Araştırma Makalesi *BAUN Fen Bil. Enst. Dergisi, 20(2), 401-412, (2018)*

DOI: 10.25092/baunfbed.468674 *J. BAUN Inst. Sci. Technol., 20(2), 401-412, (2018)*

Explicit relations for the modified degenerate Apostol-type polynomials

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Geliş Tarihi (Recived Date): 22.05.2018
Kabul Tarihi (Accepted Date): 12.07.2018

Abstract

Recently, the degenerate Bernoulli numbers and polynomials and the degenerate Euler numbers and polynomials have been studied by several authors. In this paper, we consider the modified Apostol-Bernoulli polynomials and the modified Apostol-Euler polynomials. We give explicit relation for the modified degenerate Bernoulli polynomials and the modified degenerate Euler polynomials. Also, we prove some identities between the modified Apostol-Bernoulli polynomials and the degenerate Stirling numbers of two kinds.

Keywords: *Bernoulli polynomials and numbers, Euler polynomials and numbers, Modified Bernoulli numbers and polynomials, Modified Euler numbers and polynomials, Degenerate Stirling numbers of the second kind, Degenerate μ-multiple sums, Degenerate μ-multiple alternating sums.*

Modifiye dejenere Apostol-tipi polinomlar için kesin bağıntılar

Özet

Son yıllar da dejenere Bernoulli sayıları, polinomları ve dejenere Euler sayıları, polinomlarını birçok yazarlar tarafından çalışılıyor. Bu makale de modifiye Apostol-Bernoulli polinomları ve modifiye Apostol-Euler polinomlarını tanımladık. Modifiye dejenere Bernoulli polinomları ve modifiye dejenere Euler polinomları için kesin bağıntı verdik. Ayrıca, ikinci çeşit dejenere Stirling sayıları ve modifiye Apostol-Bernoulli polinomları arasında bazı özellikler ispatlandı.

Anahtar kelimeler: *Bernoulli polinomları ve sayıları, Euler polinomları ve sayıları, Modifiye Bernoulli polinomları ve sayıları, Modifiye Euler polinomları ve sayıları, İkinci Çeşit Dejenere Stirling sayıları, Dejenere μ-katlı toplamlar.*

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1. Introduction

As usual, throughout this paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, R denotes the set of real numbers and C denotes the set of complex numbers. We begin by introducing the following definitions and notations (see also [14-17]).

It is well known the Bernoulli polynomials and Euler polynomials are defined by the generating functions respectively

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt}, |t| < 2\pi \tag{1}$$

and

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt}, |t| < \pi. \tag{2}$$

When *x=0*, $B_n = B_n(0)$ and $E_n = E_n(0)$ are called the Bernoulli numbers and the Euler numbers respectively.

The generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in N_0$ and the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in N_0$ are defined by the following generating functions (see, for detail [9, 14-17])

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{t}{\lambda e^t-1}\right)^{\alpha} e^{xt},\ \{|t| < 2\pi \ when\ \lambda = 1, |t| < |log\lambda|\ when\ \lambda \neq 1\} \tag{3}$$

and

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{2}{\lambda e^t+1}\right)^{\alpha} e^{xt},\ \{|t| < \pi \ when\ \lambda = 1, |t| < |\log(-\lambda)|\ when\ \lambda \neq 1\}. \tag{4}$$

Carlitz in [1, 2] defined degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} B_n(x|\lambda)\frac{t^n}{n!}, \tag{5}$$

when *x=0*, $B_n(\lambda) = B_n(0|\lambda)$ are called the degenerate Bernoulli numbers. From (5), we can easily derive the following equations

$$B_n(x|\lambda) = \sum_{l=0}^{n}\binom{n}{l} B_{n-l}(\lambda)(x|\lambda)_l, n > 0 \tag{i}$$

where $(x|\lambda)_n = x(x-\lambda)\cdots\big(x-\lambda(n-1)\big), (x|\lambda)_0 = 1.$

$$B_{n+1}(x+1|\lambda) - B_n(x|\lambda) = (n+1)(x|\lambda)_l. \tag{ii}$$

Dolgy *et al.* [3] studied the following modified degenerate Bernoulli polynomials $B_{n,\lambda}(x)$ which are different from Carlitz' s degenerate Bernoulli polynomials $B_n(x|\lambda)$ generated by (5) as follows

$$\frac{t}{(1+\lambda)^{t/\lambda}-1}(1+\lambda)^{xt/\lambda}=\sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^n}{n!} \tag{6}$$

which, in the special case when $x=0$ *and* $B_{n,\lambda}:=B_{n,\lambda}(0), n\in N_0$. We have the modified degenerate Bernoulli numbers $B_{n,\lambda}$.

It is easily observed from the generating function (6) that

$$\lim_{\lambda\to 0}\left[\sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^n}{n!}\right]=\lim_{\lambda\to 0}\left[\frac{t}{(1+\lambda)^{t/\lambda}-1}(1+\lambda)^{xt/\lambda}\right]$$
$$=\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{7}$$

Thus, by applying (7), we find that

$\lim_{\lambda\to 0}B_{n,\lambda}(x)=B_n(x)$.

Kwon *et al.* in [8] studied the analogously modified degenerate Euler polynomials $E_n(x|\lambda)$ generated by

$$\frac{2}{(1+\lambda)^{t/\lambda}+1}(1+\lambda)^{xt/\lambda}=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!} \tag{8}$$

which in the special case when $x=0$ *and* $E_{n,\lambda}:=E_{n,\lambda}(0), n\in N_0$ reduces to the generating function for the modified degenerate Euler numbers $E_{n,\lambda}$.

Motivated essentially by each of these works [3] and [8], we consider and investigate the generalized higher order modified degenerate Apostol-Bernoulli polynomials $B_{n,\alpha}^{(r)}(x|\lambda)$ and the generalized higher order modified degenerate Apostol-Euler polynomials $E(x|\lambda)$ by means of following generating functions

$$\sum_{n=0}^{\infty}B_{n,\alpha}^{(r)}(x|\lambda)\frac{t^n}{n!}=\left(\frac{t}{\alpha(1+\lambda)^{t/\lambda}-1}\right)^{(r)}(1+\lambda)^{xt/\lambda} \tag{9}$$

and

$$\sum_{n=0}^{\infty}E_{n,\alpha}^{(r)}(x|\lambda)\frac{t^n}{n!}=\left(\frac{2}{\alpha(1+\lambda)^{t/\lambda}+1}\right)^{(r)}(1+\lambda)^{xt/\lambda} \tag{10}$$

respectively. Here and in what follows where $r\in N$ *and* $\alpha\in R$ in particular, for $x=0$ in (9) and (10), we have the generalized higher order modified degenerate Apostol-Bernoulli numbers $B_{n,\alpha}^{(r)}(\lambda)$ and the generalized higher order modified degenerate Apostol-Euler numbers $E_{n,\alpha}^{(r)}(\lambda)$, respectively.

By applying to the generating functions (9) and (10), we get

$\lim_{\lambda\to 0}B_{n,\alpha}^{(r)}(x|\lambda)=B_{n,\alpha}^{(r)}(x)=B_n^{(r)}(x,\alpha)$

and

$$\lim_{\lambda\to 0} E_{n,\alpha}^{(r)}(x|\lambda) = E_{n,\alpha}^{(r)}(x) = E_n^{(r)}(x,\alpha)$$
respectively.

A degenerate version of the Stirling number $S_2(n,k)$ of the second kind is defined by generating function

$$\frac{1}{k!}\left((1+\lambda)^{t/\lambda}-1\right)^k = \sum_{n=0}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}. \tag{11}$$

In terms of the multinomial coefficients given by

$$\binom{l}{v_1,v_2,\cdots,v_m} := \frac{l!}{v_1!v_2!\cdots v_m!}$$

the μ-multiple power sums were defined by Luo [12] as follows

$$S_k^{(l)}(m;\mu) = \sum_{\substack{0\le v_1\le v_2\le\cdots\le v_{m=l}\\ v_1+\cdots+v_m=n}} \binom{l}{v_1,v_2,\cdots,v_m}\mu^{v_1+2v_2+\cdots+mv_m}(v_1+2v_2+\cdots+mv_m)^k \tag{12}$$

which readily yields

$$\left(\frac{1-\mu^m e^{mt}}{1-\mu e^t}\right)^l = \frac{1}{\mu^l}\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}(-l)^{n-k}S_k^{(l)}(m;\mu)\right\}\frac{t^n}{n!} \tag{13}$$

where $\mu \in C$.

Similarly, the μ-multiple alternating power sums were defined by Luo [13] as follows

$$Z_k^{(l)}(m;\mu) = (-1)^l \sum_{\substack{0\le v_1\le v_2\le\cdots\le v_{m=l}\\ v_1+\cdots+v_m=n}} \binom{l}{v_1,v_2,\cdots,v_m}(-\mu)^{v_1+2v_2+\cdots+mv_m} \quad (v_1+2v_2+\cdots+mv_m)^k \tag{14}$$

which readily yields.

$$\left(\frac{1+(-1)^{m+1}\mu^m e^{mt}}{1+\mu e^t}\right)^l = \frac{1}{\mu^l}\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}(-l)^{n-k}Z_k^{(l)}(m;\mu)\right\}\frac{t^n}{n!} \tag{15}$$

where $\mu \in C$.

From (13) and (15), we define the μ-multiple degenerate power sums and the μ-multiple degenerate alternating power sums by means of the following equations

$$\left(\frac{1-\mu^m(1+\lambda)^{mt/\lambda}}{1-\mu(1+\lambda)^{t/\lambda}}\right)^l = \frac{1}{\mu^l}\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}(-l)^{n-k}S_{k,\lambda}^{(l)}(m;\mu)\right\}\frac{t^n}{n!} \tag{16}$$

and

$$\left(\frac{1+(-1)^{m+1}\mu^{m}(1+\lambda)^{mt/\lambda}}{1+\mu(1+\lambda)^{mt/\lambda}}\right)^{l}=\frac{1}{\mu^{l}}\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}(-l)^{n-k}Z_{k,\lambda}^{(l)}(m;\mu)\right\}\frac{t^{n}}{n!}. \tag{17}$$

Keeping in view many of the above-mentioned and other related investigation by Carlitz (see [1, 2]), Dolgy *et al.* [3], Kim *et al.* [5], He *et al.* [6], Kim [7] , Kwon *et al.* [8], Kurt [9, 10], Liu and Wang [11], Luo [12, 13], Srivastava [16], Kurt [18], Yang [21], Young [22]. We systematically study the above defined the generalized higher order modified degenerate Apostol-Bernoulli polynomials and the generalized higher order modified degenerate Apostol-Euler polynomials.

In particular, we give some explicit relation between the modified degenerate Bernoulli polynomials and the modified degenerate Euler polynomials. Also, we prove identities for the modified degenerate Apostol-Bernoulli polynomials and modified degenerate Apostol-Euler polynomials.

2. Explicit relations for the modified degenerate Bernoulli and Euler polynomials

In this section, we give some explicit relationships for the modified degenerate Bernoulli and the modified degenerate Euler polynomials. We prove some identitites for these polynomials.

Also, by using the equation (9) and (10), we can obtain the following relations:

$$\alpha B_{n,\alpha}(x+1|\lambda)-B_{n,\alpha}(x|\lambda)=n(x|\lambda)_{n-1}, \tag{i}$$
$$\alpha E_{n,\alpha}(x+1|\lambda)+E_{n,\alpha}(x|\lambda)=(x|\lambda)_{n}, \tag{ii}$$

where r=1 in (9) and (10)

$$B_{n,\alpha}^{(r_1+r_2)}(x+y|\lambda)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k,\alpha}^{(r_1)}(x|\lambda)B_{k,\alpha}^{(r_2)}(y|\lambda), \tag{iii}$$
$$\alpha B_{n,\alpha}^{(r)}(x|\lambda)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k,\alpha}^{(r)}\times(x|\lambda)_{k}, \tag{iv}$$

and

$$B_{n,\alpha^2}(x|\lambda)=2^{-n}\sum_{k=0}^{n}\binom{n}{k}B_{k,\alpha}(x|\lambda)E_{n-k,\alpha}(x|\lambda), \tag{v}$$

where r=1 in (9) and (10).

Theorem 1. There is the following relation between the modified degenerate Bernoulli polynomials and the degenerate Stirling numbers of the second kind:

$$B_{n-k}^{(r)}(x|\lambda)\frac{n!}{(n-k)!}=k!\sum_{l}^{n}\binom{n}{l}B_{l}^{(r+k)}(x|\lambda)S_{2,\lambda}(n-l,k). \tag{18}$$

Proof. From (5) and (11),

$$\sum_{n=0}^{\infty}B_{n}^{(r)}(x|\lambda)\frac{t^{n}}{n!}=\left(\frac{t}{(1+\lambda)^{t/\lambda}-1}\right)^{r}(1+\lambda)^{xt/\lambda}$$

$$= \left(\frac{t}{(1+\lambda)^{t/\lambda}-1}\right)^r (1+\lambda)^{xt/\lambda} \frac{\left((1+\lambda)^{t/\lambda}-1\right)^k}{k!} \frac{k!}{\left((1+\lambda)^{t/\lambda}-1\right)^k}$$
$$= \frac{t^r}{\left((1+\lambda)^{t/\lambda}-1\right)^{r+k}} k!\,(1+\lambda)^{xt/\lambda} \sum_{m=0}^{\infty} S_{2,\lambda}(m,k) \frac{t^m}{m!}$$
$$= k!\,t^{-k} \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} B_l^{(r+k)}(x|\lambda) S_{2,\lambda}(n-l,k) \frac{t^n}{n!}$$
$$\sum_{n=0}^{\infty} B_n^{(r)}(x|\lambda) \frac{t^{n+k}}{n!} = k! \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} B_l^{(r+k)}(x|\lambda) S_{2,\lambda}(n-l,k) \frac{t^n}{n!}$$
$$\sum_{n=k}^{\infty} B_{n-k}^{(r)}(x|\lambda) \frac{n!}{(n-k)!} \frac{t^n}{n!} = k! \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} B_l^{(r+k)}(x|\lambda) S_{2,\lambda}(n-l,k) \frac{t^n}{n!}.$$

Since the right hand of this equality to n=k is zero, comparing both sides of this equality, we have (18).

Theorem 2. The following relation holds true

$$E_n^{(r)}(x|\lambda) = \sum_{j=0}^{\infty} \binom{-r}{j} \frac{j!}{2^J} \sum_{l=0}^{n} \binom{n}{l} S_{2,\lambda}(n-l,k)(x|\lambda)_l. \tag{19}$$

Proof. By using the identities

$$\left(\frac{2}{(1+\lambda)^{t/\lambda}+1}\right)^r = \left(1+\frac{(1+\lambda)^{t/\lambda}-1}{2}\right)^{(-r)} = \sum_{j=0}^{\infty} \binom{-r}{j} \left(\frac{(1+\lambda)^{t/\lambda}-1}{2}\right)^j$$
$$\sum_{n=0}^{\infty} E_n^{(r)}(x|\lambda) \frac{t^n}{n!} = \left(\frac{2}{(1+\lambda)^{t/\lambda}+1}\right)^r (1+\lambda)^{xt/\lambda}$$
$$= \sum_{j=0}^{\infty} \binom{-r}{j} \frac{j!}{2^J} \frac{\left((1+\lambda)^{t/\lambda}-1\right)^j}{j!} (1+\lambda)^{xt/\lambda}$$
$$= \sum_{j=0}^{\infty} \binom{-r}{j} \frac{j!}{2^J} \sum_{m=0}^{\infty} S_{2,\lambda}(m,k) \frac{t^m}{m!} \sum_{l=0}^{\infty} (x|\lambda)_l \frac{t^l}{l!}$$
$$= \sum_{n=0}^{\infty} \left[\sum_{l=0}^{n} \binom{n}{l} \sum_{j=0}^{\infty} \binom{-r}{j} \frac{j!}{2^J} S_{2,\lambda}(n-l,k)(x|\lambda)_l\right] \frac{t^n}{n!}.$$

Comparing the coefficients of both sides of equation, we have result.

Theorem 3. There is the following relation between the degenerate Bernoulli number and the degenerate Stirling numbers of the second kind as:

$$B_n^{(r)}(\lambda) = \sum_{j=0}^{\infty} \binom{-r}{j} \frac{j!}{2^J} \sum_{k=0}^{j} \binom{j}{k} k!\,(-1)^{j-k} S_{2,\lambda}(n+k,k) \frac{n!}{(n+k)!}. \tag{20}$$

Proof. From (5), for *x=0*

$$\sum_{n=0}^{\infty} B_n^{(r)}(0|\lambda) \frac{t^n}{n!} = \left(\frac{t}{(1+\lambda)^{t/\lambda}-1}\right)^r = \left(1+\frac{(1+\lambda)^{t/\lambda}-t-1}{t}\right)^{(-r)}$$
$$= \sum_{j=0}^{\infty} \binom{-r}{j} \left(\frac{(1+\lambda)^{t/\lambda}-t-1}{t}\right)^j = \sum_{j=0}^{\infty} \binom{-r}{j} t^{-j} \sum_{k=0}^{j} \binom{j}{k} \left((1+\lambda)^{t/\lambda}-1\right)^k (-t)^{j-k}$$
$$= \sum_{j=0}^{\infty} \binom{-r}{j} \sum_{k=0}^{j} \binom{j}{k} k! \frac{\left((1+\lambda)^{t/\lambda}-1\right)^k}{k!} (-1)^{j-k} t^{-k}$$
$$= \sum_{j=0}^{\infty} \binom{-r}{j} \sum_{k=0}^{j} \binom{j}{k} k!\,(-1)^{j-k} \sum_{n=0}^{\infty} S_{2,\lambda}(n,k) \frac{t^{n-k}}{n!}$$
$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \binom{-r}{j} \sum_{k=0}^{j} \binom{j}{k} k!\,(-1)^{j-k} S_{2,\lambda}(n,k) \frac{(n-k)!}{n!} \frac{t^{n-k}}{(n-k)!}$$
$$= \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} \binom{-r}{j} \frac{j!}{2^J} \sum_{k=0}^{j} \binom{j}{k} k!\,(-1)^{j-k} S_{2,\lambda}(n+k,k) \frac{n!}{(n+k)!}\right] \frac{t^n}{n!}.$$

Comparing the coefficients of both sides of $\frac{t^n}{n!}$, we have (20).

Theorem 4. The degenerate Euler polynomials satisfy the following relation

$$E_n(x|\lambda) = 2(x|\lambda) - \sum_{k=0}^{n} \binom{n}{k} E_{n-k}(x|\lambda).(x|\lambda)_k. \tag{21}$$

Proof. By using the following identitites and (8),

$$\frac{-2}{\left((1+\lambda)^{t/\lambda}+1\right)(1+\lambda)^{t/\lambda}} = \frac{2}{(1+\lambda)^{t/\lambda}+1} - \frac{2}{(1+\lambda)^{t/\lambda}}$$

we write

$$\frac{-2}{(1+\lambda)^{t/\lambda}+1}\frac{(1+\lambda)^{tx/\lambda}}{(1+\lambda)^{t/\lambda}} = \frac{2(1+\lambda)^{tx/\lambda}}{(1+\lambda)^{t/\lambda}+1} - \frac{2(1+\lambda)^{tx/\lambda}}{(1+\lambda)^{t/\lambda}}.$$

From last equality, we write as

$$\frac{-2}{(1+\lambda)^{t/\lambda}+1}(1+\lambda)^{tx/\lambda} = \frac{2(1+\lambda)^{tx/\lambda}}{(1+\lambda)^{t/\lambda}+1}(1+\lambda)^{t/\lambda} - 2(1+\lambda)^{t/\lambda}$$

and

$$-\sum_{n=0}^{\infty} E_n(x|\lambda)\frac{t^n}{n!} = \sum_{m=0}^{\infty} E_m(x|\lambda)\frac{t^m}{m!}\sum_{k=0}^{\infty}(1|\lambda)_k\frac{t^k}{k!} - 2\sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}.$$

By using Cauchy product and comparing the coefficient, we have result.

3. Some symmetry identities for the modified degenerate Apostol-Bernoulli polynomials

In this section, by using μ-multiple power sums, we give some symmetry identities for the modified degenerate Apostol-Bernoulli polynomials.

Theorem 5. There is the following relation between the modified degenerate Apostol-Bernoulli polynomials and the modified μ-multiple power sums:

$$a^{m-1}\sum_{\beta=0}^{\infty}\binom{\beta+m-1}{\beta}\mu^{a\beta-b}\left\{\sum_{p=0}^{n}\binom{n}{p}\sum_{r=0}^{p}\binom{p}{r}(-1)^{p+m-r-1}S_{p,\lambda}(a,\mu^b)B_{n-p,\mu^b}^{(m-1)}\left(ax+ay+\frac{a\beta}{b}\middle|\lambda\right)b^n\right\}$$

$$= b^{m-1}\sum_{\beta=0}^{\infty}\binom{\beta+m-1}{\beta}\mu^{b\beta-a}\left\{\sum_{p=0}^{n}\binom{n}{p}\sum_{r=0}^{p}\binom{p}{r}(-1)^{p+m-r-1}S_{p,\lambda}(b,\mu^a)B_{n-p,\mu^a}^{(m-1)}\left(bx+by+\frac{b\beta}{a}\middle|\lambda\right)a^n\right\} \tag{22}$$

Proof. Let

$$f(t) = \frac{t^{m-1}(1+\lambda)^{abxt/\lambda}\left(1-\mu^{ab}(1+\lambda)^{abt/\lambda}\right)(1+\lambda)^{abyt/\lambda}}{\left(1-\mu^a(1+\lambda)^{at/\lambda}\right)^m\left(1-\mu^b(1+\lambda)^{bt/\lambda}\right)^m}$$

$$=\frac{1}{b^{m-1}}\frac{(1+\lambda)^{\frac{abxt}{\lambda}}}{\left(1-\mu^{a}(1+\lambda)^{\frac{at}{\lambda}}\right)^{m}}\left(\frac{1-\mu^{ab}(1+\lambda)^{\frac{abt}{\lambda}}}{1-\mu^{b}(1+\lambda)^{\frac{bt}{\lambda}}}\right)\left(\frac{bt}{1-\mu^{b}(1+\lambda)^{bt/\lambda}}\right)^{m-1}(1+\lambda)^{abyt/\lambda}.$$

By using (9) and (13) for $l=1$,

$$=\frac{1}{b^{m-1}}\sum_{\beta=0}^{\infty}\binom{\beta+m-1}{\beta}\mu^{a\beta-b}\sum_{p=0}^{\infty}\sum_{r=0}^{p}\binom{p}{r}(-1)^{p-r}S_{p,\lambda}\left(a,\mu^{b}\right)\frac{b^{p}t^{p}}{p!}$$
$$\times(-1)^{m-1}\sum_{k=0}^{\infty}B_{k,\mu^{b}}^{(m-1)}\left(ax+ay+\frac{a\beta}{b}\middle|\lambda\right)b^{k}\frac{t^{k}}{k!}.$$

Using Cauchy product, we have

$$=\frac{1}{b^{m-1}}\sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}\left\{\sum_{\beta=0}^{\infty}\binom{\beta+m-1}{\beta}\mu^{a\beta-b}\sum_{r=0}^{p}\binom{p}{r}(-1)^{r-p+m-1}\right\}$$
$$\times S_{p,\lambda}\left(a,\mu^{b}\right)b^{p}B_{n-p,\mu^{b}}^{(m-1)}\left(ax+ay+\frac{a\beta}{b}\middle|\lambda\right)b^{n-p}\frac{t^{n}}{n!}. \tag{23}$$

In similar manner,

$$f(t)=\frac{t^{m-1}(1+\lambda)^{abyt/\lambda}\left(1-\mu^{ab}(1+\lambda)^{abt/\lambda}\right)(1+\lambda)^{abxt/\lambda}}{\left(1-\mu^{b}(1+\lambda)^{bt/\lambda}\right)^{m}\left(1-\mu^{a}(1+\lambda)^{at/\lambda}\right)^{m}}.$$

From (9) and (13), we write

$$=\frac{1}{a^{m-1}}\sum_{n=0}^{\infty}\left\{\sum_{\beta=0}^{\infty}\binom{\beta+m-1}{\beta}\mu^{b\beta-a}\sum_{p=0}^{n}\binom{n}{p}\sum_{r=0}^{p}\binom{p}{r}(-1)^{p+m-r-1}\right\}$$
$$\times S_{p,\lambda}(b,\mu^{a})B_{n-p,\mu^{a}}^{(m-1)}\left(bx+by+\frac{b\beta}{a}\middle|\lambda\right)a^{n}\frac{t^{n}}{n!}. \tag{24}$$

By comparing the coefficients of $\frac{t^{n}}{n!}$ in (23) and (24), we prove the theorem.

Theorem 6. For all $a,b,m\in N\ and\ n\in N_{0}$, we have the following symmetry identities

$$b^{m}\sum_{m_1=0}^{n}\binom{n}{m_1}B_{n-m_1,\mu^{a}}^{(m+1)}(bx|\lambda)a^{n-m_1}b^{m_1}\mu^{-mb}\sum_{p=0}^{m_1}\binom{m_1}{p}\sum_{r=0}^{p}\binom{p}{r}(-m)^{p-r}S_{p,\lambda}^{(m)}(a,\mu^{b})B_{m_1-p,\mu^{b}}(ay|\lambda)$$
$$=a^{m}\sum_{m_1=0}^{n}\binom{n}{m_1}B_{n-m_1,\mu^{b}}^{(m+1)}(ay|\lambda)b^{n-m_1}a^{m_1}\mu^{-ma}\sum_{p=0}^{m_1}\binom{m_1}{p}\sum_{r=0}^{p}\binom{p}{r}(-m)^{p-r}S_{p,\lambda}^{(m)}(b,\mu^{a})B_{m_1-p,\mu^{a}}(bx|\lambda) \tag{25}$$

Proof. Let

$$h(t)=\frac{t^{m+2}(1+\lambda)^{abxt/\lambda}\left(1-\mu^{ab}(1+\lambda)^{abt/\lambda}\right)^{m}(1+\lambda)^{abyt/\lambda}}{\left(1-\mu^{a}(1+\lambda)^{at/\lambda}\right)^{m+1}\left(1-\mu^{b}(1+\lambda)^{bt/\lambda}\right)^{m+1}}$$
$$=\frac{1}{a^{m+1}b}\left(\frac{at}{1-\mu^{a}(1+\lambda)^{\frac{at}{\lambda}}}\right)^{m+1}(1+\lambda)^{\frac{abxt}{\lambda}}\left(\frac{1-\mu^{ab}(1+\lambda)^{\frac{abt}{\lambda}}}{1-\mu^{b}(1+\lambda)^{\frac{bt}{\lambda}}}\right)\left(\frac{bt}{1-\mu^{b}(1+\lambda)^{bt/\lambda}}\right)(1+\lambda)^{\frac{abyt}{\lambda}}$$
$$=\frac{1}{a^{m+1}b}(-1)^{m+1}\sum_{k=0}^{\infty}B_{k,\mu^{a}}^{(m+1)}(bx|\lambda)\frac{a^{k}t^{k}}{k!}\frac{1}{\mu^{mb}}$$
$$\times\sum_{p=0}^{\infty}\sum_{r=0}^{p}\binom{p}{r}(-m)^{p-r}S_{p,\lambda}^{(m)}\left(a,\mu^{b}\right)\frac{b^{p}t^{p}}{p!}(-1)\sum_{q=0}^{\infty}B_{q,\mu^{b}}(ay|\lambda)\frac{b^{q}t^{q}}{q!}$$
$$=\frac{1}{a^{m+1}b}(-1)^{m}\sum_{n=0}^{\infty}\sum_{m_1=0}^{n}\binom{n}{m_1}B_{n-m_1,\mu^{a}}^{(m+1)}(bx|\lambda)a^{n-m_1}\mu^{-mb}$$
$$\times\sum_{p=0}^{m_1}\binom{m_1}{p}\sum_{r=0}^{p}\binom{p}{r}(-m)^{p-r}S_{p,\lambda}^{(m)}\left(a,\mu^{b}\right)B_{m_1-p,\mu^{b}}(ay|\lambda)b^{m_1}\frac{t^{n}}{n!}. \tag{26}$$

In a similar manner,

$$h(t)=\frac{t^{m+2}(1+\lambda)^{abyt/\lambda}\left(1-\mu^{ab}(1+\lambda)^{abt/\lambda}\right)^{m}(1+\lambda)^{abxt/\lambda}}{\left(1-\mu^{b}(1+\lambda)^{bt/\lambda}\right)^{m+1}\left(1-\mu^{a}(1+\lambda)^{at/\lambda}\right)^{m+1}}$$

$$=\frac{1}{b^{m+1}a}\left(\frac{bt}{1-\mu^{b}(1+\lambda)^{\frac{bt}{\lambda}}}\right)^{m+1}(1+\lambda)^{\frac{abyt}{\lambda}}\left(\frac{1-\mu^{ab}(1+\lambda)^{\frac{abt}{\lambda}}}{1-\mu^{b}(1+\lambda)^{\frac{bt}{\lambda}}}\right)\left(\frac{at}{1-\mu^{a}(1+\lambda)^{at/\lambda}}\right)(1+\lambda)^{\frac{abxt}{\lambda}}$$

$$=\frac{1}{b^{m+1}a}(-1)^{m}\sum_{k=0}^{\infty}B_{k,\mu^{b}}^{(m+1)}(ay|\lambda)\frac{b^{k}t^{k}}{k!}\frac{1}{\mu^{ma}}$$

$$\times\sum_{p=0}^{\infty}\sum_{r=0}^{p}\binom{p}{r}(-m)^{p-r}S_{p,\lambda}^{(m)}(b,\mu^{a})\frac{a^{p}t^{p}}{p!}\sum_{q=0}^{\infty}B_{q,\mu^{a}}(bx|\lambda)\frac{a^{q}t^{q}}{q!}$$

$$=\frac{1}{b^{m+1}a}(-1)^{m}\sum_{n=0}^{\infty}\sum_{m_1=0}^{n}\binom{n}{m_1}B_{n-m_1,\mu^{b}}^{(m+1)}(ay|\lambda)b^{n-m_1}\mu^{-ma}$$

$$\times\sum_{p=0}^{m_1}\binom{m_1}{p}\sum_{r=0}^{p}\binom{p}{r}(-m)^{p-r}S_{p,\lambda}^{(m)}(b,\mu^{a})B_{m_1-p,\mu^{a}}(bx|\lambda)a^{m_1}\frac{t^{n}}{n!}. \tag{27}$$

By comparing the coefficients of $\frac{t^n}{n!}$ in the above equation (26) and (27), we get (25).

4. Some symmetry identities for the modified degenerate Apostol-Euler polynomials

In this section, by using μ-multiple power sums, we give some symmetry identities for the modified degenerate Apostol-Euler polynomials.

Theorem 7. Let a and b be positive integers with the same parity. Then

$$\sum_{k=0}^{n}\binom{n}{k}E_{k,\mu^{a}}(bx|\lambda)\,a^{k}b^{n-k}\mu^{-b}\sum_{r=0}^{n-k}\binom{n-k}{r}(-1)^{n-k-r}Z_{n-k,\lambda}(a;\mu^{b})$$
$$=\sum_{k=0}^{n}\binom{n}{k}E_{k,\mu^{b}}(ax|\lambda)\,b^{k}a^{n-k}\mu^{-a}\sum_{r=0}^{n-k}\binom{n-k}{r}(-1)^{n-k-r}Z_{n-k,\lambda}(b;\mu^{a}). \tag{28}$$

Proof. Let

$$h(t)=\frac{2(1+\lambda)^{abxt/\lambda}}{\left(\mu^{a}(1+\lambda)^{at/\lambda}+1\right)}\frac{1+(-1)^{a+1}\left(\mu^{b}(1+\lambda)^{bt/\lambda}\right)^{a}}{\left(\mu^{b}(1+\lambda)^{bt/\lambda}+1\right)}.$$

From (10) and (17) for *l=1*, we have

$$h(t)=\sum_{k=0}^{\infty}E_{k,\mu^{a}}(bx|\lambda)\frac{a^{k}t^{k}}{k!}\frac{1}{\mu^{b}}\sum_{p=0}^{\infty}\sum_{r=0}^{p}\binom{p}{r}(-1)^{p-r}Z_{n-k,\lambda}(a;\mu^{b})\frac{b^{p}t^{p}}{p!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}E_{k,\mu^{a}}(bx|\lambda)a^{k}b^{n-k}\mu^{-b}\sum_{r=0}^{n-k}\binom{n-k}{r}(-1)^{n-k-r}Z_{n-k,\lambda}(a;\mu^{b})\frac{t^{n}}{n!}. \tag{29}$$

Since $(-1)^{a+1}=(-1)^{b+1}$, the expression for

$$h(t)=\frac{2(1+\lambda)^{abxt/\lambda}}{\left(\mu^{b}(1+\lambda)^{bt/\lambda}+1\right)}\frac{1+(-1)^{b+1}\left(\mu^{a}(1+\lambda)^{at/\lambda}\right)^{b}}{\left(\mu^{a}(1+\lambda)^{at/\lambda}+1\right)}.$$

is symmetric in a and b. Then we obtain the following power series for $h(t)$ by symmetry

$$h(t)=\sum_{k=0}^{\infty}E_{k,\mu^{b}}(ax|\lambda)\frac{b^{k}t^{k}}{k!}\frac{1}{\mu^{a}}\sum_{p=0}^{\infty}\sum_{r=0}^{p}\binom{p}{r}(-1)^{p-r}Z_{n-k,\lambda}(b;\mu^{a})\frac{a^{p}t^{p}}{p!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}E_{k,\mu^{b}}(ax|\lambda)b^{k}a^{n-k}\mu^{-a}\sum_{r=0}^{n-k}\binom{n-k}{r}(-1)^{n-k-r}Z_{n-k,\lambda}(b;\mu^{a})\frac{t^{n}}{n!}. \tag{30}$$

Equating the coefficients of $\frac{t^n}{n!}$ in (29) and (30) for $h(t)$ gives us the desired result.

Theorem 8. Let a and b be positive integers with the same parity. Then

$$\sum_{m_1=0}^{n}\binom{n}{m_1}E_{n-m_1,\mu^a}^{(m+1)}(bx|\lambda)\,a^{n-m_1}b^{m_1}\mu^{-bm}\sum_{p=0}^{m_1}\binom{m_1}{p}\sum_{r=0}^{p}\binom{p}{k}(-m)^{p-r}Z_{p,\lambda}^{(m)}(a;\mu^b)E_{m_1-r,\mu^b}(ay|\lambda)$$
$$=\sum_{m_1=0}^{n}\binom{n}{m_1}E_{n-m_1,\mu^b}^{(m+1)}(ay|\lambda)\,b^{n-m_1}a^{m_1}\mu^{-am}\sum_{p=0}^{m_1}\binom{m_1}{p}\sum_{r=0}^{p}\binom{p}{r}(-m)^{p-r}Z_{p,\lambda}^{(m)}(b;\mu^a)E_{m_1-r,\mu^a}(bx|\lambda) \tag{31}$$

Proof. Let

$$k(t)=\frac{2^{m+2}(1+\lambda)^{abxt/\lambda}}{\left(\mu^a(1+\lambda)^{at/\lambda}+1\right)^{m+1}}\frac{\left[1+(-1)^{a+1}\mu^{ab}(1+\lambda)^{bat/\lambda}\right]^m}{\left(\mu^b(1+\lambda)^{bt/\lambda}+1\right)^{m+1}}(1+\lambda)^{abyt/\lambda}$$
$$=\left(\frac{2}{\mu^a(1+\lambda)^{at/\lambda}+1}\right)^{m+1}(1+\lambda)^{abxt/\lambda}\left(\frac{1+(-1)^{a+1}\mu^{ab}(1+\lambda)^{bat/\lambda}}{\mu^b(1+\lambda)^{bt/\lambda}+1}\right)^m\frac{2}{\mu^b(1+\lambda)^{bt/\lambda}+1}(1+\lambda)^{abyt/\lambda}$$
$$=\sum_{q=0}^{\infty}E_{q,\mu^a}^{(m+1)}(bx|\lambda)\frac{b^qt^q}{q!}\frac{1}{\mu^{bm}}\sum_{p=0}^{\infty}\sum_{r=0}^{p}\binom{p}{r}(-m)^{p-r}Z_{p,\lambda}^{(m)}(a;\mu^b)\frac{b^pt^p}{p!}\sum_{s=0}^{\infty}E_{s,\mu^b}(ay|\lambda)\frac{b^st^s}{s!}$$
$$=\sum_{n=0}^{\infty}\sum_{m_1=0}^{n}\binom{n}{m_1}E_{n-m_1,\mu^a}^{(m+1)}(bx|\lambda)a^{n-m_1}b^{m_1}\mu^{-bm}$$
$$\times\sum_{p=0}^{m_1}\binom{m_1}{p}\sum_{r=0}^{p}\binom{p}{r}(-m)^{p-r}Z_{p,\lambda}^{(m)}(a;\mu^b)E_{m_1-r,\mu^b}(ay|\lambda)\frac{t^n}{n!}. \tag{32}$$

Since $(-1)^{a+1}=(-1)^{b+1}$, the expression for $k(t)$ is symmetric in a and b. In a similar manner, we have

$$k(t)$$
$$=\left(\frac{2}{\mu^a(1+\lambda)^{at/\lambda}+1}\right)^{m+1}(1+\lambda)^{abxt/\lambda}\left(\frac{1+(-1)^{a+1}\mu^{ab}(1+\lambda)^{bat/\lambda}}{\mu^b(1+\lambda)^{bt/\lambda}+1}\right)^m\frac{2}{\mu^b(1+\lambda)^{bt/\lambda}+1}(1+\lambda)^{abyt/\lambda}$$
$$=\sum_{q=0}^{\infty}E_{q,\mu^b}^{(m+1)}(ay|\lambda)\frac{b^qt^q}{q!}\frac{1}{\mu^{am}}\sum_{p=0}^{\infty}\sum_{r=0}^{p}\binom{p}{r}(-m)^{p-r}Z_{p,\lambda}^{(m)}(b;\mu^a)\frac{a^pt^p}{p!}\sum_{s=0}^{\infty}E_{s,\mu^a}(bx|\lambda)\frac{a^st^s}{s!}$$
$$=\sum_{n=0}^{\infty}\sum_{m_1=0}^{n}\binom{n}{m_1}E_{n-m_1,\mu^b}^{(m+1)}(ay|\lambda)b^{n-m_1}a^{m_1}\mu^{-am}$$
$$\times\sum_{p=0}^{m_1}\binom{m_1}{p}\sum_{r=0}^{p}\binom{p}{r}(-m)^{p-r}Z_{p,\lambda}^{(m)}(b;\mu^a)E_{m_1-r,\mu^a}(bx|\lambda)\frac{t^n}{n!}. \tag{33}$$

Equating the coefficients of $\frac{t^n}{n!}$ in (32) and (33) for $k(t)$ gives us the desired result.

Theorem 9. Let p, l, a, b and n be positive integers and a and b be of the same parity. Then

$$\sum_{p=0}^{n}\binom{n}{p}B_{n-p,\mu^a}(n|\lambda)a^n\mu^{-a}\sum_{r=0}^{p}\binom{p}{r}(-1)^{p-r}Z_{p,\lambda}(b;\mu^a)$$
$$=2^{n-1}a^n\left[B_{n,\mu^{2a}}\left(\frac{n}{2}\middle|\lambda\right)+(-1)^{b+1}\mu^{ab}B_{n,\mu^{2a}}\left(\frac{b+n}{2}\middle|\lambda\right)\right]. \tag{34}$$

Proof.

$$g(t)=\frac{at(1+\lambda)^{axt/\lambda}}{\mu^a(1+\lambda)^{at/\lambda}-1}\frac{1+(-1)^{b+1}\mu^{ab}(1+\lambda)^{bat/\lambda}}{\mu^a(1+\lambda)^{at/\lambda}-1}.$$

From (9) and (17), we have

$$g(t)=\sum_{q=0}^{\infty}B_{q,\mu^a}(n|\lambda)\frac{a^q t^q}{q!}\frac{1}{\mu^a}\sum_{p=0}^{\infty}\sum_{r=0}^{p}\binom{p}{r}(-1)^{p-r}Z_{p,\lambda}(b;\mu^a)\frac{a^p t^p}{p!}$$
$$=\sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}B_{n-p,\mu^a}(n|\lambda)a^n\mu^{-a}\sum_{r=0}^{p}\binom{p}{r}(-1)^{p-r}Z_{p,\lambda}(b;\mu^a)\frac{t^n}{n!}.$$

On the other hand, we write the function $g(t)$ as

$$g(t)=\frac{1}{2}\frac{2at\left[(1+\lambda)^{2at/\lambda}\right]^{n/2}}{\mu^{2a}(1+\lambda)^{2at/\lambda}-1}+\frac{(-1)^{b+1}\mu^{ab}2a^t\left[(1+\lambda)^{at/\lambda}\right]^{(n+b)/2}}{2\left[\mu^{2a}(1+\lambda)^{2at/\lambda}-1\right]}$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}B_{n,\mu^{2a}}\left(\frac{n}{2}\middle|\lambda\right)\frac{2^n a^n t^n}{n!}+\frac{(-1)^{b+1}\mu^{ab}}{2}\sum_{n=0}^{\infty}B_{n,\mu^{2a}}\left(\frac{n+b}{2}\middle|\lambda\right)\frac{2^n a^n t^n}{n!}$$
$$=\sum_{n=0}^{\infty}2^{n-1}a^n\left[B_{n,\mu^{2a}}\left(\frac{n}{2}\middle|\lambda\right)+(-1)^{b+1}\mu^{ab}B_{n,\mu^{2a}}\left(\frac{b+n}{2}\middle|\lambda\right)\right]\frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$, we obtain (34).

Acknowledgement

The present investigation was supported, by the Scientific Research Project Administration of Akdeniz University.

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