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Generalized Topological Notions by Operators

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Abstract — In this paper, it is introduced the notion of r-fuzzy β-$T_i, i = 0, 1, 2$ separation axioms related to a fuzzy operator β on the initial set X which is a generalization of previous fuzzy separation axioms. An r-fuzzy α-connectedness related to a fuzzy operator α on the set X is introduced which is a generalization of many types of r-fuzzy connectedness. An r-fuzzy α-compactness related to a fuzzy operator α on the set X is introduced which is a generalization of many types of fuzzy compactness.

Keywords — *Fuzzy operators, fuzzy separation axioms, fuzzy compactness, fuzzy connectedness.*

1 Introduction

It is a way to use fuzzy operators α, β on the initial set X and to use fuzzy operators θ, δ on the set Y giving generalizations of many notions and results in fuzzy topological spaces. r-fuzzy β-$T_i,\ i = 0, 1, 2$ separation axioms of the set X is a new type of fuzzy separation axioms related with a fuzzy operator β on X. It is proved that the image of r-fuzzy β-$T_i,\ i = 0, 1, 2$ is r-fuzzy δ-$T_i,\ i = 0, 1, 2$, and also the preimage of r-fuzzy δ-$T_i,\ i = 0, 1, 2$ is r-fuzzy β-$T_i,\ i = 0, 1, 2$. r-fuzzy α-connectedness is introduced related with the fuzzy operator α on X giving a generalization of many of fuzzy connectedness notions. It is proved that the image of r-fuzzy α-connected is r-fuzzy θ-connected, and some particular cases are included. r-fuzzy α-compactness is introduced using the fuzzy operator α on X giving a generalization of many of fuzzy compactness notions. It is proved that the image of r-fuzzy r-fuzzy compact is r-fuzzy θ-compact, and many special cases are deduced.

2 Preliminaries

Throughout the paper, X refers to an initial universe, I^X is the set of all fuzzy sets on X (where $I = [0, 1], I_0 = (0, 1]$, $\lambda^c(x) = 1 - \lambda(x)\ \forall x \in X$ and for all $t \in I$, $\overline{t}(x) = t\ \forall x \in X$).

(X, τ) is a fuzzy topological space ([14]), if $\tau : I^X \to I$ satisfies the following conditions:

(O1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,

(O2) $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ for all $\lambda_1, \lambda_2 \in I^X$,

(O3) $\tau(\bigvee_{j \in J} \lambda_j) \geq \bigwedge_{j \in J} \tau(\lambda_j)$ for all $\{\lambda_j\}_{j \in J} \subseteq I^X$.

By the concept of a fuzzy operator on a set X is meant a map $\gamma : I^X \times I_0 \to I^X$. Assume with respect to a fuzzy topology in Šostak sense defined on X, we have

$$\text{int}_\tau(\mu, r) \leq \gamma(\mu, r) \leq \text{cl}_\tau(\mu, r) \;\; \forall \mu \in I^X, \; \forall r \in I_0,$$

where $\text{int}_\tau, \text{cl}_\tau : I^X \times I_0 \to I^X$ are defined in Šostak sense for any $\mu \in I^X$ and each grade $r \in I_0$ as follows:

$$\text{int}_\tau(\mu, r) = \bigvee \{\eta \in I^X : \eta \leq \mu, \; \tau(\eta) \geq r\}$$

and

$$\text{cl}_\tau(\mu, r) = \bigwedge \{\eta \in I^X : \eta \geq \mu, \; \tau(\eta^c) \geq r\}$$

Let (X, τ_1) and (Y, τ_2) be two fuzzy topological spaces, α and β are fuzzy operators on X, θ and δ are fuzzy operators on Y, respectively. This type of maps α or β is called an expansion on X or a fuzzy operator on (X, τ_1), and the map θ or δ is called an expansion on Y or a fuzzy operator on (Y, τ_2) and let us fix that:

(1) β is a fuzzy operator on X such that $\beta(\mu, r) \leq \mu \;\; \forall \mu \in I^X, \; \forall r \in I_0$.

(2) α is a fuzzy operator on X such that $\alpha(\mu, r) \geq \mu \;\; \forall \mu \in I^X, \; \forall r \in I_0$.

As a special case of fuzzy operators, by the identity fuzzy operator id_X on a set X we mean that $id_X : I^X \times I_0 \to I^X$ so that $id_X(\nu, r) = \nu \;\; \forall \nu \in I^X, \forall r \in I_0$.

Recall that a fuzzy ideal $\mathcal{I}$ on X ([13]) is a map $\mathcal{I} : I^X \to I$ that satisfies the following conditions:

(1) $\lambda \leq \mu \;\Rightarrow\; \mathcal{I}(\lambda) \geq \mathcal{I}(\mu)$,

(2) $\mathcal{I}(\lambda \vee \mu) \geq \mathcal{I}(\lambda) \wedge \mathcal{I}(\mu)$.

Also, $\mathcal{I}$ is called proper if $\mathcal{I}(\overline{1}) = 0$ and there exists $\mu \in I^X$ such that $\mathcal{I}(\mu) > 0$. Define the fuzzy ideal $\mathcal{I}^\circ$ by

$$\mathcal{I}^\circ(\mu) \;=\; \begin{cases} 1 & \text{at } \mu = \overline{0}, \\ 0 & \text{otherwise} \end{cases}$$

Let us define the fuzzy difference between two fuzzy sets as follows:

$$(\lambda \,\bar{\wedge}\, \mu) \;=\; \begin{cases} \overline{0} & \text{if } \; \lambda \leq \mu, \\ \lambda \wedge \mu^c & \text{if } \; \text{otherwise.} \end{cases}$$

Definition 2.1. [4]

(1) A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be fuzzy $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous if for every $\mu \in I^Y$, any fuzzy ideal $\mathcal{I}$ on X,

$$\mathcal{I}[\alpha(f^{-1}(\delta(\mu, r)), r) \mathbin{\bar\wedge} \beta(f^{-1}(\theta(\mu, r)), r)] \geq \tau_2(\mu);\ r \in I_0.$$

We can see that the above definition generalizes the concept of fuzzy continuity ([14]) when we choose $\alpha =$ identity operator, $\beta =$ interior operator, $\delta =$ identity operator, $\theta =$ identity operator and $\mathcal{I} = \mathcal{I}^\circ$.

(2) A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be fuzzy $(\alpha, \beta, \theta, \delta, \mathcal{I}^*)$-open if for every $\lambda \in I^X$, any fuzzy ideal $\mathcal{I}^*$ on Y,

$$\mathcal{I}^*[\theta(f(\beta(\lambda, r)), r) \mathbin{\bar\wedge} \delta(f(\alpha(\lambda, r)), r)] \geq \tau(\lambda);\ r \in I_0.$$

We can see that the above definition generalizes the concept of fuzzy openness ([14]) when we choose $\alpha =$ identity operator, $\beta =$ interior operator, $\delta =$ interior operator, $\theta =$ identity operator and $\mathcal{I}^* = \mathcal{I}^\circ$.

3 r-Fuzzy β-T_i Separation Axioms

Here, we introduce and study fuzzy separation axioms related with a fuzzy operator β on the initial set X.

Definition 3.1.

(1) A set X is called r-fuzzy β-T_0 if for all $x \neq y$ in X, there exists $\lambda \in I^X, r \in I_0$ with $t \leq \beta(\lambda, r)(x);\ t \in I_0$ such that $t > \lambda(y)$ or there exists $\mu \in I^X, r \in I_0$ with $s \leq \beta(\mu, r)(y);\ s \in I_0$ such that $s > \mu(x)$.

(2) A set X is called r-fuzzy β-T_1 if for all $x \neq y$ in X, there exist $\lambda, \mu \in I^X, r \in I_0$ with $t \leq \beta(\lambda, r)(x),\ s \leq \beta(\mu, r)(y);\ t, s \in I_0$ such that $t > \lambda(y),\ s > \mu(x)$.

(3) A set X is called r-fuzzy β-T_2 if for all $x \neq y$ in X, there exist $\lambda, \mu \in I^X, r \in I_0$ with $t \leq \beta(\lambda, r)(x),\ s \leq \beta(\mu, r)(y);\ t, s \in I_0$ such that $(t \wedge s) > \sup(\lambda \wedge \mu)$.

Proposition 3.2. Every r-fuzzy β-T_i set X is an r-fuzzy β-T_{i-1}, $i = 1, 2$.

Proof. r-fuzzy β-$T_2 \Rightarrow r$-fuzzy β-T_1: Suppose that X is an r-fuzzy β-T_2 but it is not r-fuzzy β-T_1. Then, for all $x \neq y$ in X and for all $\lambda \in I^X$ with $t \leq \beta(\lambda, r)(x), r \in I_0$, suppose that $\lambda(y) \geq t; t \in I_0$. Now, for $\mu \in I^X$ with $s \leq \beta(\mu, r)(y) \leq \mu(y);\ s \in I_0$, we get that

$$\sup(\lambda \wedge \mu) \geq (\lambda \wedge \mu)(y) \geq (t \wedge s),$$

which means a contradiction to X is r-fuzzy β-T_2. Hence, X is an r-fuzzy β-T_1.

r-fuzzy β-$T_1 \Rightarrow r$-fuzzy β-T_0: Direct.

Recall that: a fuzzy operator θ is finer than a fuzzy operator β on a set X, denoted by $\beta \sqsubseteq \theta$, if $\beta(\nu, r) \leq \theta(\nu, r)\ \ \forall \nu \in I^X,\ \forall r \in I_0$.

Proposition 3.3. Let X be an r-fuzzy β-T_i, $i = 0, 1, 2$, and θ a fuzzy operator on X finer than β. Then X is also r-fuzzy θ-T_i space, $i = 0, 1, 2$.

Proof. For all the axioms r-fuzzy β-$T_i, i = 0, 1, 2$, the proof comes from that $\beta(\nu, r) \leq \theta(\nu, r)\ \forall \nu \in I^X,\ \forall r \in I_0$.

Example 3.4.

(1) Let $X = \{x, y\}$, $r \in I_0$ and

$$\beta(\nu, r) = \begin{cases} \nu & \text{at } \nu = \overline{0}, \overline{1} \\ x_1 & \text{at } x_1 \leq \nu < \overline{1}, \\ \overline{0} & \text{otherwise.} \end{cases}$$

Then, we get $\lambda = x_1 \in I^X, t = \frac{1}{4} \in I_0$ with $\beta(\lambda, r)(x) = x_1(x) = 1 \geq t$ and $\lambda(y) = x_1(y) = 0 < t$. Hence, the set X is an r-fuzzy β-T_0 set and it is neither r-fuzzy β-T_1 nor r-fuzzy β-T_2.

(2) Let $X = \{x, y\}$, $r \in I_0$ and

$$\beta(\nu, r) = \begin{cases} \nu & \text{at } \nu = \overline{0}, \overline{1} \\ x_1 & \text{at } x_1 \leq \nu < \overline{1}, \\ y_1 & \text{at } y_1 \leq \nu < \overline{1}, \\ \overline{0} & \text{otherwise.} \end{cases}$$

Then, we get $\lambda = y_1 \in I^X, t = \frac{1}{5} \in I_0$ with $\beta(\lambda, r)(y) = y_1(y) = 1 \geq t$ and $\lambda(x) = y_1(x) = 0 < t$. Similarly, we get $\mu = x_1 \in I^X, s = \frac{1}{3} \in I_0$ with $\beta(\mu, r)(x) = x_1(x) = 1 \geq s$ and $\mu(y) = x_1(y) = 0 < s$. Hence, the set X is an r-fuzzy β-T_1 set.

For $\lambda = x_1 \vee y_{\frac{1}{2}},\ \mu = y_1 \vee x_{\frac{1}{2}} \in I^X,\ t, s > \frac{1}{2} \in I_0$, we get that

$$\beta(\lambda, r)(x) = x_1(x) = 1 \geq t \quad \text{and} \quad \beta(\mu, r)(y) = y_1(y) = 1 \geq s$$

such that

$$(t \wedge s) > \frac{1}{2} = \sup(x_{\frac{1}{2}} \vee y_{\frac{1}{2}}) = \sup(\lambda \wedge \mu).$$

Hence, the set X is an r-fuzzy β-T_2 set.

(3) Let $X = \{x, y\}$, $r \in I_0$ and

$$\beta(\nu, r) = \begin{cases} \nu & \text{at } \nu = \overline{0}, \overline{1} \\ \overline{0.2} & \text{at } \overline{0.2} \leq \nu,\ \nu < x_1 \vee y_{0.2},\ \nu < x_{0.2} \vee y_1, \\ x_1 \vee y_{0.2} & \text{at } x_1 \vee y_{0.2} \leq \nu < \overline{1}, \\ x_{0.2} \vee y_1 & \text{at } x_{0.2} \vee y_1 \leq \nu < \overline{1} \\ \overline{0} & \text{otherwise.} \end{cases}$$

Then, there exist $\lambda = x_1 \vee y_{0.3},\ \mu = x_{0.3} \vee y_1$ such that $\beta(\lambda, r)(x) = 1 \geq t > 0.3 = \lambda(y)$ for $t \in I_0$ and $\beta(\mu, r)(y) = 1 \geq s > 0.3 = \mu(x)$ for $s \in I_0$, and then X is an r-fuzzy β-T_1 set.

Now, we study all possible fuzzy sets in I^X:

Then

(a) For any $\lambda = x_1 \vee y_p, \mu = x_1 \vee y_q, p, q \geq 0.2$, we get that: $\beta(\lambda, r)\,(x) = 1 \geq t,\ \beta(\mu, r)\,(y) = 0.2 \geq s;\ t, s \in I_0$ but $(t \wedge s) \leq 0.2 \leq \sup(\lambda \wedge \mu),\ p, q \geq 0.2$.

(b) For any $\lambda = x_p \vee y_1$ or $x_1 \vee y_p,\ \mu = x_q \vee y_1$ or $x_1 \vee y_q,\ p, q < 0.2$, we get that: $\beta(\lambda, r)\,(x) = \overline{0}(x) = 0 = \overline{0}(y) = \beta(\mu, r)\,(y)$.

(c) For any $\lambda = x_p, \mu = x_q$ or $\lambda = y_p, \mu = y_q$ or $\lambda = x_p, \mu = y_q,\ p, q \in I$, we get that: $\beta(\lambda, r)\,(x) = \overline{0}(x) = 0 = \overline{0}(y) = \beta(\mu, r)\,(y)$.

Hence, for every $\lambda, \mu \in I^X$ with $\beta(\lambda, r)\,(x) \geq t$ and $\beta(\mu, r)\,(y) \geq s;\ t, s \in I_0$, we have $(t \wedge s) \leq \sup(\lambda \wedge \mu)$, and thus X is not an r-fuzzy β-T_2 set.

Proposition 3.5. Let $f : X \to Y$ be an injective mapping. Assume that δ is a fuzzy operator on Y such that

$$f^{-1}(\delta(\lambda, r)) \ \leq\ \beta(f^{-1}(\lambda), r) \quad \forall \lambda \in I^Y,\ \forall r \in I_0.$$

Then, Y is an r-fuzzy δ-T_i implies that X is an r-fuzzy β-T_i, $i = 0, 1, 2$.

Proof. Since $x \neq y$ in X implies that $f(x) \neq f(y)$ in Y and Y is an r-fuzzy δ-T_1, then there exists $\lambda \in I^Y$ with $t \leq \delta(\lambda, r)(f(x)); t \in I_0$ so that $t > \lambda(f(y))$, that is,

$$t \ \leq\ [f^{-1}(\delta(\lambda, r))](x) \ \leq\ [\beta(f^{-1}(\lambda), r)](x) \ \text{ and } \ t > (f^{-1}(\lambda))\,(y),$$

which means that there exists $\mu = f^{-1}(\lambda) \in I^X$ with $t \leq \beta(\mu, r)(x);\ t \in I_0$ so that $t > \mu(y)$. Hence, X is an r-fuzzy β-T_1, and consequently X is an r-fuzzy β-T_0.

Now, for $x \neq y$ in X implies that $f(x) \neq f(y)$ in Y and Y is an r-fuzzy δ-T_2, then there exist $\lambda, \mu \in I^Y$ with $t \leq \delta(\lambda, r)(f(x)), s \leq \delta(\mu, r)(f(y));\ s, t \in I_0$ so that $(t \wedge s) > \sup(\lambda \wedge \mu)$.

Since $\sup(\lambda \wedge \mu) \geq \sup(f^{-1}(\lambda) \wedge f^{-1}(\mu))$, then $(t \wedge s) > \sup(f^{-1}(\lambda) \wedge f^{-1}(\mu))$. Also,

$$t \ \leq\ [f^{-1}(\delta(\lambda, r))](x) \ \leq\ [\beta(f^{-1}(\lambda), r)](x)$$

and

$$s \ \leq\ [f^{-1}(\delta(\mu, r))](y) \ \leq\ [\beta(f^{-1}(\mu), r)](y).$$

Hence, there exist $\nu = f^{-1}(\lambda), \rho = f^{-1}(\mu) \in I^X$ with $t \leq \beta(\nu, r)(x), s \leq \beta(\rho, r)(y);\ s, t \in I_0$ so that $(t \wedge s) > \sup(\nu \wedge \rho)$, and thus X is an r-fuzzy β-T_2.

Proposition 3.6. Let $f : X \to Y$ be a surjective mapping. Assume that δ is a fuzzy operator on Y such that

$$f(\beta(\lambda, r)) \ \leq\ \delta(f(\lambda), r) \quad \forall \lambda \in I^X,\ \forall r \in I_0.$$

Then, X is an r-fuzzy β-T_i implies that Y is an r-fuzzy δ-T_i, $i = 0, 1, 2$.

Proof. Since $p \neq q$ in Y implies that $x \neq y$ where $x = f^{-1}(p), y = f^{-1}(q)$ in X, and X is an r-fuzzy β-T_1, then there exists $\lambda \in I^X$ with $t \leq \beta(\lambda, r)(f^{-1}(p)); t \in I_0$ so that $t > \lambda(f^{-1}(q))$, that is,

$$t \ \leq\ [f(\beta(\lambda, r))](p) \ \leq\ [\delta(f(\lambda), r)](p) \ \text{ and } \ t > (f(\lambda))\,(q),$$

which means that there exists $\mu = f(\lambda) \in I^Y$ with $t \leq \delta(\mu, r)(p);\ t \in I_0$ so that $t > \mu(q)$. Hence, Y is an r-fuzzy δ-T_1, and consequently Y is an r-fuzzy δ-T_0.

Now, for $p \neq q$ in Y implies that $f^{-1}(p) \neq f^{-1}(q)$ in X and X is an r-fuzzy β-T_2, then there exist $\lambda, \mu \in I^X$ with $t \leq \beta(\lambda, r)(f^{-1}(p)), s \leq \beta(\mu, r)(f^{-1}(q));\ s, t \in I_0$ so that $(t \wedge s) > \sup(\lambda \wedge \mu)$.

Since $\sup(\lambda \wedge \mu) \ \geq \sup(f(\lambda) \wedge f(\mu))$, then $(t \wedge s) > \sup(f(\lambda) \wedge f(\mu))$. Also,

$$t \ \leq \ [f(\beta(\lambda, r))](p) \ \leq \ [\delta(f(\lambda), r)](p) \ \text{ and } \ s \ \leq \ [f(\beta(\mu, r))](q) \ \leq \ [\delta(f(\mu), r)](q).$$

Hence, there exist $\nu = f(\lambda), \rho = f(\mu) \in I^Y$ with $t \leq \delta(\nu, r)(p), s \leq \delta(\rho, r)(q);\ s, t \in I_0$ so that $(t \wedge s) > \sup(\nu \wedge \rho)$, and thus Y is an r-fuzzy δ-T_2.

Remark 3.7.

(1) For a fuzzy topological space (X, τ), by choosing $\beta =$ fuzzy interior operator, you can deduce the equivalence between the graded fuzzy separation axioms (t, s)-T_i, $i = 0, 1, 2;\ t, s \in I_0$ introduced in [5, 6] and the axioms r-fuzzy β-$T_i, i = 0, 1, 2$.

(2) For two fuzzy topological spaces (X, τ), (Y, σ), and $f : X \to Y$ a mapping, by choosing $\beta =$ fuzzy interior operator, we get that (X, τ) is (t, s)-T_i, $i = 0, 1, 2;\ t, s \in I_0$ whenever (Y, σ) is (t, s)-T_i, $i = 0, 1, 2;\ t, s \in I_0$ and f is injective fuzzy continuous (when $\delta =$ fuzzy interior operator in Proposition 3.5) as shown in [5]. This is equivalent to f is injective and $\alpha =$ identity operator, $\beta =$ interior operator, $\delta =$ interior operator, $\theta =$ identity operator and $\mathcal{I} = \mathcal{I}^\circ$ in Definition 2.1 (1).

(3) For two fuzzy topological spaces (X, τ), (Y, σ), and $f : X \to Y$ a mapping, by choosing $\delta =$ fuzzy interior operator, we get that (Y, σ) is (t, s)-T_i, $i = 0, 1, 2;\ t, s \in I_0$ whenever (X, τ) is (t, s)-T_i, $i = 0, 1, 2;\ t, s \in I_0$ and f is surjective fuzzy open (when $\beta =$ fuzzy interior operator in Proposition 3.6) as shown in [5]. This is equivalent to f is surjective and $\alpha =$ identity operator, $\beta =$ interior operator, $\delta =$ interior operator, $\theta =$ identity operator and $\mathcal{I} = \mathcal{I}^\circ$ in Definition 2.1 (2).

4 r-Fuzzy α-Connected Spaces

Here, we introduce the r-fuzzy connectedness of a space X relative to a fuzzy operator α. Assume (with respect to any fuzzy topology τ defined on X) that:

$$\lambda \ \leq \ \alpha(\lambda, r) \leq \ \mathrm{cl}_\tau(\lambda, r) \quad \forall \lambda \in I^X;\ r \in I_0.$$

Also, assume that α is a monotone operator, that is,

$$\mu \leq \nu \ \text{ implies } \ \alpha(\mu, r) \leq \alpha(\nu, r) \ \ \forall \mu, \nu \in I^X;\ r \in I_0.$$

Definition 4.1. Let X be a non-empty set. Then,

(1) the fuzzy sets $\lambda, \mu \in I^X$ are called r-fuzzy α-separated sets if

$$\alpha(\lambda, r) \wedge \mu \ = \ \lambda \wedge \alpha(\mu, r) = \overline{0};\ r \in I_0.$$

(2) X is called r-fuzzy α-connected space if it could not be found $\lambda, \mu \in I^X$, $\lambda \neq \overline{0}, \mu \neq \overline{0}$ such that λ, μ are r-fuzzy α-separated and $\lambda \vee \mu = \overline{1}$. That is, there are no r-fuzzy α-separated sets $\lambda, \mu \in I^X$ except $\lambda = \overline{0}$ or $\mu = \overline{0}$.

Definition 4.2. Let $\lambda, \mu \in I^X$, $\lambda \neq \overline{0}, \mu \neq \overline{0}$ such that:

(1) λ, μ are r-fuzzy α-separated and $\lambda \vee \mu = \overline{1}$. Then X is called r-fuzzy α-disconnected space.

(2) λ, μ are r-fuzzy α-separated and $\lambda \vee \mu = \nu$. Then ν is called r-fuzzy α-disconnected fuzzy set in I^X.

(3) λ, μ are r-fuzzy α-separated and $\lambda \vee \mu = \chi_A$, $A \subseteq X$. Then A is called r-fuzzy α-disconnected crisp set in I^X.

Remark 4.3. For a fuzzy topological space (X, τ)

(1) Taking α = fuzzy closure operator on (X, τ), then we have the r-fuzzy connectedness as given in [7].

(2) Taking α = fuzzy preclosure operator on (X, τ), then we have the r-fuzzy preconnectedness as given in [2].

(3) Taking α = fuzzy strongly semi-closure operator on (X, τ), then we have the r-fuzzy strongly connectedness as given in [10].

(4) Taking α = fuzzy semi-closure operator on (X, τ), then we have the 1-type of r-fuzzy strongly connectedness as given in [10].

(5) Taking α = fuzzy semi-preclosure operator on (X, τ), then we have the r-fuzzy semi-preconnectedness as given in [2].

(6) Taking α = fuzzy strongly preclosure operator on (X, τ), then we have the r-fuzzy strongly preconnectedness as given in [2].

Example 4.4. Let $X = \{x, y\}$, $r \in I_0$,

$$\alpha(\nu, r) = \begin{cases} \nu & \text{at } \nu = \overline{0}, \overline{1} \\ x_1 & \text{at } \overline{0} < \nu \leq x_1, \\ y_1 & \text{at } \overline{0} < \nu \leq y_1, \\ \overline{1} & \text{otherwise,} \end{cases}$$

Now, at $\lambda \neq \overline{0}, \lambda \leq x_1, \mu \neq \overline{0}, \mu \leq y_1, r \leq \frac{1}{4}$, then we have $\alpha(\lambda, r) \wedge \mu = x_1 \wedge \mu = \overline{0}$ and $\alpha(\mu, r) \wedge \lambda = y_1 \wedge \lambda = \overline{0}$, and thus λ, μ are r-fuzzy α-separated sets for $\lambda \neq \overline{0}, \lambda \leq x_1, \mu \neq \overline{0}, \mu \leq y_1$.

At $\lambda = x_1$ and $\mu = y_1$, we get r-fuzzy α-separated sets with $\overline{1} = \lambda \vee \mu$. Hence, X is an r-fuzzy α-disconnected space.

Proposition 4.5. Let (X, τ) be a fuzzy topological space. Then the following are equivalent.

(1) (X, τ) is r-fuzzy α-connected.

(2) $\lambda \wedge \mu = \overline{0}$, $\tau(\lambda) \geq r, \tau(\mu) \geq r;\ r \in I_0$, and $\overline{1} = \lambda \vee \mu$ imply $\lambda = \overline{0}$ or $\mu = \overline{0}$.

(3) $\lambda \wedge \mu = \overline{0}$, $\tau_c(\lambda) \geq r, \tau_c(\mu) \geq r;\ r \in I_0$, and $\overline{1} = \lambda \vee \mu$ imply $\lambda = \overline{0}$ or $\mu = \overline{0}$.

Proof. (1) $\Rightarrow$ (2): Let $\lambda, \mu \in I^X$ with $\tau(\lambda) \geq r, \tau(\mu) \geq r;\ r \in I_0$ such that $\lambda \wedge \mu = \overline{0}$ and $\overline{1} = \lambda \vee \mu$. Then, $\lambda = \mu^c$ and $\mu = \lambda^c$, and then

$$\overline{0} = \lambda \wedge \mu = \mu^c \wedge \lambda^c = \mathrm{cl}_\tau(\mu^c, r) \wedge \lambda^c \ \geq\ \alpha(\mu^c, r) \wedge \lambda^c \quad \text{and}$$
$$\overline{0} = \lambda \wedge \mu = \mu^c \wedge \lambda^c = \mu^c \wedge \mathrm{cl}_\tau(\lambda^c, r) \ \geq\ \mu^c \wedge \alpha(\lambda^c, r);\ r \in I_0,$$

which means that λ^c, μ^c are fuzzy α-separated so that $\lambda^c \vee \mu^c = \mu \vee \lambda = \overline{1}$. But (X, τ) is r-fuzzy α-connected implies that $\lambda^c = \overline{0}$ or $\mu^c = \overline{0}$, and thus $\lambda = \overline{0}$ or $\mu = \overline{0}$.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): Let $\lambda, \mu \in I^X$, $\lambda \neq \overline{0}$, $\mu \neq \overline{0}$ such that $\lambda \vee \mu = \overline{1}$. Taking $\nu = \mathrm{cl}_\tau(\lambda, r)$ and $\rho = \mathrm{cl}_\tau(\mu, r);\ r \in I_0$, then $\nu \vee \rho = \overline{1}$ and $\tau_c(\nu) \geq r, \tau_c(\rho) \geq r;\ r \in I_0$.

Now, suppose that (3) is not satisfied. That is, $\nu \neq \overline{0}$, $\rho \neq \overline{0}$ and $\nu \wedge \rho = \overline{0}$. Then,

$$\alpha(\lambda, r) \wedge \mu \ \leq\ \mathrm{cl}_\tau(\lambda, r) \wedge \mathrm{cl}_\tau(\mu, r) \ =\ \nu \wedge \rho \ =\ \overline{0} \quad \text{and}$$
$$\alpha(\mu, r) \wedge \lambda \ \leq\ \mathrm{cl}_\tau(\lambda, r) \wedge \mathrm{cl}_\tau(\mu, r) \ =\ \nu \wedge \rho \ =\ \overline{0},$$

which means that λ, μ are r-fuzzy α-separated sets, $\lambda \neq \overline{0}$, $\mu \neq \overline{0}$ with $\lambda \vee \mu = \overline{1}$. Hence, (X, τ) is not r-fuzzy α-connected space.

Proposition 4.6. Let X be a non-empty set and $\lambda \in I^X$. Then the following are equivalent.

(1) λ is r-fuzzy α-connected.

(2) If μ, ρ are r-fuzzy α-separated sets with $\lambda \leq \mu \vee \rho$, then $\lambda \wedge \mu = \overline{0}$ or $\lambda \wedge \rho = \overline{0}$.

(3) If μ, ρ are r-fuzzy α-separated sets with $\lambda \leq \mu \vee \rho$, then $\lambda \leq \mu$ or $\lambda \leq \rho$.

Proof. (1) $\Rightarrow$ (2): Let μ, ρ be r-fuzzy α-separated with $\lambda \leq \mu \vee \rho$, that is, $\alpha(\mu, r) \wedge \rho = \alpha(\rho, r) \wedge \mu = \overline{0};\ r \in I_0$ so that $\lambda \leq \mu \vee \rho$. Then, from that α is a monotone fuzzy operator, we get that

$$\alpha((\lambda \wedge \mu), r) \wedge (\lambda \wedge \rho) \leq \alpha(\lambda, r) \wedge \alpha((\mu, r) \wedge (\lambda \wedge \rho) = (\alpha(\lambda, r) \wedge \lambda) \wedge (\alpha((\mu, r) \wedge \rho) = \lambda \wedge \overline{0} = \overline{0}$$

and

$$\alpha((\lambda \wedge \rho), r) \wedge (\lambda \wedge \mu) \leq (\alpha(\lambda, r) \wedge \lambda) \wedge (\alpha(\rho, r) \wedge \mu) = \lambda \wedge \overline{0} = \overline{0};\ r \in I_0.$$

That is, $\lambda \wedge \mu$ and $\lambda \wedge \rho$ are r-fuzzy α-separated sets so that $\lambda = (\lambda \wedge \mu) \vee (\lambda \wedge \rho)$. But λ is r-fuzzy α-connected implies that $(\lambda \wedge \mu) = \overline{0}$ or $(\lambda \wedge \rho) = \overline{0}$.

(2) $\Rightarrow$ (3): If $\lambda \wedge \mu = \overline{0}$, then $\lambda = \lambda \wedge (\mu \vee \rho) = \lambda \wedge \rho$, and thus $\lambda \leq \rho$. Also, if $\lambda \wedge \rho = \overline{0}$, then $\lambda = \lambda \wedge \mu$, and then $\lambda \leq \mu$.

(3) $\Rightarrow$ (1): Let μ, ρ be r-fuzzy α-separated sets such that $\lambda = \mu \vee \rho$. Then, from (3), $\lambda \leq \mu$ or $\lambda \leq \rho$. If $\lambda \leq \mu$, then $\rho = \lambda \wedge \rho \leq \mu \wedge \rho \ \leq\ \alpha(\mu, r) \wedge \rho = \overline{0}$. Also, if $\lambda \leq \rho$, then $\mu = \lambda \wedge \mu \leq \rho \wedge \mu \ \leq\ \alpha(\rho, r) \wedge \mu = \overline{0}$. Hence, λ is r-fuzzy α-connected.

Theorem 4.7. Let $f : X \to Y$ be a mapping such that

$$\alpha(f^{-1}(\nu), r) \leq f^{-1}(\theta(\nu, r)) \ \ \forall \nu \in I^Y, \ r \in I_0,$$

where α is a fuzzy operator on X and θ is a fuzzy operator on Y. Then, $f(\lambda) \in I^Y$ is r-fuzzy θ-connected if $\lambda \in I^X$ is r-fuzzy α-connected.

Proof. Let $\mu, \rho \in I^Y$, $\mu \neq \overline{0}$, $\rho \neq \overline{0}$ be r-fuzzy θ-separated sets in I^Y with $f(\lambda) = \mu \vee \rho$. That is, $\theta(\mu, r) \wedge \rho = \theta(\rho, r) \wedge \mu = \overline{0}$; $r \in I_0$. Then, $\lambda \leq f^{-1}(\mu) \vee f^{-1}(\rho)$, and

$$\begin{aligned} \alpha(f^{-1}(\mu), r) \wedge f^{-1}(\rho) &\leq f^{-1}(\theta(\mu, r)) \wedge f^{-1}(\rho) \\ &= f^{-1}(\theta(\mu, r) \wedge \rho) \\ &= f^{-1}(\overline{0}) \ = \ \overline{0}, \end{aligned}$$

$$\begin{aligned} \alpha(f^{-1}(\rho), r) \wedge f^{-1}(\mu) &\leq f^{-1}(\theta(\rho, r)) \wedge f^{-1}(\mu) \\ &= f^{-1}(\theta(\rho, r) \wedge \mu) \\ &= f^{-1}(\overline{0}) \ = \ \overline{0}. \end{aligned}$$

Hence, $f^{-1}(\mu)$ and $f^{-1}(\rho)$ are r-fuzzy α-separated sets in X so that $\lambda \leq f^{-1}(\mu) \vee f^{-1}(\rho)$. But λ is r-fuzzy α-connected means, from (3) in Proposition 4.6, that $\lambda \leq f^{-1}(\mu)$ or $\lambda \leq f^{-1}(\rho)$, which means that $f(\lambda) \leq \mu$ or $f(\lambda) \leq \rho$. Thus, again from (3) in Proposition 4.6, we get that $f(\lambda)$ is r-fuzzy θ-connected.

Corollary 4.8. (Theorem 2.12 in [7]) Let $(X, \tau_1), (Y, \tau_2)$ be two fuzzy topological spaces. If $f : X \to Y$ is a fuzzy continuous mapping and $\lambda \in I^X$ is r-fuzzy connected in X, then $f(\lambda)$ is an r-fuzzy connected in Y.

Proof. Let $\alpha =$ fuzzy closure operator and $\theta =$ fuzzy closure operator. Then, the result follows from Theorem 4.7.

Corollary 4.9. (Theorems 2.12, 3.11 in [10]) Let $(X, \tau_1), (Y, \tau_2)$ be two fuzzy topological spaces. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be S-irresolute (resp. irresolute). If $\lambda \in I^X$ is r-fuzzy strongly connected (resp. 1-type of r-fuzzy strongly connected) in X, then $f(\lambda)$ is r-fuzzy strongly connected (resp. 1-type of r-fuzzy strongly connected) in Y.

Proof. Let $\alpha =$ fuzzy strongly semi-closure (resp. semi-closure) operator and $\theta =$ fuzzy strongly semi-closure (resp. semi-closure) operator. Then, the result follows from Theorem 4.7.

Corollary 4.10. Let $(X, \tau_1), (Y, \tau_2)$ be two fuzzy topological spaces. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be fuzzy semi-pre-irresolute. If $\lambda \in I^X$ is r-fuzzy semi-preconnected in X, then $f(\lambda)$ is r-fuzzy semi-preconnected in Y.

Proof. Let $\alpha =$ fuzzy semi-preclosure operator and $\theta =$ fuzzy semi-preclosure operator. Then, the result follows from Theorem 4.7.

Corollary 4.11. (Theorem 5.10 in [2]) Let $(X, \tau_1), (Y, \tau_2)$ be two fuzzy topological spaces. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be fuzzy strongly pre-irresolute (resp. pre-irresolute). If $\lambda \in I^X$ is r-fuzzy s preconnected (resp. preconnected) in X, then $f(\lambda)$ is r-fuzzy s preconnected (preconnected) in Y.

Proof. Let α = fuzzy strongly preclosure (resp. preclosure) operator and θ = fuzzy strongly preclosure (resp. preclosure) operator. Then, the result follows from Theorem 4.7.

Corollary 4.12. Let $(X, \tau_1), (Y, \tau_2)$ be two fuzzy topological spaces. Let $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be fuzzy semi-continuous (resp. precontinuous, strongly semi-continuous, strongly precontinuous and semi-precontinuous) mapping. If $\lambda \in I^X$ is 1-type of r-fuzzy strongly connected (resp. r-fuzzy preconnected, r-fuzzy strongly connected, r-fuzzy strongly preconnected and r-fuzzy semi-preconnected) in X, then $f(\lambda)$ is r-fuzzy connected in Y.

Proof. Let α = fuzzy semi-closure (resp. preclosure, strongly semi-closure, strongly preclosure and semi-preclosure) operator and θ = fuzzy closure operator. Then, the result follows from Theorem 4.7.

Proposition 4.13. Any fuzzy point $x_t, t \in I_0$ is r-fuzzy α-connected, and consequently $x_1 \forall x \in X$ is r-fuzzy α-connected.

Proof. Clear.

Definition 4.14. Let X be a non-empty set and $\lambda \in I^X$. Then, λ is r-fuzzy α-component if λ is maximal r-fuzzy α-connected set in X, that is, if $\mu \geq \lambda$ and μ is r-fuzzy α-connected set, then $\lambda = \mu$.

Proposition 4.15. Let $\lambda \neq \overline{0}$ be r-fuzzy α-connected in X and $\lambda \leq \mu \leq \alpha(\lambda, r)$; $r \in I_0$. Then, μ is r-fuzzy α-connected.

Proof. Let ν, ρ be r-fuzzy α-separated sets such that $\mu = \nu \vee \rho$. That is, $\alpha(\nu, r) \wedge \rho = \alpha(\rho, r) \wedge \nu = \overline{0}$; $r \in I_0$. Since $\lambda \leq \mu$, then $\lambda \leq (\nu \vee \rho)$. From λ is r-fuzzy α-connected, and from (3) in Proposition 4.6, we have $\lambda \leq \nu$ or $\lambda \leq \rho$. If $\lambda \leq \nu$, then

$$\rho = \mu \wedge \rho \leq \alpha(\lambda, r) \wedge \rho \leq \alpha(\nu, r) \wedge \rho = \overline{0}.$$

If $\lambda \leq \rho$, then

$$\nu = \mu \wedge \nu \leq \alpha(\lambda, r) \wedge \nu \leq \alpha(\rho, r) \wedge \nu = \overline{0}.$$

Hence, μ is r-fuzzy α-connected.

5 Fuzzy α-Compact Spaces

This section is devoted to introduce the notion of r-fuzzy α-compact spaces.

Definition 5.1. Let (X, τ) be a fuzzy topological space, α a fuzzy operator on X, and $\mu \in I^X$, $r \in I_0$. Then, μ is called r-fuzzy α-compact if for each family $\{\lambda_j \in I^X : \tau(\lambda_j) \geq r,\ j \in J\}$ with $\mu \leq \bigvee_{j \in J} \lambda_j$, there exists a finite subset $J_0 \subseteq J$ such that $\mu \leq \bigvee_{j \in J_0} \alpha(\lambda_j, r)$.

Remark 5.2. For a fuzzy topological space (X, τ):

(1) if α = fuzzy identity operator, we get the r-fuzzy compactness as given in [1].

(2) if $\alpha =$ fuzzy closure operator, we get the r-fuzzy almost compactness as given in [1].

(3) if $\alpha =$ fuzzy interior closure operator, we get the r-fuzzy near compactness as given in [1].

(4) if $\alpha =$ fuzzy semi-closure (resp. preclosure, strongly semi-closure, strongly preclosure and semi-preclosure) operator, we get the r-fuzzy semi-compactness (resp. precompactness, strongly semi-compactness, strongly precompactness and semi-precompactness [11]).

Theorem 5.3. Let (X, τ) and (Y, σ) be two fuzzy topological spaces, α a fuzzy operator on X, θ is a fuzzy operators on Y. If $f : X \to Y$ is fuzzy $(\alpha, \mathrm{int}_\tau, \theta, id_Y, \mathcal{I}^\circ)$-continuous and $\mu \in I^X$ is r-fuzzy compact in X, then $f(\mu)$ is r-fuzzy θ-compact in Y.

Proof. Let $\{\lambda_j \in I^Y : \sigma(\lambda_j) \geq r,\ j \in J\}$ be a family with $f(\mu) \leq \bigvee_{j\in J} \lambda_j$. Since f is fuzzy $(\alpha, \mathrm{int}_\tau, \theta, id_Y, \mathcal{I}^\circ)$-continuous, we get that there exists $\mu_j = \mathrm{int}_\tau(f^{-1}(\theta(\lambda_j, r)), r) \in I^X$ with $\tau(\mu_j) \geq r$ $\forall j \in J$ such that

$$\alpha(f^{-1}(\lambda_j), r) \leq \mu_j \leq f^{-1}(\theta(\lambda_j, r)).$$

Also, since $f^{-1}(\lambda_j) \leq \alpha(f^{-1}(\lambda_j), r)$, then

$$f^{-1}(\lambda_j) \leq \mu_j \leq f^{-1}(\theta(\lambda_j, r)),$$

which means that

$$\mu \leq \bigvee_{j\in J} f^{-1}(\lambda_j) \leq \bigvee_{j\in J} (\mu_j) \leq f^{-1}(\bigvee_{j\in J} \theta(\lambda_j, r)),$$

that is, $\mu \leq \bigvee_{j\in J} (\mu_j)$. By r-fuzzy compactness of μ, there exists a finite set $J_0 \subseteq J$ such that $\mu \leq \bigvee_{j\in J_0} (\mu_j)$, and thus

$$f(\mu) \leq \bigvee_{j\in J_0} f(\mu_j) \leq \bigvee_{j\in J_0} \theta(\lambda_j, r),$$

and therefore $f(\mu)$ is r-fuzzy θ-compact.

Corollary 5.4. ([11]) Let (X, τ) and (Y, σ) be two fuzzy topological spaces. Let $f : X \to Y$ be a fuzzy continuous mapping and $\mu \in I^X$ an r-fuzzy compact set in X, then $f(\mu)$ is r-fuzzy compact in Y.

Proof. Let $\alpha =$ fuzzy identity operator on X, $\theta =$ fuzzy identity operator and $\mathcal{I} = \mathcal{I}^\circ$, then the result follows from Theorem 5.3.

Corollary 5.5. ([11]) Let (X, τ) and (Y, σ) be two fuzzy topological spaces. Let $f : X \to Y$ be a fuzzy weakly continuous mapping ([8]) and $\mu \in I^X$ an r-fuzzy compact set in X, then $f(\mu)$ is r-fuzzy almost compact in Y.

Proof. Let $\alpha =$ fuzzy identity operator on X, $\theta =$ fuzzy closure operator and $\mathcal{I} = \mathcal{I}^\circ$, then the result follows from Theorem 5.3.

Corollary 5.6. ([11]) Let (X, τ) and (Y, σ) be two fuzzy topological spaces. Let $f : X \to Y$ be a fuzzy almost continuous mapping ([9]) and $\mu \in I^X$ an r-fuzzy compact set in X, then $f(\mu)$ is r-fuzzy nearly compact in Y.

Proof. Let $\alpha =$ fuzzy identity operator on X, $\theta =$ fuzzy interior closure operator and $\mathcal{I} = \mathcal{I}^\circ$, then the result follows from Theorem 5.3.

Corollary 5.7. Let (X, τ) and (Y, σ) be two fuzzy topological spaces. Let $f : X \to Y$ be a fuzzy semi-continuous [12] (resp. precontinuous [8], strongly semi-continuous [3], strongly precontinuous [2] and semi-precontinuous [8]) mapping, and $\mu \in I^X$ an r-fuzzy compact set in X, then $f(\mu)$ is r-fuzzy semi-compact (resp. precompact, strongly semi-compact, strongly precompact and semi-precompact) in Y.

Proof. Let $\alpha =$ fuzzy identity operator on X, $\theta =$ fuzzy semi-closure (resp. preclosure, strongly semi-closure, strongly preclosure and semi-preclosure) operator and $\mathcal{I} = \mathcal{I}^\circ$, then the result follows from Theorem 5.3.

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