hat{R}(F^{p-q}X, F^{p-q}Y)u) + (F^{p-q}(\hat{R}(F^{p-q}X, Y)u)) + \big(F^{p-q}(R(X, F^{p-q}Y)u)\big) - ((F^{p-q})^2(\hat{R}(X,Y)u))\}^V = 0$,
then we get $N_{(F^p)^H(F^p)^H}(X^H, Y^H) = 0$. The theorem is proved.
where $\hat{R}$ denotes the curvature tensor of the affine connection $\hat{\nabla}$ defined by $\hat{\nabla}_X Y = \nabla_Y X + [X,Y]$ (see Yano & Ishihara; 1973, p.88-89).

Theorem 8 The Nijenhuis tensor $N_{(F^p)^H(F^p)^H}(X^H, Y^V)$ of the horizontal lift of F^p vanishes if the Nijenhuis tensor of the F^{p-q} is zero and $\nabla F^{p-q} = 0$.

Proof. $N_{(F^p)^H(F^p)^H}(X^H, Y^V) = [(F^p)^H X^H, (F^p)^H Y^V] - (F^p)^H[(F^p)^H X^H, Y^V]$

$$
\begin{aligned}
&-(F^p)^H[X^H,(F^p)^H Y^V]+(F^p)^H(F^p)^H[X^H,Y^V]\\
&=[F^{p-q}X,F^{p-q}Y]^V-(F^{p-q}[F^{p-q}X,Y])^V\\
&-(F^{p-q}[X,F^{p-q}Y])^V+((F^{p-q})^2[X,Y])^V\\
&+(\nabla_{F^{p-q}Y}F^{p-q}X)^V-\left(F^{p-q}(\nabla_Y F^{p-q}X)\right)^V\\
&-\left(F^{p-q}(\nabla_{F^{p-q}Y}X)\right)^V+((F^{p-q})^2\nabla_Y X)^V\\
&=\left(N_{F^{p-q}F^{p-q}}(X,Y)\right)^V+(\nabla_{F^{p-q}Y}F^{p-q})X\\
&-\left(F^{p-q}\left((\nabla_Y F^{p-q})X\right)\right)^V
\end{aligned}
$$

Theorem 9 The Nijenhuis tensor $N_{(F^p)^H(F^p)^H}(X^V,Y^V)$ of the horizontal lift of F^p vanishes.

Proof. Because of $[X^V,Y^V]=0$ for $X,Y\in M$, easily we get

$$N_{(F^p)^H(F^p)^H}(X^V,Y^V)=0.$$

Theorem 10 The Sasakian metric ${}^S g$ is pure with respect to $(F^p)^H$ if $F^{p-q}=I$, where
$I=$identity tensor field of type $(1,1)$.

Proof. $S(\tilde{X},\tilde{Y})={}^S g((F^p)^H\tilde{X},\tilde{Y})-{}^S g(\tilde{X},(F^p)^H\tilde{Y})$ if $S(\tilde{X},\tilde{Y})=0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form X^V,Y^V or X^H,Y^H then $S=0$.

$i)$

$$
\begin{aligned}
S(X^V,Y^V)&={}^S g((F^p)^H X^V,Y^V)-{}^S g(X^V,(F^p)^H Y^V)\\
&={}^S g((F^{p-q}X)^V,Y^V)-{}^S g(X^V,(F^{p-q}Y)^V)\\
&=\left(g(F^{p-q}X,Y)\right)^V-\left(g(X,F^{p-q}Y)\right)^V\}
\end{aligned}
$$

$ii)$

$$
\begin{aligned}
S(X^V,Y^H)&={}^S g((F^p)^H X^V,Y^H)-{}^S g(X^V,(F^p)^H Y^H)\\
&=-{}^S g(X^V,(F^{p-q}Y)^H)\\
&=0
\end{aligned}
$$

$iii)$

$$
\begin{aligned}
S(X^H,Y^H)&={}^S g((F^p)^H X^H,Y^H)-{}^S g(X^H,(F^p)^H Y^H)\\
&=({}^S g(F^{p-q}X)^H,Y^H)-{}^S g(X^H,(F^{p-q}Y)^H)\\
&=(g(F^{p-q}X),Y)^V-(g(X,(F^{p-q}Y)^H))^V
\end{aligned}
$$

Theorem 11 Let $\phi_\varphi$be the Tachibana operator and the structure $(F^p)^H-(F^{p-q})^H=0$ defined by Definition 4 and (1.14), respectively. if $L_Y F^{p-q}=0$ and $F^{p-q}=I$, then all results with respect to $(F^p)^H$ are zero, where $X,Y\in\Im_0^1(M)$, the horizontal lifts $X^H,Y^H\in\Im_0^1\left(T(M^n)\right)$ and the vertical lift $X^V,Y^V\in\Im_0^1\left(T(M^n)\right)$

$$
\begin{aligned}
&i)\phi_{(F^p)^H X^H}Y^H=\left((L_Y F^{p-q})X\right)^H-(\hat{R}(Y,F^{p-q}X)u)^V\\
&\qquad+(F^{p-q}(\hat{R}(Y,X)u))^V,\\
&ii)\phi_{(F^p)^H X^H}Y^V=-\left((L_Y F^{p-q})X\right)^V+\left((\nabla_Y F^{p-q})X\right)^V,\\
&iii)\phi_{(F^p)^H X^V}Y^H=-\left((L_Y F^{p-q})X\right)^V-(\nabla_{F^{p-q}X}Y)^V+\left(F^{p-q}(\nabla_X Y)\right)^V,\\
&iv)\phi_{(F^p)^H X^V}Y^V=0,
\end{aligned}
$$

Proof. $i)$

$$\begin{aligned}\phi_{(F^p)^H X^H} Y^H &= -(L_{Y^H}(F^p)^H)X^H \\ &= -L_{Y^C}(F^p)^H X^H + (F^p)^H L_{Y^H} X^H \\ &= -[Y, F^{p-q}X]^H + \gamma\hat{R}[Y, F^{p-q}X] \\ &\quad +(F^{p-q}[Y,X])^H - (F^{p-q})^H(\hat{R}(Y,X)u)^V \\ &= \left((L_Y F^{p-q})X\right)^H - (\hat{R}(Y, F^{p-q}X)u)^V \\ &\quad +(F^{p-q}(\hat{R}(Y,X)u))^V,\end{aligned}$$

$ii)$

$$\begin{aligned}\phi_{(F^p)^H X^H} Y^V &= -(L_{Y^V}(F^p)^H)X^H \\ &= -L_{Y^V}(F^p X)^H + (F^p)^H L_{Y^V} X^H \\ &= -[Y, F^{p-q}X]^V + (\nabla_Y F^{p-q}X)^V \\ &\quad +(F^{p-q}[Y,X])^V - \left(F^{p-q}(\nabla_Y X)\right)^V \\ &= -\left((L_Y F^{p-q})X\right)^V + \left((\nabla_Y F^{p-q})X\right)^V\end{aligned}$$

$iii)$

$$\begin{aligned}\phi_{(F^p)^H X^V} Y^H &= -(L_{Y^H}(F^p)^H)X^V \\ &= -L_{Y^H}(F^p X)^V + (F^p)^H L_{Y^H} X^V \\ &= [Y, F^{p-q}X]^V - (\nabla_{F^{p-q}X} Y)^V \\ &\quad +(F^{p-q}[Y,X])^H + \left(F^{p-q}(\nabla_X Y)\right)^V \\ &= -\left((L_Y F^{p-q})X\right)^V - (\nabla_{F^{p-q}X} Y)^V + \left(F^{p-q}(\nabla_X Y)\right)^V\end{aligned}$$

$iv)$

$$\begin{aligned}\phi_{(F^p)^H X^V} Y^V &= -(L_{Y^V}(F^p)^H)X^V \\ &= -L_{Y^V}(F^{p-q}X)^V + (F^{p-q})^H L_{Y^V} X^V \\ &= 0\end{aligned}$$

2.4 The Structure $(F^p)^H - (F^{p-q})^H = 0$ on Cotangent Bundle

In this section, we establish by calculating Nijenhuis tensors of the horizontal lifts of $F(p, -(p-q))$ −structure. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of $F(p, -(p-q))$ −structure in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the lifts of the structure.

Let F, G be two tensor fields of type $(1,1)$ on the manifold M. If F^H denotes the horizontal lift of F, we have (Yano & Ishihara; 1973).

$$F^H G^H + G^H F^H = (FG + GF)^H$$

Taking F and G identical, we get

$$(F^H)^2 = (F^2)^H \tag{2.6}$$

Multiplying both sides by F^H and making use of the same (2.6), we get

$$(F^H)^3 = (F^3)^H$$

and so on. Thus it follows that

$$(F^H)^4 = (F^4)^H \quad \text{and} \quad (F^H)^5 = (F^5)^H \tag{2.7}$$

and so on. Since F gives on M the $F(p, -(p-q))$ −structure, we have

$$F^p - F^{p-q} = 0 \tag{2.8}$$

Taking horizontal lift, we obtain

$$(F^p)^H - (F^{p-q})^H = 0 \tag{2.9}$$

In view of (2.7), we can write

$$(F^H)^p - (F^H)^{p-q} = 0. \tag{2.10}$$

Theorem 12 The Nijenhuis tensor $N_{(F^p)^H (F^p)^H}(X^H, Y^H)$ of the horizontal lift F^p vanishes if $F^{p-q} = I$ on M.

Proof. The Nijenhuis tensor $N(X^H, Y^H)$ for the horizontal lift of F^p is given by

$$\begin{aligned}
N_{(F^p)^H,(F^p)^H}(X^H, Y^H) &= [(F^p)^H X^H, (F^p)^H Y^H] - (F^p)^H[(F^p)^H X^H, Y^H] \\
&\quad -(F^p)^H[X^H, (F^p)^H Y^H] + (F^p)^H (F^p)^H [X^H, Y^H] \\
&= [(F^{p-q})^H X^H, (F^{p-q})^H Y^H] - (F^{p-q})^H[(F^{p-q})^H X^H, Y^H] \\
&\quad -(F^{p-q})^H[X^H, (F^{p-q})^H Y^H] + (F^{p-q})^H (F^{p-q})^H [X^H, Y^H] \\
&= \{[F^{p-q}X, F^{p-q}Y] - F^{p-q}[(F^{p-q}X), Y] - F^{p-q}[X, F^{p-q}Y] \\
&\quad +(F^{p-q})^2[X, Y]\}^H + \gamma\{R(F^{p-q}X, F^{p-q}Y) \\
&\quad -R((F^{p-q}X), Y)F^{p-q} - R(X, F^{p-q}Y)(F^{p-q})^2 \\
&\quad +R(X, Y)(F^{p-q})^2\}
\end{aligned}$$

Let us suppose that $F^{p-q} = I$ on M. Thus, the equation becomes

$$\begin{aligned}
N_{(F^p)^H,(F^p)^H}(X^H, Y^H) &= \{[X, Y] - [X, Y] - [X, Y] + [X, Y]\}^H \\
&\quad +\gamma\{R(X, Y) - R(X, Y) - R(X, Y) + R(X, Y).
\end{aligned}$$

Therefore, it follows

$$N_{(F^p)^H,(F^p)^H}(X^H, Y^H) = 0$$

Theorem 13 The Nijenhuis tensor $N_{(F^p)^H (F^p)^H}(X^H, \omega^V)$ of the horizontal lift F^p vanishes if $\nabla F^{p-q} = 0$.

Proof.

$$\begin{aligned}
N_{(F^p)^H,(F^p)^H}(X^H, \omega^V) &= [(F^p)^H X^H, (F^p)^H \omega^V] - (F^p)^H[(F^p)^H X^H, \omega^V] \\
&\quad -(F^p)^H[X^H, (F^p)^H \omega^V] + (F^p)^H (F^p)^H [X^H, \omega^V] \\
&= (\nabla_{F^{p-q}X}(\omega \circ F^{p-q}))^V - ((\nabla_{F^{p-q}X}) \circ F^{p-q})^V \\
&\quad -((\nabla_X(\omega \circ F^{p-q})) \circ F^{p-q})^V + ((\nabla_X \omega) \circ (F^{p-q})^2)^V \\
&= \{(\omega \circ (\nabla_{F^{p-q}X} F^{p-q}) - (\omega \circ (\nabla_X F^{p-q}) F^{p-q}\}^V
\end{aligned}$$

where $F \in \Im_1^1(M)$, $X \in \Im_0^1(M)$, $\omega \in \Im_1^0(M)$. The theorem is proved.

Theorem 14 *The Nijenhuis tensor $N_{(F^p)^H,(F^p)^H}(\omega^V, \theta^V)$ of the horizontal lift F^p vanishes.*

Proof. Because of $[\omega^V, \theta^V] = 0$ and $\omega \circ F^{p-q} \in \Im_1^0(M^n)$ on $T^*(M^n)$, the equation becomes

$$N_{(F^p)^H,(F^p)^H}(\omega^V, \theta^V) = 0.$$

Theorem 15 Let $(F^p)^H$ be a tensor field of type (1,1) on $T^*(M^n)$. If the Tachibana operator ϕ_φ applied to vector and covector fields according to horizontal lifts of F^p defined by (2.10) on $T^*(M^n)$, then we get the following results.

$$i)\phi_{(F^p)^H X^H} Y^H = -((L_Y F^{p-q})X)^H - (pR(Y, F^{p-q}X))^V + ((pR(Y,X))F^{p-q})^V, \tag{2.11}$$

$$ii)\phi_{(F^p)^H X^H} \omega^V = (\nabla_{F^{p-q}X}\omega)^V - ((\nabla_X \omega) \circ F^{p-q})^V,$$

$$iii)\phi_{(F^p)^H \omega^V} X^H = -(\omega \circ (\nabla_X F^{p-q}))^V,$$

$$iv)\phi_{(F^p)^H \omega^V} \theta^V = 0,$$

where horizontal lifts $X^H, Y^H \in \Im_0^1(T^*(M^n))$ of $X, Y \in \Im_0^1(M^n)$ and the vertical lift $\omega^V, \theta^V \in \Im_0^1(T^*(M^n))$ of $\omega, \theta \in \Im_1^0(M^n)$ are given, respectively.

Proof. $i)$

$$\begin{aligned}\phi_{(F^p)^H X^H} Y^H &= -(L_{Y^H}(F^p)^H)X^H \\ &= -L_{Y^H}(F^p)^H X^H + (F^p)^H L_{Y^H} X^H \\ &= -((L_Y F^{p-q})X)^H - (pR(Y, F^{p-q}X))^V \\ &\quad +((pR(Y,X))F^{p-q})^V\end{aligned}$$

$ii)$

$$\begin{aligned}\phi_{(F^p)^H X^H} \omega^V &= -(L_{\omega^V}(F^p)^H)X^H \\ &= -L_{\omega^V}(F^p)^H X^H + (F^p)^H L_{\omega^V} X^H \\ &= -L_{\omega^V}(F^{p-q}X)^H - (F^{p-q})^H (\nabla_X \omega)^V \\ &= (\nabla_{F^{p-q}X}\omega)^V - ((\nabla_X \omega) \circ F^{p-q})^V,\end{aligned}$$

$iii)$

$$\begin{aligned}\phi_{(F^p)^H \omega^V} X^H &= -(L_{X^H}(F^p)^H)\omega^V \\ &= -(\nabla_X(\omega \circ F^{p-q}))^V + ((\nabla_X \omega) \circ F^{p-q})^V \\ &= -(\omega \circ (\nabla_X F^{p-q}))^V\end{aligned}$$

$iv)$

$$\begin{aligned}\phi_{(F^p)^H \omega^V} \theta^V &= -(L_{\theta^V}(F^p)^H)\omega^V \\ &= -L_{\theta^V}(F^p)^H \omega^V + (F^p)^H L_{\theta^V} \omega^V \\ &= 0\end{aligned}$$

Definition 5 A Sasakian metric ${}^S g$ is defined on $T^*(M^n)$ by the three equations

$${}^S g(\omega^V, \theta^V) = (g^{-1}(\omega, \theta))^V = g^{-1}(\omega, \theta) o \pi, \tag{2.12}$$

$${}^S g(\omega^V, Y^H) = 0, \tag{2.13}$$

$${}^S g(X^H, Y^H) = (g(X,Y))^V = g(X,Y) \circ \pi. \tag{2.14}$$

For each $x \in M^n$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $\pi^{-1}(x) = T_x^*(M^n)$ by

$$g^{-1}(\omega, \theta) = g^{ij}\omega_i \theta_j, \tag{2.15}$$

where $X, Y \in \Im_0^1(M^n)$ and $\omega, \theta \in \Im_1^0(M^n)$. Since any tensor field of type (0,2) on $T^*(M^n)$ is completely determined by its action on vector fields of type X^H and ω^V (see [15], p.280), it follows that ${}^S g$ is completely determined by equations (2.12), (2.13) and (2.14).

Theorem 16 Let $(T^*(M^n), {}^S g)$ be the cotangent bundle equipped with Sasakian metric ${}^S g$ and a tensor field $(F^5)^H$ of type (1,1) defined by (2.11). Sasakian metric ${}^S g$ is pure with respect to $(F^p)^H$ if $F^{p-q} = I$ (I = identity tensor field of type (1,1)).

Proof. We put

$$S(\tilde{X}, \tilde{Y}) = {}^S g((F^p)^H \tilde{X}, \tilde{Y}) - {}^S g(\tilde{X}, (F^p)^H \tilde{Y}).$$

If $S(\tilde{X}, \tilde{Y}) = 0$, for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form ω^V, θ^V or X^H, Y^H, then $S = 0$. By virtue of $(F^p)^H - (F^{p-q})^H = 0$ and (2.12), (2.13), (2.14), we get

$i)$

$$\begin{aligned} S(\omega^V, \theta^V) &= {}^S g((F^p)^H \omega^V, \theta^V) - {}^S g(\omega^V, (F^p)^H \theta^V) \\ &= {}^S g((F^{p-q})^H \omega^V, \theta^V) - {}^S g(\omega^V, (F^{p-q})^H \theta^V) \\ &= ({}^S g((\omega \circ F^{p-q})^V, \theta^V) - {}^S g(\omega^V, (\theta \circ F^{p-q})^V)). \end{aligned}$$

$ii)$

$$\begin{aligned} S(X^H, \theta^V) &= {}^S g((F^p)^H X^H, \theta^V) - {}^S g(X^H, (F^p)^H \theta^V) \\ &= {}^S g((F^{p-q})^H X^H, \theta^V) - {}^S g(X^H, (F^{p-q})^H \theta^V) \\ &= ({}^S g((F^{p-q} X)^H, \theta^V) - {}^S g(X^H, (\omega \circ F^{p-q})^V)) \\ &= 0. \end{aligned}$$

$iii)$

$$\begin{aligned} S(X^H, Y^H) &= {}^S g((F^p)^H X^H, Y^H) - {}^S g(X^H, (F^p)^H Y^H) \\ &= {}^S g((F^{p-q})^H X^H, Y^H) - {}^S g(X^H, (F^{p-q})^H Y^H) \\ &= ({}^S g((F^{p-q} X)^H, Y^H) - {}^S g(X^H, (F^{p-q} Y)^H)). \end{aligned}$$

Thus, $F^{p-q} = I$, then ${}^S g$ is pure with respect to $(F^p)^H$.

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References

Çayır, H. (2015). Some notes on lifts of almost paracontact structures. *American Review of Mathematics and Statistics, 3*(1), 52–60.

Çayır, H. (2016a). Lie derivatives of almost contact structure and almost paracontact structure with respect to X^v and X^h on tangent bundle T(M). *Proceedings of the Institute of Mathematics and Mechanics, 42*(1), 38–49.

Çayır, H. (2016b). Tachibana and Vishnevskii operators applied to X^v and X^h in almost paracontact structure on tangent bundle T(M). *New Trends in Mathematical Sciences, 4*(3), 105–115.

Çayır, H., & Köseoğlu, G. (2016). Lie derivatives of almost contact structure and almost paracontact structure with respect to X^c and X^v on tangent bundle T(M). *New Trends in Mathematical Sciences, 4*(1), 153–159.

Goldberg, S. I., & Yano, K. (1971). Globally framed *f*-manifolds. *Illinois Journal of Mathematics, 15*, 456–476.

Ishihara, S., & Yano, K. (1964). On integrability conditions of a structure *f* satisfying $f^3 + f = 0$. *Quarterly Journal of Mathematics, 15*, 217–222.

Kobayashi, S., & Nomizu, K. (1963). *Foundations of differential geometry* (Vol. 1). John Wiley & Sons.

Salimov, A. A. (2013). *Tensor operators and their applications*. Nova Science Publishers.

Salimov, A. A., & Çayır, H. (2013). Some notes on almost paracontact structures. *Comptes Rendus de l'Académie Bulgare des Sciences, 66*(3), 331–338.

Upadhyay, M. D., & Garg, A. (1985). Integrability conditions of a F(p, −(p − q))-structure satisfying $F^p - F^{p-q} = 0$ (F ≠ 0; p, q odd; I). *The Nepali Mathematical Sciences Report, 10*(2), 71–78.

Upadhyay, M. D., & Gupta, V. C. (1976). Integrability conditions of a F(K, −(K − 2))-structure satisfying $F^K - F^{K-2} = 0$ (F ≠ 0, I). *Revista de la Universidad Nacional de Tucumán, 20*(1–2), 31–44.

Yano, K. (1963). On a structure defined by a tensor field *f* of type (1,1) satisfying $f^3 + f = 0$. *Tensor, 14*, 99–109.

Yano, K., & Ishihara, S. (1973). *Tangent and cotangent bundles*. Marcel Dekker.

Yano, K., & Patterson, E. M. (1967a). Vertical and complete lifts from a manifold to its cotangent bundles. *Journal of the Mathematical Society of Japan, 19*, 91–113.

Yano, K., & Patterson, E. M. (1967b). Horizontal lifts from a manifold to its cotangent bundle. *Journal of the Mathematical Society of Japan, 19*, 185–198