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Elzaki Transform Series Decomposition Method for Solving Bratu Problems

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Abstract

In this paper, the Elzaki transform series decomposition method is used numerically to solve nonlinear Bratu differential equations. The method combines the Elzaki transform, series expansion and Adomian polynomials, and hence converges absolutely to the exact solution. Additionally, two numerical examples were presented to illustrate the efficacy and dependability of the suggested approach. Applications for the work can be found in mathematics as well as a number of other scientific and engineering disciplines. Calculations were done using maple 2018 software.

Keywords: Elzaki transform; Adomian polynomials; Convergence analysis; Bratu equations; Approximate solution.

Bratu Problemlerini Çözmek için Elzaki Dönüşüm Serisi Ayrıştırma Yöntemi

Öz

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Bu çalışmada, doğrusal olmayan Bratu diferansiyel denklemlerini sayısal olarak çözmek için Elzaki dönüşümü seri ayrıştırma yöntemi kullanılmıştır. Uygulanan yöntem Elzaki dönüşümünü, seri açılımını ve Adomian polinomlarını birleştirir ve bu nedenle kesin çözüme mutlak olarak yakınsar. Ayrıca, önerilen yaklaşımın etkinliğini ve güvenilirliğini göstermek için iki sayısal örnek sunulmuştur. Çalışmanın uygulamaları matematikte olduğu kadar bir dizi diğer bilimsel ve mühendislik disiplininde de bulunabilir. Hesaplamalar Maple 2018 yazılımı kullanılarak yapılmıştır.

Anahtar Kelimeler: Elzaki dönüşümü; Adomian polinomları; Yakınsama analizi; Bratu denklemleri; Yaklaşık çözüm.

1. Introduction

Consider the nonlinear Bratu differential equation of the form:

$$\mathbf{V}''(\boldsymbol{\xi}) + \boldsymbol{\gamma}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{V}} = \mathbf{0}; \quad \mathbf{0} < \boldsymbol{\xi} < \mathbf{1}; \quad \boldsymbol{\gamma}, \boldsymbol{\alpha} \in \mathfrak{R} \tag{1}$$

with initial conditions

$$\mathbf{V}(\mathbf{0}) = \mathbf{V}'(\mathbf{0}) = \mathbf{0}\,. \tag{2}$$

Establishing the fundamental theory of thermal combustion, equation (1) formulates a one-dimensional Bratu-type problem by optimizing the solid fuel igniting model. The nonlinear Bratu differential equation is central to modeling thermal combustion, as it represents the delicate balance between heat produced by chemical reactions and heat lost through diffusion. Acting as a simplified yet effective mathematical model, it encapsulates the core dynamics of thermal combustion, including the interaction between exponentially increasing heat generation and diffusive heat loss, which can lead to critical phenomena such as ignition and thermal runaway. The Bratu equation (1), provides a foundational framework for optimizing solid fuel ignition by accurately capturing this balance, enabling engineers to identify safe operating conditions, predict ignition thresholds, and design combustion systems with greater precision. Nonlinear differential equation development has been very beneficial in certain fields such as fluid mechanics, chemical mechanics, and plasma physics [1]. Nonlinear differential equations are fascinating to scientists because they can be solved precisely. Among the numerous domains where the Bratu equation is used are thermal reaction, radiative heat transfer, chemical reactor theory, fuel ignition model of thermal combustion, Chandrasekhar model of universe expansion, and non-deformable material of constant density during the ignition period [1]. The Bratu equation has been applied recently in technical fields, such as the production of nanofibers using electrospinning. Nonlinear elliptic equations with such boundary conditions have been used to explain thermal explosions in combustion theory. The Bratu equation is frequently used as a benchmark to validate the precision and strength of numerical methods. This has led to a great deal of research on Bratu-type differential equations (see [1-16] for more information). Due to their nonlinear nature, several analytical and numerical approaches have been proposed to obtain accurate approximate and exact solutions. One study applied the perturbation method to solve Bratu's type equation, demonstrating that perturbation techniques can effectively approximate solutions of nonlinear differential equations under certain assumptions [1]. Another investigation employed the spline method to obtain numerical solutions of Bratu-type equations, highlighting the capability of spline interpolation in approximating nonlinear boundary value problems [2]. The method of successive differentiation has also been implemented to generate approximate solutions to Bratu and Bratu-type equations by systematically constructing higher-order derivatives of the solution [3]. In addition, the variational iteration method has been utilized to solve Bratu's equation, showing that iterative correction functionals can provide efficient approximations with good convergence behavior [4]. A further contribution introduced a sixth-order Runge–Kutta seven-stage scheme for solving second-order initial value problems related to Bratu-type equations, demonstrating improved numerical accuracy and computational efficiency [5]. Perturbation-based techniques have also been enhanced through the development of a new improved variational homotopy perturbation method, which was applied to generate approximate solutions for Bratu-type problems while improving convergence properties [6]. Similarly, the optimal homotopy analysis method has been used to obtain analytic approximate solutions of the Bratu problem, illustrating the flexibility of homotopy-based analytical approaches [7]. The method of weighted residuals has also been applied for the numerical solution of Bratu problems, providing another framework for approximating nonlinear differential equations [8]. Furthermore, the Adomian decomposition method has been used to treat Bratu-type equations effectively by decomposing nonlinear terms into a rapidly convergent series [9]. Wavelet-based numerical strategies have also been explored, including the Chebyshev wavelets method, which has been employed to solve Bratu's problem and demonstrates the effectiveness of orthogonal wavelet bases in approximating nonlinear differential equations [10]. In the area of integral transforms, a comparative study of the Mohand and Elzaki transforms has been carried out to evaluate their efficiency in solving differential equations [11]. Additional investigations have examined the properties and applications of the Elzaki transform, establishing it as a useful analytical tool for solving a variety of differential models [12]. Recent studies have also proposed alternative analytical techniques for addressing Bratu equations. One approach applied the Taylor series technique to obtain approximate solutions, showing its simplicity and effectiveness in solving nonlinear problems [13]. Another study demonstrated the application of shifted Legendre polynomials for the numerical solution of nonlinear Bratu differential equations, providing improved approximation accuracy [14]. Furthermore, an efficient numerical method for generating closed-form solutions for nonlinear Bratu differential equations has been developed, contributing to the analytical understanding of such nonlinear systems [15]. More recently, the Caputo–Fabrizio fractional Elzaki decomposition technique has been applied to the fractional telegraph equation, combining fractional calculus with transform and decomposition methods to address complex fractional differential models [16].

Although these techniques work well, there remains a big gap in the development of a method that avoids the complexity of the previously stated methods by smoothly combining the simplicity and elegance of series decomposition with the computational efficiency of integral transforms. In particular, the Elzaki Transform has not been widely applied to this category of problems, and its capabilities remain largely unexplored when combined with a simple series-based decomposition technique for solving nonlinear Bratu equation under the given initial conditions. In this work, we obtain the numerical solution of Bratu's nonlinear differential equation (1), subject to the initial conditions (2), by employing the Elzaki Transform Series Decomposition Method (ETSDM). The proposed method demonstrates strong efficiency, and the preliminary results obtained are both promising and dependable. The Numerical solution is expressed as an infinite series that converges accurately to the exact solution.

2. Materials and Methods

2.1. Preliminaries of Elzaki Transform

The Elzaki transform of a function $V(\xi)$ is defined in [12] as

$$\mathfrak{E}[\mathrm{V}(\xi)] = v(s) = s\int_0^\infty \mathrm{V}(\xi)e^{-\frac{\xi}{s}}d\xi$$

where the Elzaki transform operator is represented by $\mathfrak{E}$.

Thus, $\mathrm{V}(\xi)$ is called the inverse Elzaki transform of $v(s)$ and this is written as

$$\mathrm{V}(\xi) = \mathfrak{E}^{-1}[v(s)]$$

Hence, we have the following properties [12]:

$$\mathfrak{E}\{a\mathrm{V}_1(\xi) + b\mathrm{V}_2(\xi)\} = a\mathfrak{E}\{\mathrm{V}_1(\xi)\} + b\mathfrak{E}\{\mathrm{V}_2(\xi)\} \tag{3}$$

$$\mathfrak{E}(\xi^m) = m!\, s^{m+2} \tag{4}$$

$$\mathfrak{E}^{-1}(s^m) = \frac{\xi^{m-2}}{(m-2)!} \tag{5}$$

$$\mathfrak{E}\{V'(\xi)\} = \frac{1}{s}v(s) - sV(0) \tag{6}$$

$$\mathfrak{E}\{V''(\xi)\} = \frac{1}{s^2}v(s) - V(0) - sV'(0) \tag{7}$$

$$\mathfrak{E}\{V^{(m)}(\xi)\} = \frac{1}{s^m}v(s) - \sum_{k=0}^{m-1} s^{2-m+k}V^{(k)}(0) \tag{8}$$

2.2. Elzaki Transform Series Decomposition Method (ETSDM)

Examine the following general nonlinear, nonhomogeneous differential equation:

$$LV(\xi) + RV(\xi) + NV(\xi) = g(\xi) \tag{9}$$

Where L stands for the linear differential operator of highest order, the linear differential operator of order less than L is denoted by R, a series expansion is assumed for the source term g and the non-linear differential operator N.

On both sides of (9), the Elzaki transform is applied to yield:

$$\mathfrak{E}[LV(\xi)] + \mathfrak{E}[RV(\xi)] + \mathfrak{E}[NV(\xi)] = \mathfrak{E}[g(\xi)] \tag{10}$$

By utilizing the Elzaki transform's differentiation property (8) in (10), we can achieve

$$\frac{\mathfrak{E}[V(\xi)]}{s^m} - \sum_{k=0}^{m-1} \frac{V(\xi)^{(k)}(0)}{s^{m-(k+2)}} + \mathfrak{E}[RV(\xi)] + \mathfrak{E}[NV(\xi)] = \mathfrak{E}[g(\xi)] \tag{11}$$

Where

$$\sum_{k=0}^{m-1} \frac{V(\xi)^{(k)}(0)}{s^{m-(k+2)}} = \sum_{k=0}^{m-1} \frac{V(0)^{(k)}(0)}{s^{m-(k+2)}}$$

Simplifying (11) further we have that

$$\mathfrak{E}[V(\xi)] - s^m \sum_{k=0}^{m-1} \frac{V(\xi)^{(k)}(0)}{s^{m-(k+2)}} + s^m(\mathfrak{E}[RV(\xi)] + \mathfrak{E}[NV(\xi)] - \mathfrak{E}[g(\xi)]) = 0 \tag{12}$$

Applying the inverse Elzaki transform on (12) yields

$$V(\xi) = G(\xi) - \mathfrak{E}^{-1}\{s^m(\mathfrak{E}[RV(\xi)] + \mathfrak{E}[NV(\xi)])\} \tag{13}$$

with

$$G(\xi) = \mathfrak{E}^{-1}\left[s^m\left(\sum_{k=0}^{m-1}\frac{\mathrm{V}(\xi)^{(k)}(0)}{s^{m-(k+2)}} + \mathfrak{E}[g(\xi)]\right)\right] \tag{14}$$

Where the term resulting from the source term and the specified initial conditions is indicated by $G(\xi)$.

As an infinite series, (13) has the following solution:

$$\mathrm{V}(\xi) = \sum_{n=0}^{\infty}\mathrm{V}_n(\xi) \tag{15}$$

Decomposing the nonlinear term yields:

$$N\mathrm{V}(\xi) = \sum_{n=0}^{\infty}A_n(\mathrm{V}_0, \mathrm{V}_1, \dots, \mathrm{V}_\mathrm{n}) \tag{16}$$

where A_n denotes the Adomian polynomials, which can be calculated by applying the technique described in [9]:

$$\mathrm{A}_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i\mathrm{V}_i\right)\right]_{\lambda=0}, n = 0,1,2,3,\dots \tag{17}$$

Substituting (15) and (16) into (13) gives

$$\sum_{n=0}^{\infty}\mathrm{V}_{n+1}(\xi) = G(\xi) - \mathfrak{E}^{-1}\{\mathfrak{E}s^m([R\sum_{n=0}^{\infty}\mathrm{V}_n(\xi)] + [\sum_{n=0}^{\infty}A_n])\} \tag{18}$$

The series solution can be obtained by continuously simplifying (18).

$$\left.\begin{aligned} \mathrm{V}_0(\xi) = G(\xi) = \mathfrak{E}^{-1}\left[s^m\left(\sum_{k=0}^{m-1}\frac{\mathrm{V}(\xi)^{(k)}(0)}{s^{m-(k+2)}} + \mathfrak{E}[g(\xi)]\right)\right] \\ \mathrm{V}_{n+1}(\xi) = -\mathfrak{E}^{-1}\{\mathfrak{E}s^m([R\mathrm{V}_n(\xi)] + [A_n])\}: n \geq 0 \end{aligned}\right\} \tag{19}$$

Thus, the series solution is ascertained by applying the recursive relation (19).

$$\mathrm{V}(\xi) = \sum_{n=0}^{\infty}\mathrm{V}_n(\xi) = \mathrm{V}_0(\xi) + \mathrm{V}_1(\xi) + \mathrm{V}_2(\xi) + \mathrm{V}_3(\xi) + \mathrm{V}_4(\xi) + \cdots$$

2.3. Convergence Analysis

A complete normed linear space addressing the convergence of the Elzaki transform series decomposition method (ETSDM) is provided in this section. A sequence of function recurrences is produced by applying the approach to the nonlinear differential equation. It is presumed that the solution to the given problem is located at the limit of this sequence.

Theorem 2.3.1.

Let $X = (C[0, T], \|.\|)$ be a complete normed linear space of all continuous functions endowed with the maximum norm with $\xi \longmapsto \mathrm{V}(\xi)$. Set $\mathrm{V}_{n+1}(\xi) = \mathrm{V}_0(\xi) - \mathfrak{E}^{-1}(\xi)\left[\mathfrak{E}(\xi)s^m[R(\mathrm{V}_n(\xi)) + N(\mathrm{V}_n(\xi))]\right], n > 0$, where $N[\mathrm{V}_n(\xi)] = \sum_{n=0}^{\infty} A_n(\mathrm{V}_0, \mathrm{V}_1, \dots, \mathrm{V}_\mathrm{n})$. Define a mapping $T: X \longrightarrow X$ as follows $T(\xi) \coloneqq \mathrm{V}_0(\xi) - \mathfrak{E}^{-1}(\xi)\left[\mathfrak{E}(\xi)s^m[R(\mathrm{V}_n(\xi)) + N(\mathrm{V}_n(\xi))]\right], n > 0$ and suppose R and N are Lipschitzian mappings such that there exists two Lipschitzian constants $k_1, k_2 < 1$ and s^m with $0 < k < 1$, where $k = (k_1 + k_2)s^m\,; s \in \mathbb{Z}^+, m = -n\ \forall\ n \in \mathbb{N}$. Then the recursive relation (19), has a unique solution.

Proof.

There is at most one solution. Assuming that V^* and V as exact and approximate solutions respectively such that $\mathrm{V}^*(\xi) \neq \mathrm{V}(\xi)$. Since

$$\|R\mathrm{V}^* - R\mathrm{V}\| \leq k_1\|\mathrm{V}^* - \mathrm{V}\|, \forall\ \mathrm{V}^*, \mathrm{V} \in X \tag{20}$$

$$\|N\mathrm{V}^* - N\mathrm{V}\| \leq k_2\|\mathrm{V}^* - \mathrm{V}\|, \forall\ \mathrm{V}^*, \mathrm{V} \in X \tag{21}$$

for all V^* and V as exact and approximate solutions respectively. We want to show the uniqueness of T. To do this, we use the following estimate and the properties mentioned above to get:

$$\begin{aligned}\|T\mathrm{V}^* - T\mathrm{V}\| = \Big|\mathrm{V}_0(\xi) - \mathfrak{E}^{-1}(\xi)\left[\mathfrak{E}(\xi)s^m\left[R({\mathrm{V}^*}_n(\xi)) + N({\mathrm{V}^*}_n(\xi))\right]\right] \\ - \left[\mathrm{V}_0(\xi) - \mathfrak{E}^{-1}(\xi)\left[\mathfrak{E}(\xi)s^m[R(\mathrm{V}_n(\xi)) + N(\mathrm{V}_n(\xi))]\right]\right]\Big|\end{aligned}$$

$$= \Big|-\mathfrak{E}^{-1}(\xi)\left[\mathfrak{E}(\xi)s^m\left[R({\mathrm{V}^*}_n(\xi)) + N({\mathrm{V}^*}_n(\xi))\right]\right] + \mathfrak{E}^{-1}(\xi)\left[\mathfrak{E}(\xi)s^m[R(\mathrm{V}_n(\xi)) + N(\mathrm{V}_n(\xi))]\right]\Big|$$

$$= \Big|\mathfrak{E}^{-1}(\xi)\left[\mathfrak{E}(\xi)s^m[R(\mathrm{V}_n(\xi)) + N(\mathrm{V}_n(\xi))]\right] - \mathfrak{E}^{-1}(\xi)\left[\mathfrak{E}(\xi)s^m\left[R({\mathrm{V}^*}_n(\xi)) + N({\mathrm{V}^*}_n(\xi))\right]\right]\Big|$$

$$\leq \max_{\xi \in [0,T]} \left|\mathfrak{E}^{-1}(\xi)\left[\mathfrak{E}(\xi)s^m[R(\mathrm{V}_n(\xi)) - R({\mathrm{V}^*}_n(\xi))]\right] + \mathfrak{E}^{-1}(\xi)\left[\mathfrak{E}(\xi)s^m[N(\mathrm{V}_n(\xi)) - N({\mathrm{V}^*}_n(\xi))]\right]\right|$$

$$= \max_{\xi \in [0,T]} |\mathfrak{E}^{-1}(\xi)[\mathfrak{E}(\xi)s^m]|\|R(\mathrm{V}_n(\xi)) - R({\mathrm{V}^*}_n(\xi))\| + |\mathfrak{E}^{-1}(\xi)[\mathfrak{E}(\xi)s^m]|\|N(\mathrm{V}_n(\xi)) - N({\mathrm{V}^*}_n(\xi))\|$$

$$\leq \max_{\xi \in [0,T]} (k_1 + k_2)|\mathfrak{E}^{-1}(\xi)[\mathfrak{E}(\xi)s^m]|\|\mathrm{V}_n(\xi) - {\mathrm{V}^*}_n(\xi)\|$$

$$\leq \max_{\xi\in[0,T]} (k_1+k_2)s^m \|V_n(\xi) - V^*{}_n(\xi)\|$$

$\leq \max_{\xi\in[0,T]} (k_1+k_2)s^m \|V^*(\xi) - V(\xi)\|$, since $0 < k < 1$, we get the contradiction $\|V^*(\xi) - V(\xi)\| = \|V(\xi) - V^*(\xi)\| \Rightarrow V(\xi) = V^*(\xi)$.

This completes the proof.

Theorem 2.3.2.

Let $X = (C[0,T], \|.\|)$ be a complete normed linear space of all continuous functions endowed with the maximum norm with $\xi \longmapsto V(\xi)$. Set $V_{n+1}(\xi) = V_0(\xi) - \mathfrak{E}^{-1}(\xi)\big[\mathfrak{E}(\xi)s^m[R(V_n(\xi)) + N(V_n(\xi))]\big], n > 0$, where $N[V_n(\xi)] = \sum_{n=0}^{\infty} A_n(V_0, V_1, \dots, V_n)$. Define a mapping $T: X \longrightarrow X$ as follows $T(\xi) \coloneqq V_0(\xi) - \mathfrak{E}^{-1}(\xi)\big[\mathfrak{E}(\xi)s^m[R(V_n(\xi)) + N(V_n(\xi))]\big], n > 0$ and suppose R and N are Lipschitzian mappings such that there exists two Lipschitzian constants $k_1, k_2 < 1$ and s^m with $0 < k < 1$, where $k = (k_1+k_2)s^m$; $s \in \mathbb{Z}^+, m = -n \; \forall \, n \in \mathbb{N}$. Then the recursive relation (19), is convergent.

Proof.

To prove that recursive relation (19) is convergent, we will show that for any given point $m^* \in X$, the n^{th} partial sum of the series $\sum_{k=0}^{n} V_k(\xi)$ is Cauchy. To see this, let $M_n = \sum_{k=0}^{n} W_k(\xi)$ be the n^{th} partial sum of the series in (19).

Now

$$\|M_n - M_m\| = \max_{\xi\in[0,T]} |M_n - M_m| = \max_{\xi\in[0,T]} \left|\sum_{k=m+1}^{n} V_k(\xi)\right|$$

$$\|M_n - M_m\| = \max_{\xi\in[0,T]} \left|\sum_{k=m+1}^{n} (V_0(\xi) - \mathfrak{E}^{-1}(\xi)\big[\mathfrak{E}(\xi)s^m[R(V_{k-1}(\xi)) + N(V_{k-1}(\xi))]\big]\right|$$

$$\leq \max_{\xi\in[0,T]} \left|\mathfrak{E}^{-1}(\xi)\big[\mathfrak{E}(\xi)s^m[R(\textstyle\sum_{k=m+1}^{n} V_{k-1}(\xi))]\big] + \mathfrak{E}^{-1}(\xi)\big[\mathfrak{E}(\xi)s^m[N(\textstyle\sum_{k=m+1}^{n} V_{k-1}(\xi))]\big]\right|$$

$$\leq \max_{\xi\in[0,T]} \left|\mathfrak{E}^{-1}(\xi)\big[\mathfrak{E}(\xi)s^m[\textstyle\sum_{k=m+1}^{n} R(M_{n-1}) - R(M_{m-1})]\big] + \mathfrak{E}^{-1}(\xi)\big[\mathfrak{E}(\xi)s^m[\textstyle\sum_{k=m+1}^{n} N(M_{n-1}) - N(M_{m-1})]\big]\right|$$

$$\leq \max_{\xi\in[0,T]} \left|\mathfrak{E}^{-1}(\xi)\big[\mathfrak{E}(\xi)s^m[R(M_{n-1}) - R(M_{m-1})]\big] + \mathfrak{E}^{-1}(\xi)\big[\mathfrak{E}(\xi)s^m[N(M_{n-1}) - N(M_{m-1})]\big]\right|$$

$$\leq \max_{\xi\in[0,T]} (k_1+k_2)[\mathfrak{E}^{-1}(\xi)[\mathfrak{E}(\xi)s^m\|M_{n-1} - M_{m-1}\|]]$$

$$\leq (k_1 + k_2)s^m \|M_{n-1} - M_{m-1}\|$$

$$= k\|M_{n-1} - M_{m-1}\|.$$

Let $n = m + 1$, we have by induction that $\|M_{m+1} - M_m\| = (k)^m \|M_1 - M_0\|$ and this is a contraction.

Since $0 < k < 1$, it results that $(k)^m \to 0$ as $m \to \infty$.

In the sequel,

$$\|M_{m+1} - M_m\| = |k|^m \|M_1 - M_0\|$$

Showing that $\{M_m\}_{m=0}^{\infty}$ is a Cauchy sequence.

Since $X = (C[O,T], \|.\|)$ is a complete normed linear space, then $\{M_m\}_{m=0}^{\infty}$ converges to some $m^* \in X$.

3. Numerical Application

This section demonstrates how the suggested methodology was applied to solve two Bratu-type problems. Results demonstrate the effectiveness and dependability of the suggested approach.

Example 3.1. [1]: Consider the Bratu-type problem (1), by choosing $\gamma = -2$ and $\alpha = 1$ to obtain:

$$\frac{d^2V}{d\xi^2} - 2e^V = 0, \quad 0 < \xi < 1, \tag{22}$$

$$V(0) = V'(0) = 0$$

The closed form solution of the system (22), is given as:

$$V^*(\xi) = -2\ln(\cos\xi)$$

Solution.

Applying the Elzaki transform on both sides of (22), gives:

$$\mathfrak{E}\left(\frac{d^2V}{d\xi^2} - 2e^V\right) = \mathfrak{E}(0)$$

Using (7), (16), (19) and the given initial conditions gives the following:

$$V_{n+1} = 2\mathfrak{E}^{-1}[s^2\mathfrak{E}(A_n)] : n \geq 0$$

For $n = 0$

$$V_1 = 2\mathfrak{E}^{-1}[s^2\mathfrak{E}(A_0)] = 2\mathfrak{E}^{-1}\left[s^2\mathfrak{E}\left(e^{V_0}\right)\right] = \xi^2 \quad (V_0 = 0)$$

For $n = 1$

$$V_2 = 2\mathfrak{E}^{-1}[s^2\mathfrak{E}(A_1)] = 2\mathfrak{E}^{-1}\left[s^2\mathfrak{E}\left(V_1e^{V_0}\right)\right] = \frac{\xi^4}{6}$$

For $n = 2$

$$V_3 = 2\mathfrak{E}^{-1}[s^2\mathfrak{E}(A_2)] = 2\mathfrak{E}^{-1}\left[s^2\mathfrak{E}\left(e^{V_0}\left[V_2 + \frac{1}{2!}{V_1}^2\right]\right)\right] = \frac{2\xi^6}{45}$$

For $n = 3$

$$V_4 = 2\mathfrak{E}^{-1}[s^2\mathfrak{E}(A_3)] = 2\mathfrak{E}^{-1}\left[s^2\mathfrak{E}\left(e^{V_0}\left[V_3 + V_1V_2 + \frac{1}{3!}{V_1}^3\right]\right)\right] = \frac{17\xi^8}{1260}$$

Therefore, the approximative solution is provided as $V(\xi) = \xi^2 + \frac{\xi^4}{6} + \frac{2\xi^6}{45} + \frac{17\xi^8}{1260} + \cdots$ which is the same as the exact solution.

Example 3.2. [1]: Consider the Bratu-type problem (1), by choosing $\gamma = -1$ and $\alpha = 2$ to obtain:

$$\frac{d^2V}{d\xi^2} - e^{2V} = 0, \quad 0 < \xi < 1, \tag{23}$$

$$V(0) = V'(0) = 0$$

The closed form solution of the system (23), is given as:

$$V^*(\xi) = -\ln(\cos\xi)$$

Solution.

Applying the Elzaki transform on both sides of (23), gives:

$$\mathfrak{E}\left(\frac{d^2V}{d\xi^2} - e^{2V}\right) = \mathfrak{E}(0)$$

Using (7), (16), (19) and the initial conditions gives the following:

$$V_{n+1} = \mathfrak{E}^{-1}[s^2\mathfrak{E}(A_n)] : n \geq 0$$

For $n = 0$

$$V_1 = \mathfrak{E}^{-1}[s^2\mathfrak{E}(A_0)] = \mathfrak{E}^{-1}[s^2\mathfrak{E}(e^{2V_0})] = \frac{\xi^2}{2} \ (V_0 = 0)$$

For $n = 1$

$$V_2 = \mathfrak{E}^{-1}[s^2\mathfrak{E}(A_1)] = \mathfrak{E}^{-1}[s^2\mathfrak{E}(2V_1e^{2V_0})] = \frac{\xi^4}{12}$$

For $n = 2$

$$V_3 = \mathfrak{E}^{-1}[s^2\mathfrak{E}(A_2)] = \mathfrak{E}^{-1}\left[s^2\mathfrak{E}\left(e^{2V_0}\left[2V_2 + \frac{1}{2!}4{V_1}^2\right]\right)\right] = \frac{\xi^6}{45}$$

For $n = 3$

$$V_4 = \mathfrak{E}^{-1}[s^2\mathfrak{E}(A_3)] = \mathfrak{E}^{-1}\left[s^2\mathfrak{E}\left(e^{V_0}\left[2V_3 + 4V_1V_2 + \frac{1}{3!}8{V_1}^3\right]\right)\right] = \frac{17\xi^8}{2520}$$

Therefore, the approximative solution is provided as

$V(\xi) = \frac{\xi^2}{2} + \frac{\xi^4}{12} + \frac{\xi^6}{45} + \frac{17\xi^8}{2520} + \cdots$ which is the same as the exact solution.

4. Numerical results

The proposed method is applied for different values of γ and α, the numerical results are compared with the method available in the literature and presented in tabular form as follows:

Table 1: Comparison of numerical outcomes for example 3.1. **(**$\gamma = -2$ and $\alpha = 1$**).**

ξ	Exact solution	Approximate solution	Absolute Error by the proposed method	Absolute error by Perturbation Method [1]
0.1	0.01001671124	0.01001671124	0.00000000e-00	1.117×10^{-8}

0.2	0.04026954565	0.04026954565	0.00000000e-00	7.2635×10^{-7}
0.3	0.09138328521	0.09138328521	0.00000000e-00	0.00000849054
0.4	0.1644575533	0.1644575533	0.00000000e-00	0.0000494120
0.5	0.2611638145	0.2611638145	0.00000000e-00	0.0001968626
0.6	0.3839002149	0.3839002149	0.00000000e-00	0.0006183771
0.7	0.5360233017	0.5360233017	0.00000000e-00	0.0016503540
0.8	0.7221811037	0.7221811037	0.00000000e-00	0.0039113549
0.9	0.9487774909	0.9487774909	0.00000000e-00	0.0084672048

Table 2: Comparison of numerical outcomes for example 3.2. ($\gamma = -1$ and $\alpha = 2$).

ξ	Exact solution	Approximate solution	Absolute Error by the proposed method	Absolute error by Perturbation Method [1]
0.1	0.00500835562	0.00500835562	0.00000000e-00	2.3386×10^{-8}
0.2	0.02013477282	0.02013477282	0.00000000e-00	0.00000149142
0.3	0.04569164261	0.04569164261	0.00000000e-00	0.00001688860
0.4	0.08222877663	0.08222877663	0.00000000e-00	0.00009410781
0.5	0.1305819072	0.1305819072	0.00000000e-00	0.0003551506
0.6	0.1919501074	0.1919501074	0.00000000e-00	0.0010464518
0.7	0.2680116508	0.2680116508	0.00000000e-00	0.0025969723
0.8	0.3610905518	0.3610905518	0.00000000e-00	0.0056791045
0.9	0.4743887455	0.4743887455	0.00000000e-00	0.0112663280

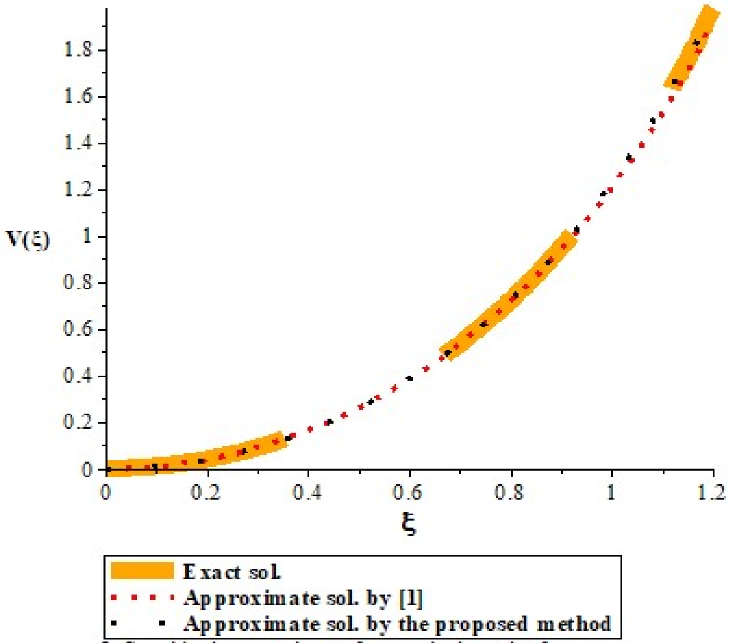

Figure 1: Graphical comparison of numerical results for example 3.1

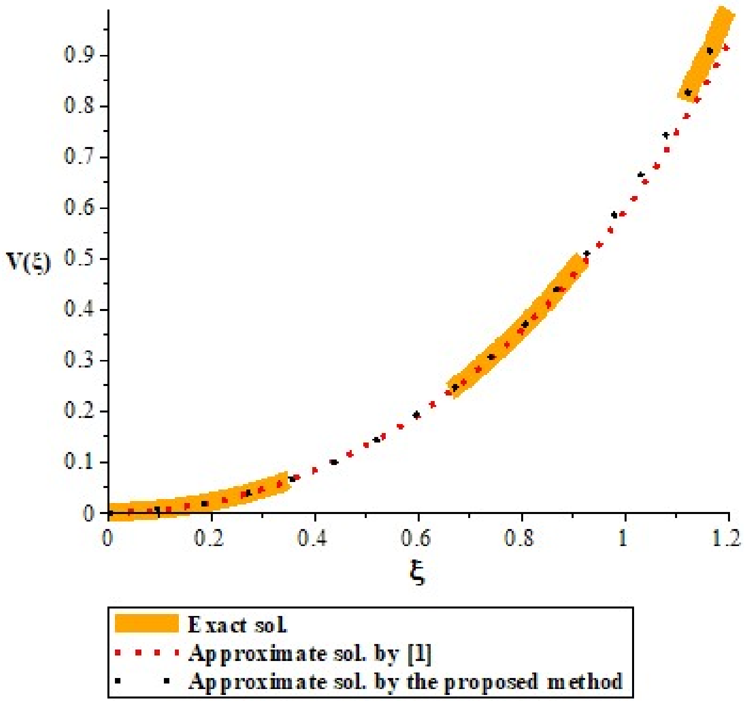

Figure 2: Graphical comparison of numerical results for example 3.2

5. Discussion of Results

The numerical results presented in Tables 1 and 2 clearly demonstrate the effectiveness and reliability of the Elzaki Transform Series Decomposition Method (ETSDM) in solving nonlinear Bratu differential equations. For both Example 3.1 and Example 3.2, the approximate solutions obtained through the proposed method coincide with the exact analytical solutions for all considered values of ξ in the interval $0 \leq \xi \leq 0.9$. The computed absolute errors using ETSDM are zeros, indicating that the truncated series generated by the method converges very rapidly to the exact solution. This high level of accuracy confirms the theoretical convergence results established in section 2.3 and highlights the strong capability of the method in handling nonlinear exponential terms that commonly appear in Bratu-type equations. A comparison with the perturbation method reported in the literature further emphasizes the superiority of the proposed technique. As shown in Tables 1 and 2, the perturbation method produces small but noticeable errors that increase gradually as $\xi \to 1$. For instance, in Example 3.1, the perturbation error grows from 1.117×10^{-8} at $\xi = 0.1$ to approximately 8.47×10^{-3} at $\xi = 0.9$. A similar trend is observed in Example 3.2, where the perturbation error increases from 2.3386×10^{-8} to approximately 1.13×10^{-2}. In contrast, the ETSDM maintains negligible errors across the entire computational domain, demonstrating significantly improved accuracy and numerical stability.

Another important observation from the results is that only a few terms of the series expansion are required to obtain highly accurate approximations. In the numerical examples considered, the first four components of the decomposition series are sufficient to reproduce the exact power-series representation of the solution. This indicates that the ETSDM has strong convergence properties and can significantly reduce computational effort compared with some traditional numerical techniques such as spline methods, weighted residual methods, or high-order Runge–Kutta schemes, which often require finer discretization or iterative correction procedures. Overall, the results confirm that the Elzaki Transform Series Decomposition Method provides an accurate, stable, and computationally efficient framework for solving nonlinear Bratu differential equations. The close agreement between the approximate and exact solutions, together with the negligible numerical error and rapid convergence of the series solution, demonstrates that the proposed method is a reliable alternative to the existing analytical and numerical techniques for this class of nonlinear boundary value problems.

6. Conclusion

This study has established the effectiveness of the Elzaki transform series decomposition method for generating solutions to nonlinear Bratu differential equations. The method successfully integrates the Elzaki transform, series expansion, and Adomian polynomials, yielding rapidly convergent series as indicated graphically in figure 1 and 2. Furthermore, Tables 1 and 2 highlights the technique's superior performance compared to the existing methods in the literature, confirming its practical utility and reliability for this class of problems.

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